Strong symmetry defined by twisting modules, applied to quasi–hereditary algebras with triangular decomposition and vanishing radical cube

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Abstract

For a finite dimensional algebra with triangular decomposition, a new kind of modules is defined, the twisting modules. Using the structure of these modules, for some algebras with vanishing radical cube the characteristic tilting module is described and its endomorphism ring is computed. This covers both Temperley–Lieb algebras and q–Schur algebras of finite representation type. Our basic tool is a new symmetry condition, stronger than the symmetry provided by the existence of a triangular decomposition in general.

1 Introduction

Finite dimensional quasi–hereditary algebras occur rather frequently in applications: (classical or quantized) Schur algebras whose modules are the polynomial representations of (classical or quantum) GL_n have this structure as well as the algebras associated with blocks of the BGG–category $\mathcal{O}$ of a semisimple complex Lie algebra and as Temperley–Lieb algebras (see [12]) which are of interest both in statistical mechanics and for constructing the Jones knot polynomial. Triangular decompositions for finite dimensional algebras are the analogue of PBW theorems for (quantized) enveloping algebras and also of crossed products (e.g. the Drinfeld double) for Hopf algebras. Such decompositions exist in many of the situations just mentioned.

Various questions of interest in applications can be asked for the class of algebras with triangular decompositions, or more generally, for quasi–hereditary algebras. A basic problem is to compute quiver and relations of such algebras. This is needed for example in order to apply the sophisticated methods of representation theory of finite dimensional algebras. For many of the algebras mentioned above, the quiver (which has as vertices the isomorphism classes of simple modules and arrows corresponding to extensions between simples) can be computed. For example in the case of category $\mathcal{O}$, knowing the quiver is just an equivalent form of the (proven) Kazhdan–Lusztig conjecture. However, no general methods seem to be known to compute the relations.

Moreover, one would like to be able to compute Ringel's characteristic tilting modules, their endomorphism rings, and the quadratic duals of algebras with triangular decompositions. However, at present these questions seem to be too hard for this quite large class of algebras in general. Thus there are two possible strategies. One strategy is to look for additional properties, in particular symmetry conditions, which are satisfied in (most of) the applications in order to make the class of algebras smaller. The other one is to study subclasses which are easier to access, but contain at least reasonably many examples from applications. As examples of related investigations we mention Dyer's study of blocks of category $\mathcal{O}$ and related algebras [7, 8], and the equivalences Khoroshkin has found between certain categories of D–modules and certain representations of quivers [13].

In this note we study such a subclass. We require the algebras to be quadratic and — a strong restriction — to have vanishing radical cube. This covers all Temperley–Lieb algebras, and a few of the other applications (for example, (quantized) Schur algebras of finite representation type). We give a general construction by quiver and relations of the algebras in this class, and find a condition, satisfied for Temperley–Lieb algebras, under which we prove that the characteristic tilting module is induced from an injective module over the exact Borel subalgebra. (Note that this is not true for the dual extensions defined in [25] and studied in [4]). In this way we compute the characteristic tilting modules and determine their endomorphism rings. We also compute quadratic duals. Verifying the condition for Temperley–Lieb algebras, we also classify the blocks of these algebras.

The condition which we use and which is satisfied in all these examples is an isomorphism between certain induced and certain coinduced modules. This seems to be a 'strong symmetry condition' of the kind one has to look for when following the first of the above strategies.

Such symmetry conditions are also of interest with respect to the following **uniqueness problem** (which at present is completely open): A finite dimensional algebra A having a triangular decomposition $A \simeq C \otimes_S B$ is not uniquely determined by the data C, B, S. In view of the analogy with crossed products for Hopf algebras one would like to find a special such algebra A, analogous to the Drinfeld double, with better symmetry properties. Our strong symmetry condition might be such a symmetry property (see propositions 3.3, 3.4, and section 5, where we use strong symmetry to classify the blocks of Temperley–Lieb algebras).

Our main tool is what we call twisting modules. They are defined in section three (for algebras with triangular decomposition in general). In section four we look at the related concept of twisting relations. In section five we formulate our 'strong symmetry condition'. In section six we give a general construction of algebras with triangular decomposition. In section seven we classify the blocks of Temperley–Lieb algebras. In section eight we produce characteristic tilting modules by induction from an exact Borel subalgebra, and we compute their endomorphism rings. Quadratic duals are computed in section nine. Proofs of all results are given in section ten.

2 Algebras with triangular decomposition

For simplicity we will work with finite dimensional algebras over an algebraically closed field k. Under this assumption, a finite dimensional algebra is Morita equivalent to a basic one (that is, having a product of copies of the field k as maximal semisimple quotient). And a basic k–algebra A can be written as quotient of the path algebra kQ where Q is the opposite quiver of the algebra. In the path algebra kQ, the product of arrows $\alpha : i \to j$ and $\beta : l \to m$ is written as $\alpha\beta$ and defined to be zero if $j \neq l$ and to be the path of length two starting in i and going via α and β to m otherwise. We note that the vertices of Q by definition are the isomorphism classes of simple A–modules. The arrows $i \to j$ form a basis of the k–vector space $Ext^1_A(L(j), L(i))$ where $L(i)$ and $L(j)$ are simple modules representing the classes i and j, respectively.

Modules always are finite dimensional left modules.

First, we recall the definition of a quasi–hereditary algebra as defined by Cline, Parshall, and Scott [2].

Definition 2.1 *Let A be a finite dimensional algebra over a field, and I the set of isomorphism classes of simple A–modules. Choose representatives $L(i)$ of the elements of I. Let $\leq$ be a partial order on I. Then $(A, \leq)$ is called* **quasi–hereditary** *if and only if the following assertions are true:*

(a) For each $i \in I$, there exists a finite dimensional A–module $\Delta(i)$ with an epimorphism $\Delta(i) \to L(i)$ such that the composition factors $L(j)$ of the kernel satisfy $j < i$.

(b) For each $i \in I$, a projective cover $P(i)$ of $L(i)$ maps onto $\Delta(i)$ such that the kernel has a finite filtration with factors $\Delta(j)$ satisfying $j > i$.

The module $\Delta(i)$ is called **standard module** of index i. The objects of the full subcategory $\mathfrak{F}(\Delta)$ of $A - mod$ by definition are the A–modules having a finite filtration with standard modules as factors. These modules are called Δ–good. Injective A–modules are filtered by modules $\nabla(i)$ (which are dual to the standard modules of the quasi–hereditary algebra $(A^{op}, \leq)$).

An algebra $(A, \leq)$ is called **directed** if it is quasi–hereditary with simple standard modules. Equivalently, the Cartan matrix of C is unitriangular with respect to the ordering of rows and columns given by $\leq$. Thus, the projective covers $P(i)$ of $L(i)$ and $P(j)$ of $L(j)$ satisfy $Hom_A(P(i), P(j)) = 0$ unless $i \geq j$ and $End_A(P(i)) = End_A(L(i))$ for all indices i and j.

Exact Borel subalgebras and Δ–subalgebras of quasi–hereditary algebras over fields have been introduced in [14]. Their existence has been shown for the blocks of category $\mathcal{O}$ [14] and for many other quasi–hereditary algebras (see for example [5]). They are defined as follows:

Let $(A, \leq)$ be quasi–hereditary over a field and S a semisimple subalgebra of A which has the same number of simple modules (up to isomorphism) as A. Fix a bijection between the two index sets of simple A– and simple S–modules. A subalgebra B of A

which contains S as maximal semisimple subalgebra (that is, $B/rad(B) \simeq S \subset B$) is called an **exact Borel subalgebra** of $(A, \leq)$ if $(B, \leq)$ is directed, and the induction functor $A \otimes_B -$ is exact and produces standard modules from simple modules: $A \otimes_B L(B,i) \simeq \Delta(i)$. A Δ–**subalgebra** C of $(A, \leq)$ is a subalgebra of A which contains S as maximal semisimple subalgebra, and which has the property that for each primitive idempotent $e(i) \in S$ the epimorphism $A \cdot e(i) \to \Delta(i)$ restricts to an isomorphism $C \cdot e(i) \simeq \Delta(i)$ of C–modules. Thus, standard modules over A are indecomposable projective over C. The algebra C is a Δ–subalgebra of $(A, \leq)$ if and only if C^{op} is an exact Borel subalgebra of $(A^{op}, \leq)$. In particular, C is directed with respect to the partial order $\geq$ which is opposite to $\leq$.

We will use the following properties of these two kind of subalgebras: If BV is an exact Borel subalgebra of A, then the **induction functor** $A \otimes_B -$ is exact. If C is a Δ–subalgebra of A, then the **coinduction functor** $Hom_C(A^{op}, -)$ is exact. (Here, A^{op} is a C–module by restriction of the A–module structure.) Applying these functors to a B–module M or a C–module N, we produce an induced module from M or a coinduced module from N, respectively.

The main theorem in [15] implies that $(A, \leq)$ has both an exact Borel subalgebra B and a Δ–subalgebra C which intersect in S if and only if the multiplication in A induces an isomorphism $C \otimes_S B \simeq A$ of left C– and right B–modules. If in addition $C \simeq B^{op}$ by an isomorphism fixing S elementwise, then an isomorphism $A \simeq C \otimes_S B$ will be called a **triangular decomposition** of $(A, \leq)$.

Exact Borel subalgebras are in analogy to Lie theoretic Borel subalgebras, since there is directedness (which replaces solvability), an exact induction functor (that is, a PBW theorem), and Verma modules (that is, standard modules) are induced from simple modules over the subalgebra.

In [16] it has been shown that the existence of a triangular decomposition is **Morita invariant**. Hence we may work with basic algebras without loss of generality.

3 Twisting modules

Our main technical tool in this paper is twisting modules which will be defined and explained in this section. Because of their potential use for defining symmetry conditions for algebras with triangular decomposition in general we give the most general definition. We define a whole bunch of twisting modules. Later on, we will make use of the smallest and the biggest twisting modules. They are closely related to twisting relations (to be explained below) which arise naturally in triangular definitions.

Let $A \simeq C \otimes_S B$ be an algebra with triangular decomposition where C (and thus $B \simeq C^{op}$) is assumed to be basic (which is no loss of generality), but A is arbitrary. Fix an index i and a B–module M (necessarily indecomposable) having simple socle $L(B,i)$ and Loewy length at most two. Induction produces an A–module $M(A) := A \otimes_B M$ which has a Δ–good filtration with one factor $\Delta(i)$ and all other factors $\Delta(j)$ satisfying $j < i$. Dually, fixing a C–module N having simple top $L(C,i)$ and Loewy length at most two, we produce by coinduction an A–module $N(A) := Hom_C(A^{op}, N)$ which is ∇–good with one factor $\nabla(i)$ and all other factors $\nabla(j)$ satisfying $j < i$. (Here we use the following chain of isomorphisms of A–modules: $\nabla(A,j) \simeq Hom_k(\Delta(A^{op},j),k) \simeq Hom_k(A^{op} \otimes_{C^{op}} L(C^{op},j),k) \simeq Hom_{C^{op}}(L(C^{op},j), Hom_k(A^{op},k)) \simeq Hom_C(A^{op}, L(C,j))$ — note that inside A^{op} the elements of C form an algebra which is isomorphic to C^{op}.) Since each $\nabla(j)$ is injective as a B–module, restricting $N(A)$ to B gives a direct sum decomposition ${}_BN(A) \simeq \nabla(i) \oplus \bigoplus_j \nabla(j)$. Hence $Hom_A(M(A), N(A)) \simeq Hom_B(M, N(A))$ contains an element ψ (unique up to non–zero scalar multiple) which is the B–injective envelope $M \to \nabla(i)$. We denote the image of the A–homomorphism ψ (defined up to A–isomorphism) by $TM(M,N)$ and call it the **twisting module** associated with M and N.

Now we discuss two special cases which will be of particular interest.

Pick three indices i, j, and l, and elements $\alpha \in Ext^1_B(L(B,i), L(B,j))$ and $\beta \in Ext^1_C(L(C,j), L(C,l))$. These elements define short exact sequences whose middle terms are the modules, say, $X(\alpha)$ and $Y(\beta)$, both of length at most two over the algebras B and C, respectively. The twisting module $TM(X(\alpha), Y(\beta))$ alternatively can be constructed by using such extensions in the following way.

Adjointness provides us with an isomorphism $Hom_B(L(B,l), \Delta(A,j)) \simeq Hom_A(\Delta(A,l), \Delta(A,j))$, hence the first space equals k if and only if $l = j$, and is zero otherwise. This implies that as a B–module, $\Delta(A,j)$ has exactly one copy of $L(B,j)$ in the socle. The inclusion produces a map $\varphi : Ext^1_B(-, L(B,j)) \to Ext^1_B(-, \Delta(A,j))$ which sends α to $\varphi(\alpha)$. Exactness of induction yields a canonical isomorphism $Ext^1_B(L(B,i), -) \simeq Ext^1_A(\Delta(A,i), -)$. Composing φ with this isomorphism, α defines an extension

$$0 \to \Delta(A,j) \to E(\alpha) \to \Delta(A,i) \to 0$$

whose middle term decomposes if and only if α is zero. The middle term $E(\alpha)$ is induced from the B–module $X(\alpha)$ of length at most two (containing $L(B,j)$ in the socle) defined by α.

By duality, β produces an extension

$$0 \to \nabla(A,l) \to F(\beta) \to \nabla(A,j) \to 0$$

The module $F(\beta)$ restricted to B is the direct sum of two indecomposable injective B–modules $\nabla(A,j)$ and $\nabla(A,l)$. Hence $Hom_A(E(\alpha), F(\beta)) \simeq Hom_B(X(\alpha), F(\beta))$ contains in particular the map which gives the injective envelope of $L(B,j) \subset X(\alpha) \to \nabla(A,j)$ (in case $\alpha = 0$, that is, when $X(\alpha)$ decomposes, one has to change the notation here). Let ψ be the A–homomorphism defined by this injective envelope via the above isomorphism of homomorphism spaces. Then the **twisting module** $TM(\alpha, \beta)$ is just the image of ψ in $F(\beta)$. It coincides of course with the twisting module associated with $X(\alpha)$ and $Y(\beta)$.

If α is zero, the twisting module is a quotient of $\Delta(A,j)$. If β is zero, the twisting module is a submodule of $\nabla(A,j)$. If both α and β are zero, then the twisting module equals $L(A,j)$.

Immediately from the construction of $TM(\alpha, \beta)$ and properties of $\Delta(A, i)$ and $\nabla(A, l)$ we get:

Proposition 3.1 *The twisting module $TM(\alpha, \beta)$ has top $L(i)$ and socle $L(l)$.* $\blacksquare$

For blocks of category $\mathcal{O}$, and some related algebras (all with triangular decomposition) such special twisting modules can be constructed in an alternative way using Jantzen's translation functors [10, 11] as will be indicated below (in the section on Temperley–Lieb algebras). They play a role in Kazhdan–Lusztig theory, since subtle information like the validity of Vogan's conjecture depends on them (see [3] for a treatment of such questions within the theory of quasi–hereditary algebras). We do not go into detail here.

The other kind of twisting module which we need is the following: Pick an index i, let $I(B, i)$ be the injective B–module with socle $L(B, i)$ and let $M(i)$ be the unique maximal submodule of $I(B, i)$ with Loewy length at most two. Similarly, let $N(i)$ be the unique maximal quotient of the indecomposable projective C–module $C(i)$ with Loewy length at most two. As before we construct the twisting module $TM(M(i), N(i))$, denote it by $TM(i)$ and call it the **universal twisting module** at i. The module $M(i)$ contains any other M, which we may use for constructing a twisting module, and $N(i)$ maps onto any other N, which we may use. Thus we get:

Proposition 3.2 *Each twisting module is a subquotient of a universal twisting module $TM(i)$ for some (unique) index i. The top and the socle of the universal twisting module $TM(i)$ both are isomorphic to a direct sum of simple modules $\bigoplus_j L(j)$ where j runs through indices with $Ext^1_B(L(j), L(i)) \neq 0$ with multiplicity given by the dimension of this extension space.* $\blacksquare$

4 Twisting relations

The construction of algebras with triangular decomposition given in the next section relies very much on what we call twisting relations. They are defined as follows: Let A be an algebra with a fixed triangular decomposition $A \simeq C \otimes_S B$ (with $B \simeq C^{op}$). Then A is generated (as a k–vector space) by elements of the form $c \otimes b$ where c is in C and $b \in B$. Products $(1 \otimes b) \times (c \otimes 1)$ (with b and c running through sets of generators of $rad(B)/rad^2(B)$ and $rad(C)/rad^2(C)$ respectively) can be rewritten as linear combinations of basis elements $c' \otimes b'$, thus there are relations $(1 \otimes b) \times (c \otimes 1) = \sum_i c_i \otimes b_i$ which we call **twisting relations**. In contrast to the relations in C and B which are of course satisfied in A, these twisting relations do not only depend on C and B. They contain more subtle information: indeed, two non–isomorphic algebras A and A' may share a common triangular decomposition, in the sense that there are directed algebras C and B such that there are triangular decompositions $A \simeq C \otimes B \simeq A'$. The difference between A and A' is to be found in the twisting relations.

Let C (and thus $B \simeq C^{op}$) be given by quiver and relations. To fix the twisting relations in A it is enough to define the multiplication of a B–arrow with a C–arrow. Since most of the examples we want to cover are quadratic algebras, we always will use quadratic twisting relations which rewrite such a product as a linear combination of products of C–arrows with B–arrows. Which products do occur with non–zero coefficient in such a linear combination? The next proposition will tell us that this can be read off from twisting modules associated with arrows.

Twisting modules do not depend on the choice of a presentation of the algebra A by quiver and relations. Thus we may choose a special such presentation. In many applications one can choose a presentation of A in such a way that B equals A^+ (the subalgebra of A generated by all vertices and the arrows $i \to j$ with $j > i$) and C equals A^- (the subalgebra of A generated by all vertices and the arrows $i \to j$ with $j < i$). We always assume such a presentation to be admissible (that is, the ideal of relations contains only linear combinations of paths of length at least two). This situation occurs for example for blocks of category $\mathcal{O}$ and also for Temperley Lieb algebras as we will see later on, however it does not always occur for Schur algebras.

Proposition 4.1 *Suppose A is basic and has a triangular decomposition $A \simeq C \otimes_S B$ with $B = A^+$ and $C = A^-$.*

(a) The twisting relations for A are quadratic if and only if all twisting modules $TM(\alpha, \beta)$ (where α and β independently run through all B– and C–arrows respectively) have Loewy length at most three.

(b) Suppose all twisting relations are quadratic. Let $i < j > l$. Then there exists a B–arrow $0 \neq \alpha \in Ext^1_B(L(j), L(i))$ and a C–arrow $0 \neq \beta \in Ext^1_C(L(l), L(j))$ such that $(1 \otimes \alpha) \times (\beta \otimes 1) : i \to j \to l$ occurs with non–zero coefficient in the twisting relation of $i \to h \to l$ if and only if $L(h)$ is a composition factor of $TM(\alpha, \beta)$.

We keep the assumptions of the proposition. Much more information can be read off from the universal twisting module at i: For each triple j, l, h with $j < l > h$ and $j > i < h$ such that there are arrows $j \to l$ and $l \to h$ there is a twisting relation rewriting the path $j \to l \to h$. The path $j \to i \to h$ occurs on the right hand side of this relation with a coefficient, say $\lambda(j, l, h) \in k$. Thus for fixed l we get a matrix $(\lambda(j, l, h))_{j,h}$ which we call the i–th **twisting matrix** at l (compare [5] for a special case of twisting matrices).

Proposition 4.2 *Suppose A is basic, has a triangular decomposition $A \simeq C \otimes_S B$ with $B = A^+$ and $C = A^-$ and all twisting relations are quadratic. Fix two indices i and l and choose j such that $i < j < l$ and there is a B–arrow $j \to l$ and a C–arrow $j \to i$. Let m be the number of all such paths (where j varies).*

Then $L(l)$ occurs as a composition factor with multiplicity m of the universal twisting module $TM(i)$ modulo its socle if and only if the i–th twisting matrix at l is invertible.

5 The strong symmetry condition

Now we can formulate our **strong symmetry condition**.

Definition 5.1 *Suppose the algebra A is basic, has a triangular decomposition $A \simeq C \otimes_S B$ with $B = A^+$ and $C = A^-$ and all twisting relations are quadratic. Then A is said to have* **strong symmetry** *if and only if (for all indices) all twisting matrices are invertible.*

Our results in the previous section now translate as follows:

Corollary 5.1 *Let A be as above and have vanishing radical cube. Then A has strong symmetry if and only if each universal twisting module is both induced and coinduced.* ■

Note that the entries of the twisting matrices depend on the choice of generators of A. We have, however, shown that the invertibility of the matrices is an invariant of the algebra A, once the triangular decomposition is fixed.

6 A general construction

The definition of M–twisted double incidence algebras of posets given in [5] can be generalized to any directed algebra given by a quiver with relations. In this section we restate this definition and show how algebras with triangular decomposition and vanishing radical cube fit into this construction. In order to cover algebras with multiple arrows we have to change the terminology used in [5] a bit. We note that our assumption on the vanishing of radical cube (although we will see that it covers several interesting examples) is rather restrictive if one considers all the examples mentioned in the introduction. For example, for blocks of category $\mathcal{O}$, vanishing radical cube should not be expected in case of $\mathfrak{sl}_3$ or any larger example (except for 'very singular' blocks).

We assume that we are given an algebra C defined by the quiver $\overline{Q} = (Q_0, Q_1)$ and certain relations which we do not have to specify. We denote by Q the underlying graph of the quiver $\overline{Q}$ and by $\overline{Q}^{op}$ the opposite quiver of $\overline{Q}$, the arrows of which will be denoted by α' if α is an arrow in $\overline{Q}$. For an arrow α we denote by $s(\alpha)$ and $t(\alpha)$ the starting vertex and the terminal vertex respectively.

First we define twisting labels. Assume we are given an arrow $\alpha : w \to z$ in $\overline{Q}$ and an arrow $\beta : z \to x$ in $\overline{Q}^{op}$. Let y_i run through all vertices (with multiplicities if multiple arrows occur) such that there is a pair of arrows $w \to y_i$ in $\overline{Q}^{op}$ and $y_i \to x$ in $\overline{Q}$. With each such pair of arrows we associate an element $l(\alpha, \beta, y_i) \in k$, the **label**. The collection of all labels is called the labelling M of C.

Definition 6.1 *Let C be a directed algebra with a labelling M. We define a new algebra $\mathcal{A}(C, M)$ which is given by the quiver obtained from $\overline{Q}$ and the opposite quiver $\overline{Q}^{op}$ by* *forming the union of the two quivers, and identifying the vertices i in $\overline{Q}$ and i in $\overline{Q}^{op}$, and imposing the following three types of relations:*

(1) the relations of the algebra C;

(2) the relations of the algebra C^{op}; and

(3) the **twisting relations***: for each pair α, β in $\overline{Q}$ with $t(\alpha) = t(\beta)$, put the relation*

$$\alpha\beta' = \sum_i l(\alpha, \beta, y_i)\gamma'\delta$$

where the summation runs over all y_i such that there are the arrows $\gamma : y_i \longrightarrow s(\alpha)$ and $\delta : y_i \longrightarrow s(\beta)$ in the quiver $\overline{Q}$.

It is clear that this new algebra $\mathcal{A}(C, M)$ is a finite dimensional k–algebra, and we call it the M–**twisted double** of C.

The construction is made in order to produce algebras with a triangular decomposition $C \otimes C^{op}$. However, in general $\mathcal{A}(C, M)$ need not have this property (see [5]). In fact, the multiplication map $C \otimes_S C^{op} \to \mathcal{A}(C, M)$ always is surjective, but not always injective. However, in the situation we are interested in, this problem does not arise.

Proposition 6.1 *Let C be a basic directed algebra with $rad^2(C) = 0$ and maximal semisimple subalgebra S. Then for any chosen labelling M, the M–twisted double A of C has a triangular decomposition $A \simeq C \otimes_S C^{op}$. Moreover, A has vanishing radical cube and quadratic twisting relations.*

Conversely, any basic algebra A with a triangular decomposition $A \simeq C \otimes_S C^{op}$, quadratic twisting relations, and vanishing radical cube is of this form.

7 Temperley–Lieb algebras

Temperley–Lieb algebras $A_n(\delta)$ (where n is a natural number and $\delta \neq 0$ a field element) can be defined as quotients of Hecke algebras of type A_n, or as algebras generated by diagrams (that is, homotopy classes of n non–intersecting strings, which are multiplied by concatenation, where the parameter δ is used). For more information the reader is referred to [19] (chapters 7 and 9), [9] or [23]. We classify the blocks of these algebras in order to provide examples of algebras as defined in the previous section. This reproves a result of Westbury [23] (and closes a gap in the proof given in [23]).

Proposition 7.1 *For any n and any $\delta \neq 0$, the Temperley–Lieb algebra $A_n(\delta)$ has the following properties:*

(a) It has vanishing radical cube.

(b) Each standard module $\Delta(i)$ and each costandard module $\nabla(i)$ is either simple or has length two.

(c) It has a triangular decomposition with respect to a total order $\leq$ which will be denoted $1 < \ldots < m$ where m is the number of simple modules.

It is convenient now to change notation of simple modules and denote the simple modules of a fixed block by indices $1, \dots, m$ for a certain natural number m. According to [19] and to [9], 2.2 and 2.3, the indices occuring in a block form an orbit under the action of the affine Weyl group $A_1^{(1)}$. We will use this fact in the proof of the next proposition when translating the assertion to be shown there into an assertion on algebraic groups.

Note that Goodman and Wenzl [9] give precise rules for distributing the simple modules among the blocks. We do not repeat these rules here.

The proposition precisely gives us the Cartan matrices of the blocks. In contrast to an assertion in [23] this is however not enough to classify the blocks. There is still a choice inside a finite set of non–isomorphic algebras (depending on whether in the twisting relations the coefficients $scalar(i)$ mentioned in the proof are zero or not).

This problem is solved by the next proposition. We give an overkill proof for the proposition, which has the advantage of showing many other natural examples of twisting relations with non–zero coefficients.

Proposition 7.2 *Let A be a block of a Temperley–Lieb algebra, and i an index which is not minimal. Then the twisting module $TM(i, i+1, i) := TM(\alpha_{i+1}, \beta_{i+1})$ has length four, and contains a composition factor $L(i-1)$. In particular, the strong symmetry condition is satisfied.*

As a consequence we get the classification of blocks:

Corollary 7.3 *A block of the Temperley–Lieb algebra is Morita equivalent to an algebra with the following quiver (the natural number $m \geq 1$ depending on the block)*

$$A:\quad \begin{tikzcd} \bullet \arrow[r, shift left, "\alpha_1"] & \bullet \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & \bullet \arrow[l, shift left, "\beta_2"] & \cdots & \bullet \arrow[r, shift left, "\alpha_m"] & \bullet \arrow[l, shift left, "\beta_m"] \end{tikzcd}$$

modulo the ideal generated by the following relations: $\beta_1 \cdot \alpha_1 = 0$ and for each i with $1 \leq i \leq m-1$: $\alpha_i \cdot \alpha_{i+1} = 0$, $\beta_{i+1} \cdot \beta_i = 0$, $\beta_i \cdot \alpha_i = \alpha_{i+1} \cdot \beta_{i+1}$. $\blacksquare$

As B. Westbury told us (private communication), another way of closing the gap in [23] is to use the double centralizer property which relates blocks of the Temperley–Lieb algebras to (well–known) blocks of $U_q(\mathfrak{sl}(2))$. In fact, the indecomposable projective modules of the blocks of $U_q(\mathfrak{sl}(2))$ are known. And the double centralizer property implies that the Temperley–Lieb algebra is the endomorphism ring of certain modules over these blocks.

We note that the same structure of blocks occurs for q–Schur algebras of finite representation type [24].

8 Characteristic tilting modules as induced modules and Ringel duals

For any quasi–hereditary algebra $(A, \leq)$ Ringel has shown in [21] that for each i (indexing simple modules) there exists a unique indecomposable module $T(i)$ which is both Δ–good and ∇–good such that all standard modules $\Delta(j)$ occuring in a Δ–good filtration of $T(i)$ satisfy $j \leq i$, and $\Delta(i)$ occurs precisely once in this filtration. The direct sum $T = \oplus_i T(i)$ is a tilting module, that is, it does not admit self–extensions and it generates a certain derived category. The module T is called the **characteristic tilting module** of $(A, \leq)$.

In this section we show for the algebras we are studying that $T(i)$ is an induced module of the form $A \otimes_B I(B, i)$ where $I(B, i)$ is indecomposable injective as B–module (hence isomorphic to $\nabla(A, i)$). We also compute the endomorphism ring of T which usually is called the **Ringel dual** of A, and which has been shown by Ringel [21] to be quasi–hereditary. We note that over an algebra with triangular decomposition and singular twisting matrices, the module $T(i)$ need not be an induced module and the Ringel dual may have a different structure (for a study of such situations see [6]).

Theorem 8.1 *Let A be basic with a triangular decomposition $A \simeq C \otimes_S B$ and $B = A^+$ and $C = A^-$. Assume that $rad^2(C) = 0$ and that all twisting relations are quadratic. Let $I(B, i)$ be the indecomposable injective B–envelope of $L(i)$.*

Then the following assertions are equivalent:

(I) The induced module $A \otimes_B I(B, i)$ is both Δ–good and ∇–good, and indecomposable (hence a direct summand of the characteristic tilting module of A).

(II) All i–th twisting matrices are invertible.

(III) The induced module $A \otimes_B I(B, i)$ is isomorphic to the coinduced module $Hom_C(A^{op}, C(i))$.

Note that these conditions are satisfied for Temperley–Lieb algebras.

By definition, the **Ringel dual** of $(A, \leq)$ is the endomorphism ring of the characteristic tilting module of $(A, \leq)$. (This is given only up to Morita equivalence, we choose the basic algebra in this class.)

Theorem 8.2 *Let A be basic with a triangular decomposition $A \simeq C \otimes_S B$ with $B = A^+$ and $C = A^-$. Assume that $rad^2(C) = 0$ and that all twisting relations are quadratic. Assume moreover, that A satisfies the strong symmetry condition.*

Then the Ringel dual $\mathcal{R}(A)$ has a triangular decomposition $\mathcal{R}(A) = B \otimes_S C$ with C and B as before and satisfying $C = \mathcal{R}(A)^+$ and $B = \mathcal{R}(A)^-$. Moreover, for each B–arrow $\alpha : i \to j$ the twisting module $X = TM(\alpha, \alpha^{op})$ over $\mathcal{R}(A)$ has Loewy length three with $rad(X)/soc(X)$ being semisimple of the form $L(j) \oplus \bigoplus_l L(l)$ with l running through the sinks of C–arrows $i \to l$ (with multiplicities).

9 Quadratic duals

As interesting as Ringel duals of quasi–hereditary algebras are quadratic duals and Ext–algebras. In this section we determine the quadratic duals of algebras with triangular decomposition having vanishing radical cube, and we observe that the quadratic duals are precisely the Ext–algebras. It turns out that taking quadratic duals commutes with forming twisted doubles in the following sense: The quadratic dual of an algebra with radical square zero is hereditary and vice versa. Now we show that the quadratic dual of the twisted double of a radical square zero algebra is the twisted double of a hereditary algebra.

An application of Bergman's diamond lemma [1] yields the first part of the following proposition. The second part is a special case of the main result in [17]; it follows also from section three of [5].

Proposition 9.1 *The twisted double of a hereditary algebra (with any labelling) is quasi–hereditary, hence it has a triangular decomposition. Moreover, it has global dimension two, if the hereditary algebra is not semisimple (and zero otherwise).* ∎

Since quadratic algebras of global dimension two are Koszul, the Ext–algebra of such a twisted double of a hereditary algebra coincides with the quadratic dual.

The quadratic dual itself has the following description (which is proved by direct computation as in section two of [5]). Here, M^t denotes the transpose of the matrix M. The matrix $-M$ is obtained from M by multiplying each entry by -1.

Proposition 9.2 *Let C be finite dimensional hereditary and M any labelling. Then the quadratic dual of the M–twisted double of C is the $(-M^t)$–twisted double of $C^{op}/rad^2(C^{op})$.* ∎

This also determines the quadratic dual (coinciding with the Ext–algebra) of algebras with triangular decomposition and vanishing radical cube.

10 Proofs

This section contains the proofs of all the results mentioned before. In each case we keep the notation of the result being proved without repeating it explicitly.

Proof of 4.1. The assumption on A implies that all paths occuring in twisting relations have length at least two (otherwise the presentation of A would not be admissible). Fix $\alpha \in Ext^1_B(L(j), L(i))$ and $\beta \in Ext^1_C(L(l), L(j))$. Let P be the (indecomposable) projective cover of the twisting module $TM(\alpha, \beta)$. Using the twisting relation as in (b) we write $i \to h \to l$ as $a(\alpha\beta) + \gamma$ where a is a scalar (possibly zero) and γ is a linear combination of paths different from $\alpha\beta$. The projective cover map sends the element $i \to h \to l$ of P to an element of $TM(\alpha, \beta)$. Now the residue class of $\alpha\beta$ in the twisting module being non–zero implies the residue class of $i \to h \to l$ to be non–zero, which means that there is a composition factor $L(h)$ (generated by the residue class of $i \to h$) in the twisting module. Conversely, if there is no such composition factor, the residue class of $i \to h \to l$ and hence the coefficient of $\alpha\beta$ in the twisting relation must be zero for all choices of α and β. Similarly, a relation involving paths of length greater than two produces a uniserial submodule of the twisting module, which is of Loewy length greater than two. ∎

Proof of 4.2. Let $M(B,i)$ be the unique largest submodule of $I(B,i)$ having Loewy length two. Each triple i,j,l as in the assumption gives rise to a composition factor $L(j)$ in $A \otimes_B M(B,i)$, and there are precisely m such composition factors. Thus the question is, which of these composition factors are in the image of the map $\psi : M(i) \to N(i)$.

Write $A \otimes_B M(B,i)$ as a quotient of a direct sum of indecomposable projective modules. Assume there is an element $j_1 \to l$ in this sum (a linear combination of arrows) such that the images in $TM(i)$ of all products $j_1 \to l \to j_2$ with a certain arrow $l \to j_2$ (with $j_2 < l$ being the starting vertex of an arrow $j_2 \to i$ and $j_2 < i$) are mapped to zero by ψ. Considering the twisting module associated with the element $j_1 \to i$ in B and the element $i \to j_2$ in C we conclude (by the previous proposition), that in the twisting relation for $j_1 \to l \to j_2$ the element $j_1 \to i \to j_2$ occurs with coefficient zero. But this element is a linear combination of rows of the i–th twisting matrix at l, which thus must be singular. Conversely, if the twisting matrix is singular we can in the same way find such an element which is mapped to zero by ψ. ∎

Proof of 6.1. Considering k–dimensions one sees that A is quasi–hereditary if and only if it has a triangular decomposition. Since the multiplication map $C \otimes_S C^{op} \to A$ always is surjective, it suffices to show that A has the same k–dimension as $C \otimes_S C^{op}$. The diamond lemma [1] provides a basis of A (note that because of $rad^2(C) = 0$ there are no ambiguities to resolve) which proves this assertion.

The triangular decomposition and the twisting relations imply that A has vanishing radical cube.

Conversely, any algebra with triangular decomposition and quadratic twisting relations can be written as $\mathcal{A}(C,M)$. The vanishing of radical cube of A implies the vanishing of radical square of C. ∎

Proof of 7.1. In [19] and in [9] (section two) it is shown that Temperley–Lieb algebras have vanishing radical cube. Thus (a) holds true.

We prove (b) by using certain facts shown in [19] and in [9]. A more computational approach to the statement is contained in [23].

In theorem 2.3 of [9] (see also the discussion preceding that theorem) the radical of a Temperley–Lieb algebra is described as bimodule over certain simple subalgebras. This description implies that all multiplicities $[\Delta(i) : L(j)]$ must be zero or one.

In [9] the Bratteli diagrams of the inclusions $\Lambda = A_n(\delta) \subset \Gamma = A_{n+1}(\delta)$ (if they are semisimple) are given. The partial order of partitions indexing simple modules in these diagrams is the same as the partial order used for defining quasi–heredity. Since standard modules are defined combinatorially and coincide with simple modules in the semisimple case, the Bratteli diagrams carry information on the general case, too. More precisely, from the Bratteli diagram one reads off the following facts: If $\Delta(\Gamma, 1)$ is restricted to Λ, then it coincides with $\Delta(\Lambda, 1)$. If for $1 < i < n$ the standard module $\Delta(\Gamma, i)$ is restricted to Λ, then it has the same composition factors as $\Delta(\Lambda, i-1)$ and $\Delta(\Lambda, i)$ together. For $i = n$ one of the two preceeding situations occurs (depending on n even or odd). Now proceeding inductively (first on the index n of the algebra $A_n(\delta)$, then on the index of simple modules) one checks that each standard module is either simple or of length two: This is clear for $\Delta(1)$ which is simple because of $A_n(\delta)$ being quasi–hereditary. Restricting $\Delta(2)$ gives a module which has length three or two. In the first case the above statement on composition multiplicities zero or one implies that $L(2)$ if restricted has length two. Hence $\Delta(2)$ (unrestricted) has length one or two. Next $\Delta(3)$ restricted has length between two and four. Length two is no problem. Length three splits into two cases; in one of them — one factor with multiplicity two — the fact that $[\Delta(3) : L(2)]$ is zero or one implies that $L(3)$ if restricted must have length two; in the other case — three different factors — we are in the first case we had for $\Delta(2)$ which again implies what we want. Finally length four implies that $\Delta(2)$ restricted has length two which implies that $L(2)$ restricted has length two. Then $L(3)$ restricted must contain the other copy of the second Λ–simple.

This shows that the basic algebras of the blocks of the Temperley Lieb algebra have quivers which are of the following form:

$$\text{A:} \quad \begin{tikzcd} \bullet \arrow[r, shift left, "\alpha_1"] & \bullet \arrow[l, shift left, "\beta_1"] \arrow[r, shift left, "\alpha_2"] & \bullet \arrow[l, shift left, "\beta_2"] & \cdots & \bullet \arrow[r, shift left, "\alpha_m"] & \bullet \arrow[l, shift left, "\beta_m"] \end{tikzcd}$$

Moreover, because of the assertion in (b), the following relations must be satisfied: $\alpha_i \alpha_{i+1} = 0$ and $\beta_{i+1}\beta_i = 0$ (for all $1 \leq i \leq m-1$). Also, there are relations $\alpha_{i+1}\beta_{i+1} = scalar(i)\beta_i\alpha_i$ for all $1 \leq i \leq m-1$ where $scalar(i)$ (depending on i) is zero or one.

This does not fix the algebra up to isomorphism, but all algebras having this quiver and satisfying these relations have a triangular decompositions with B being generated by the vertices and the arrows α_i and C being generated by the vertices and the arrows β_i. This proves (c). ∎

Proof of 7.2. It is enough to show that there is an indecomposable module M of length four having top and socle $L(i)$ and two more composition factors $L(i-1)$ and $L(i+1)$.

The Temperley–Lieb algebra is a quotient of a Hecke algebra, and the quotient map is generic, that is, it is defined independently of the value of the parameter q (that is, of δ). In the semisimple case, the Temperley–Lieb algebra is the direct factor of the Hecke algebra such that its simple modules are associated with partitions with at most two parts. Thus in general, the Temperley–Lieb algebra is the largest quotient of the Hecke algebra which has precisely these simple modules. Hence it is enough to show that such a module exists for the Hecke algebra.

The Hecke algebra can be written as eAe where A is a q–Schur algebra. Therefore it is enough to construct an analogue of M over the q–Schur algebra (the Schur functor $e \cdot -$ then will send this module to M since it preserves simple modules and it preserves top and socle).

Now M exists if an analogue exists over the classical Schur algebra, since finite dimensional modules can be quantized.

But over the classical Schur algebra, such a twisting module can be constructed using Jantzen's translation functor (see [11], 7.19). Here we have to use the parametrization of simple modules in the block of the Temperley–Lieb algebra as orbits under the action of the affine Weyl group as given in [9]. ■

Proof of 8.1. The induced and the coinduced module appearing in (III) have the same composition factors with the same multiplicities, hence the same dimension. Since the universal twisting module at i, $TM(i)$, has the same socle as the coinduced module and the same top as the induced module it follows that (III) holds if and only if the map ψ provides an isomorphism between the induced module and $TM(i)$ if and only if $TM(i)$ coincides with the coinduced module. Thus the equivalence of (II) and (III) follows from proposition 4.2.

Suppose (II) and (III) are satisfied. The triangular decomposition implies the exactness of $A \otimes_B -$, and this induction functor sends simple modules to standard modules. Hence $A \otimes I(B,i)$ is Δ–good. Moreover, $I(B,i) \simeq \nabla(A,i)$ has Loewy length two, socle $L(i)$ and top $\oplus_l L(l)$ where l runs through the sinks of the C–arrows $i \to l$. Hence $A \otimes I(B,i)$ occurs in an exact sequence of A–modules

$$(*) \quad 0 \to \Delta(i) \to A \otimes I(B,i) \to \oplus_l \Delta(l) \to 0$$

Condition (III) implies that $A \otimes_B I(B,i)$ is ∇-good, too, and it equals $TM(i)$, the universal twisting module at i. In particular, the socle of $A \otimes I(B,i)$ equals the socle of $TM(i)$ which is the socle of $\Delta(i)$.

This implies that $A \otimes I(B,i)$ is indecomposable. In fact, in a direct sum decomposition $A \otimes I(B,i) = X \oplus Y$, one summand, say X, must contain the unique composition factor $L(i)$, hence X contains the submodule $\Delta(i)$ which is generated by that composition factor, thus it contains all of $soc(A \otimes I(B,i))$, and Y equals 0.

Conversely, assume (I) is satisfied. Because of the exact sequence $(*)$, the socle of $A \otimes I(B,i)$ contains the socle of $\Delta(i)$. Hence for each direct summand $L(j)$ of the socle of $\Delta(i)$, there must be a factor $\nabla(j)$ in the ∇–good filtration of $A \otimes I(B,i)$. Counting composition factors shows that $A \otimes_B I(B,i)$ modulo these factors must be isomorphic to $\nabla(i)$. Hence the socle of $A \otimes I(B,i)$ equals the socle of $\Delta(i)$ which implies that $A \otimes I(B,i)$ is contained in, hence equal to, the coinduced module. ■

Proof of 8.2. From theorem 8.1 we know that the characteristic tilting module of A is $\oplus_i T(i)$ where by definition $T(i) := A \otimes_B I(B,i)$. By adjointness we have that as a vector space $Hom_A(A \otimes_B I(B,i), A \otimes_B I(B,j))$ is isomorphic to $Hom_B(I(B,i), A \otimes_B I(B,j))$. Moreover, $T(j) = A \otimes_B I(B,j)$ as a B–module is the direct sum of the injective

B–modules $\nabla(j) \oplus \bigoplus_l \nabla(l)$ where l runs through the sinks of C–arrows $j \to l$ (with multiplicities). Now the B–homomorphisms $\nabla(i) \to \nabla(l)$ are precisely all the linear combinations of B–arrows $i \to l$. Hence the following maps form a k–basis of the algebra $\mathcal{R}(A)$ (where i and j independently run through all indices):

(1) the identity map on $T(i)$,

(2) the map $T(i) \to T(j)$ defined by a C–arrow $i \to j$: it sends $\nabla(i)$ to $\nabla(j)$ via this arrow (hence $i > j$),

(3) the map $T(i) \to T(j)$ defined by a B–arrow $i \to j$: it sends $\nabla(i)$ identically to $\nabla(i)$ (which occurs in the ∇–good filtration of $T(j)$) (hence $i < j$),

(4) the map $T(i) \to T(j)$ defined by a B–arrow $h \to j$ and a C–arrow $i \to j$ as a composition of maps as in (2) and (3).

The maps in (2) and (3) are irreducible in the sense that they cannot be written in a non–trivial way as a linear combination of products of maps in this list, whilst the maps of type (4) can. The maps of type (2) and (3) define the arrows in the quiver of $\mathcal{R}(A)$. Moreover, via the maps in (2) and (3), the algebras B and C become subalgebras of $\mathcal{R}(A)$ which intersect in S (the maximal semisimple subalgebra of $\mathcal{R}(A)$, which is generated by the maps of type (1)).

By [21], the Ringel dual is quasi–hereditary with respect to the partial order $\geq$ opposite to the given one $\leq$. Its standard modules are of the form $Hom_A(T, \nabla(i))$. This (or the above basis) implies that $B \otimes_S C$ is a triangular decomposition of $\mathcal{R}(A)$.

From the above basis we read off that the Ringel dual has quadratic twisting relations again (and has vanishing radical cube).

In order to prove the statement on twisting modules, it is enough to show the following: Let $\alpha = \beta\gamma$ be the composition of two non–zero maps $\beta : T(i) \to T(l)$ (with $l > i$) of type (3), and $\gamma : T(l) \to T(i)$ of type (2). Then α is not zero. This is because the first map, β, sends $\nabla(i)$ isomorphically to a certain copy of $\nabla(i)$ occuring in a ∇–good filtration of $T(l)$. Hence it sends $top(T(i)) = top(\nabla(i))$ to the top of this copy of $\nabla(i)$. The second map, however, sends $top(\nabla(i))$ isomorphically to $soc(\Delta(i))$ (here we use the Δ–filtrations of $T(l)$ and of $T(i)$). ■

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