Biextensions by indecomposable modules of derived regular length 2

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1 Introduction and main results

In this paper 'algebra' means a connected, basic, finite-dimensional algebra (associative, with 1) over an algebraically closed field k. For an algebra A we denote by $A-\mathrm{mod}$ the category of finite-dimensional left A-modules and by $D^b(A)$ the derived category of bounded complexes over $A-\mathrm{mod}$.

By Drozd's 'Tame and Wild Theorem' ([Dd], see also [CB1] and [G-V]) the class of algebras may be divided into two disjoint parts. Firstly, there are the tame algebras for which the indecomposable modules occur, in each dimension d, up to isomorphism in a finite number of discrete and a finite number of one-parameter families. Secondly, there are the wild algebras whose representation theory is at least as complicated as the study of finite-dimensional vector spaces with two non-commuting endomorphisms, for which the classification up to isomorphism is a well-known unsolved problem.

We are concerned with the problem of deciding when a given algebra is tame. Frequently, using deformations and coverings, we may reduce it to the tameness problem for algebras whose ordinary quiver is directed. For this class of algebras we may often solve it by starting with known classes of tame algebras and applying iterated one-point extensions and coextensions.

At present, the most developed is the representation theory of tame algebras of polynomial growth (see [GP], [Sk] for surveys and more references). In this theory an important role is played by the representation-infinite tilted algebras of Euclidean type and their suitable enlargements ([AS], [AST], [Ri2]). The representation theory of tame non-polynomial growth algebras is rather poor and only few classes of algebras are understood ([BR], [CB2], [DS], [Er], [WW]).

In [Ri1] it is shown that, if H is a tame hereditary algebra of Euclidean type $\widetilde{\mathbb{D}}_n$ and R is an indecomposable regular H-module of regular length 2 in a tube of rank $n-2$, then the one-point extension $H[R]$ is tame but not of polynomial growth. In general, it is open what happens if one considers the multiple one-point extension $H[R_1,\dots,R_t] := H[R_1]\dots[R_t]$ where the R_i are H-modules like R above. The reason is that if one wants to work by induction, then the module categories of the intermediate algebras $A[R_1,\dots,R_i]$ are not well-understood and consequently the usual one-point extension technique based on the calculation of the vector space category $(H[R_1,\dots,R_i]-\mathrm{mod}, \mathrm{Hom}_{H[R_1,\dots,R_i]}(R_{i+1},-))$ fails.

The aim of this paper is to solve more general problems. In our approach we apply derived categories as introduced in [Ha] and recent results on fiber sum functors and generalized one-point extensions proved by the first named author in [Dr1], [Dr3].

As preparation we need to introduce a generalization of one-point extension and coextension algebras. For this purpose let $R'_1, \dots, R'_s$ and $R_1, \dots, R_t$ be two sequences of modules over an algebra B. We put $R' := \oplus_{i=1}^{s} R'_i$ which we consider as an B-k^s-bimodule. Analogously we consider $R := \oplus_{i=1}^{t} R_i$ as an B-k^t-bimodule. The *biextension algebra* $[R'_1, \dots, R'_s]B[R_1, \dots, R_t]$ of B by the two sequences $R'_1, \dots, R'_s$ and $R_1, \dots, R_t$ is by definition the matrix algebra

$$\begin{pmatrix} k^t & 0 & 0 \\ R & B & 0 \\ \mathrm{D}(R') \otimes_B R & \mathrm{D}(R') & k^s \end{pmatrix}$$

equipped with the obvious addition and multiplication. For $s = 0$ (resp. $t = 0$) one obtains the usual iterated one-point extension $B[R_1, \dots, R_t]$ (resp. iterated one-point coextension $[R'_1, \dots, R'_s]B$). If E is the right adjoint of the canonical restriction functor $B[R_1, \dots, R_t]-\mathrm{mod} \to B-\mathrm{mod}$ and L is the left adjoint of the canonical restriction functor $[R'_1, \dots, R'_s]B-\mathrm{mod} \to B-\mathrm{mod}$, then $[R'_1, \dots, R'_s]B[R_1, \dots, R_t] = [E(R'_1), \dots, E(R'_s)](B[R_1, \dots, R_t]) = ([R'_1, \dots, R'_s]B)[L(R_1), \dots, L(R_t)]$.

We shall describe now the main results of the paper. Let $\vec{\Delta}$ be a finite, directed, connected quiver with underlying graph Δ. Following [Ha] an algebra B is said to be piecewise hereditary of type $\vec{\Delta}$ if there is a triangle equivalence $F : D^b(B) \to D^b(H)$ where H is the finite-dimensional hereditary algebra $k\vec{\Delta}$. For Δ an Euclidean graph the structure of $D^b(H)$ is known rather precisely. Namely, the Auslander-Reiten quiver consists of a sequence $\mathcal{P}[\nu]$, $\nu \in \mathbb{Z}$, of directed components of type $\mathbb{Z}\vec{\Delta}$ and a sequence $\mathcal{R}[\nu]$, $\nu \in \mathbb{Z}$, of families of stable tubes of tubular type Δ^0 where Δ^0 is the Dynkin diagram attached naturally to Δ. By abuse of language, we also denote by $\mathcal{P}[\nu]$ and $\mathcal{R}[\nu]$ the corresponding full subcategories of $D^b(H)$ which are known to be standard. Note that the $\mathcal{R}[\nu]$ are abelian length categories.

After embedding $B-\mathrm{mod}$ into $D^b(B)$ in the usual way, we call *derived regular* just those modules $X \in B-\mathrm{mod}$ such that $F(X)$ lies in one of the subcategories $\mathcal{R}[\nu]$. For a derived regular module X we denote its length as object of $\mathcal{R}[\nu]$ as *derived regular length* of X.

We will denote by $\tau_{D^b(H)}$ the Auslander-Reiten translation in $D^b(H)$ and by T the shift functor. Using τ, we will usually skip the index. The $\mathcal{P}[\nu]$ and $\mathcal{R}[\nu]$ are arranged to satisfy $T\mathcal{P}[\nu] = \mathcal{P}[\nu + 1]$ and $T\mathcal{R}[\nu] = \mathcal{R}[\nu + 1]$.

Two objects X, Y of an additive category are said to be *Hom-orthogonal* if there is no non-zero map from X to Y and from Y to X. With these notations our first theorem is the following:

Theorem A. *Let B be a piecewise hereditary algebra of type $\vec{\Delta}$ where Δ is $\widetilde{\mathbb{A}}_n$ or $\widetilde{\mathbb{D}}_n$. Suppose $\{R'_1, \dots, R'_s\}$ and $\{R_1, \dots, R_t\}$ are sets of indecomposable derived regular B-modules of derived regular length 2 such that their images under F lie in non-homogeneous*

tubes in case $\widetilde{\mathbb{A}}_n$ and in one T-orbit of tubes of rank $n-2$ in case $\widetilde{\mathbb{D}}_n$. Then the biextension algebra $[R'_1,\ldots,R'_s]B[R_1,\ldots,R_t]$ is of tame representation type provided that for all $\nu\in\mathbb{Z}$ the following conditions are satisfied:
(i) The set $F^{-1}\mathcal{R}[\nu]\cap\{R_1,\ldots,R_t\}$ consists of pairwise Hom-orthogonal modules.
(ii) The set $F^{-1}\mathcal{R}[\nu]\cap\{R'_1,\ldots,R'_s\}$ consists of pairwise Hom-orthogonal modules.
(iii) There is no non-zero homomorphism from any module in $F^{-1}\mathcal{R}[\nu]\cap\{R_1,\ldots,R_t\}$ to any module in $F^{-1}\mathcal{R}[\nu+1]\cap\{R'_1,\ldots,R'_s\}$.

It is easy to construct examples which are wild if one of the three conditions on the modules R_i and R'_i is not satisfied. In this sense the theorem is optimal. Nevertheless, our second main theorem will show how one can weaken the third condition on the cost of allowing to factor out from the biextension algebra $[R'_1,\ldots,R'_s]A[R_1,\ldots,R_t]$ an ideal which only relates the extension and coextension vertices. Let us introduce these ideals systematically:

The tensor product $\mathrm{D}(R')\otimes_B R$ appearing in the lower left corner of the biextension algebra carries the structure of a k^s-k^t-bimodule. Any subbimodule W yields an ideal

$$\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ W & 0 & 0\end{pmatrix}$$

of the algebra $[R'_1,\ldots,R'_s]A[R_1,\ldots,R_t]$. We will denote this ideal by $J(W)$. To define the particular subbimodule we are interested in, let us decompose R as $\oplus_{\nu\in\mathbb{Z}}R_\nu$ and R' as $\oplus_{\nu\in\mathbb{Z}}R'_\nu$ where R_ν is the coproduct over all R_i such that FR_i is in $\mathcal{R}(\nu)$ and R'_ν is the coproduct over all R'_j such that FR'_j is in $\mathcal{R}(\nu)$. Then the subbimodule $\oplus_{\nu\in\mathbb{Z}}\mathrm{D}(R'_{\nu+1})\otimes_B R_\nu$ of the bimodule $\mathrm{D}(R')\otimes_B R$ allows to formulate the following generalization of Theorem A:

Theorem B. *Let B be a piecewise hereditary algebra of type $\vec{\Delta}$ where Δ is $\widetilde{\mathbb{A}}_n$ or $\widetilde{\mathbb{D}}_n$. Suppose $\{R'_1,\ldots,R'_s\}$ and $\{R_1,\ldots,R_t\}$ are sets of indecomposable derived regular B-modules of derived regular regular length 2 such that their images under F lie in non-homogeneous tubes in case $\widetilde{\mathbb{A}}_n$ and in one T-orbit of tubes of rank $n-2$ in case $\widetilde{\mathbb{D}}_n$. Then the algebra $[R'_1,\ldots,R'_s]B[R_1,\ldots,R_t]/J(\oplus_{\nu\in\mathbb{Z}}\mathrm{D}(R'_{\nu+1})\otimes_B R_\nu)$ is of tame representation type provided that for all $\nu\in\mathbb{Z}$ the following conditions are satisfied:*
(i) The set $\mathcal{R}[\nu]\cap\{FR_1,\ldots,FR_t\}$ consists of pairwise Hom-orthogonal modules.
(ii) The set $\mathcal{R}[\nu]\cap\{FR'_1,\ldots,FR'_s\}$ consists of pairwise Hom-orthogonal modules.
(iii) For all $\nu\in\mathbb{Z}$ the set $\mathcal{R}[\nu]\cap\{FR_1,\ldots,FR_t\}$ does not contain any object which is isomorphic to an object in the set $T^{-1}\tau^-(\mathcal{R}[\nu+1]\cap\{FR'_1,\ldots,FR'_s\})$.

The first two conditions are identical with those appearing in Theorem A. We used slightly different formulations in order to emphasize that in Theorem A all the three conditions can be verified in the module category $B-$mod once one can identify the B-modules which are mapped to the $\mathcal{R}[\nu]$ by F. In contrast, for verifying condition (iii) in Theorem B one really has to work in the derived category $D^b(H)$ because its endofunctors τ^- and T^{-1} are used. Thus Theorem A seems to be easier applicable in concrete situations. On the other hand it, is easy to see that Theorem A is an immediate corollary of Theorem

B. Namely, the isomorphism $\mathrm{D}(R_j') \otimes_B R_i \cong \mathrm{D}\,\mathrm{Hom}_B(R_i, R_j')$ shows that under condition (iii) of Theorem A the bimodule $\oplus_{\nu\in\mathbb{Z}} \mathrm{D}(R'_{\nu+1}) \otimes_B R_\nu$ and therefore its induced ideal is zero.

In order to apply Theorem A in practice, one should use the rather precise description of piecewise hereditary algebras of Euclidean type and their module categories presented in [AS]. It is shown that a representation infinite algebra B is piecewise hereditary of Euclidean type if and only if it is a domestic branch enlargement of a tame concealed algebra.

The sections 2-5 are devoted to the proof of Theorem B. Actually, we will only deal with the case $\widetilde{\mathbb{D}}_n$ since the proof for case $\widetilde{\mathbb{A}}_n$ is rather parallel. In section 2 we first translate the problem to the tameness of naturally associated multiple vector space categories with relations. In section 3 we introduce some useful operations on these categories. Finally, in section 4 we translate the question back to one special algebra C whose tameness is proved in section 5. For this we observe that C can be obtained inductively by generalized one-point extensions. This allows to use ordinary vector space categories whose tameness can be established.

We use the notation from [GR] and [Ri2]. Morphisms in an aggregate K are composed from right to left. By $\mathrm{ind}\,K$ we denote a spectroid of K. Usually we do not distinguish between an indecomposable object of K and its isomorphism class. For a k-algebra A we denote by $A-$mod the category of all finite-dimensional left A-modules and by $A-$ind a spectroid of $A-$mod.

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2 Multiple vector space categories and biextension algebras

2.1. A vector space category is a pair (K, M) consisting of an aggregate K and a k-functor $M : K \to k-\mathrm{mod}$. The attached subspace category $\check{\mathcal{U}}(K, M)$ and factorspace category $\check{\mathcal{V}}(K, M)$ are again aggregates. The objects of $\check{\mathcal{U}}(K, M)$ are the triples $U = (U_\omega, \gamma_U, U_0)$ where $U_\omega \in k-\mathrm{mod}$, $U_0 \in K$ and $\gamma_U \in \mathrm{Hom}_k(U_\omega, M(U_0))$. Analogously, the objects of $\check{\mathcal{V}}(K, M)$ are the triples $U = (U_0, \delta_U, U_\alpha)$ where $U_\alpha \in k-\mathrm{mod}$, $U_0 \in K$ and $\delta_U \in \mathrm{Hom}_k(M(U_0), U_\alpha)$. The morphisms from U to V in the subspace category (resp. factorspace category) are pairs $f = (f_\omega, f_0)$ (resp. $f = (f_0, f_\alpha)$) where $f_\omega : U_\omega \to V_\omega$ (resp. $f_\alpha : U_\alpha \to V_\alpha$) is k-linear and f_0 is a morphism in K satisfying $\gamma_V f_\omega = M(f_0)\gamma_U$ (resp. $f_\alpha \delta_U = \delta_V M(f_0)$).

A generalization of vector space categories was introduced in [Si1]. We will use a variant of this generalization which we call multiple vector space category. A *multiple vector space category* is a triple (M_ω, K, M_α) consisting of an aggregate K and two k-functors $M_\omega : K \to k^t-\mathrm{mod}$, $M_\alpha : K \to k^s-\mathrm{mod}$ where $s, t \in \mathbb{N}_0$ and k^t, k^s are considered as k-algebras via the componentwise multiplication. The category

$\check{\mathcal{M}}(M_\omega, K, M_\alpha)$ of *representations of the multiple vector space category* (M_ω, K, M_α) has as objects the quintuples $U = (U_\omega, \gamma_U, U_0, \delta_U, U_\alpha)$ where where $U_\omega, U_\alpha \in k-\mathrm{mod}$, $U_0 \in K$ and $\gamma_U \in \mathrm{Hom}_k(U_\omega, M_\omega(U_0))$, $\delta_U \in \mathrm{Hom}_k(M_\alpha(U_0), U_\alpha)$. The morphisms from U to V in $\check{\mathcal{M}}(M_\omega, K, M_\alpha)$ are triples $f = (f_\omega, f_0, f_\alpha)$ where $f_\omega : U_\omega \to V_\omega$, $f_\alpha : U_\alpha \to V_\alpha$ are k-linear and f_0 is a morphism in K satisfying $\gamma_V f_\omega = M(f_0)\gamma_U$ and $f_\alpha \delta_U = \delta_V M(f_0)$. Of course, $\check{\mathcal{M}}(M_\omega, K, M_\alpha)$ is again an aggregate.

If $s + t = 1$, then (M_ω, K, M_α) is an ordinary vector space category and $\check{\mathcal{U}}(K, M_\omega) = \check{\mathcal{M}}(M_\omega, K, M_\alpha)$ for $t = 1$ and $s = 0$ whereas $\check{\mathcal{V}}(K, M_\alpha) = \check{\mathcal{M}}(M_\omega, K, M_\alpha)$ for $t = 0$ and $s = 1$.

A multiple vector space category (M_ω, K, M_α) is said to be *faithful* provided the functor $M := M_\omega \oplus M_\alpha$ is faithful. If M fails to be faithful, one may pass to the reduced multiple vector space category $(M_\omega, K, M_\alpha)_{red} := (M_\omega, K/\operatorname{Ker} M, M_\alpha)$ which is obviously faithful. The full additive subcategory K' of K defined by the property that its indecomposable objects U satisfy $M(U) = 0$ contributes only the trivial indecomposables $(0, 0, U, 0, 0)$ to the category $\check{\mathcal{M}}(M_\omega, K, M_\alpha)$. If we consider the categorical complement $K^* := K \setminus K'$, then the canonical functor which maps a morphism $f = (f_\omega, f_0, f_\alpha)$ in $\check{\mathcal{M}}(M_\omega, K^*, M_\alpha)$ to the morphism $(f_\omega, f_0 + \operatorname{Ker} M, f_\alpha)$ in $\check{\mathcal{M}}(M_\omega, K, M_\alpha)_{red}$ is full and dense, its kernel being contained in the radical of $\check{\mathcal{M}}(M_\omega, K^*, M_\alpha)$. Hence the categories $\check{\mathcal{M}}(M_\omega, K^*, M_\alpha)$ and $\check{\mathcal{M}}(M_\omega, K, M_\alpha)_{red}$ are representation equivalent.

2.2. If B is an algebra and R a B-module, then the one-point extension of B by R is the triangular matrix algebra $B[R] := \begin{pmatrix} k & 0 \\ R & B \end{pmatrix}$ and the one-point coextension of B by R is the triangular matrix algebra $[R]B := \begin{pmatrix} B & 0 \\ D(R) & k \end{pmatrix}$. It is well-known that $B[R]-\mathrm{mod}$ can be identified with the subspace category $\check{\mathcal{U}}(B-\mathrm{mod}, \mathrm{Hom}_B(R, -))$ of the vector space category $(B-\mathrm{mod}, \mathrm{Hom}_B(R, -))$ and $[R]B-\mathrm{mod}$ with the factorspace category of $\check{\mathcal{V}}(B-\mathrm{mod}, D(R)\otimes_B(-))$ of the vector space category $(B-\mathrm{mod}, D(R)\otimes_B(-))$.

This generalizes to the biextension algebra $[R'_1, \ldots, R'_s]B[R_1, \ldots, R_t]$ of B given by the two sequences $R'_1, \ldots, R'_s$ and $R_1, \ldots, R_t$ of B-modules in the obvious way. Namely, $[R'_1, \ldots, R'_s]B[R_1, \ldots, R_t]-\mathrm{mod}$ can be identified with $\check{\mathcal{M}}(\mathrm{Hom}_B(R, -), B-\mathrm{mod}, D(R') \otimes_B (-))$ where R is the B-k^t-bimodule $\oplus_{i=1}^t R_i$ and R' is the B-k^s-bimodule $\oplus_{i=1}^s R'_i$.

2.3. In Theorem B we have to deal with a factor algebra of a biextension algebra and therefore we have to translate the property of being a module over this factor algebra into the language of representations of a multiple vector space category. Let (M_ω, K, M_α) be a multiple vector space category and Θ be a set of natural transformations $\theta : M_\omega \to M_\alpha$. We call $(M_\omega, K, M_\alpha; \Theta)$ a *multiple vector space category with relations*. Its category of representations $\check{\mathcal{M}}(M_\omega, K, M_\alpha; \Theta)$ is defined as the full subcategory of all U in $\check{\mathcal{M}}(M_\omega, K, M_\alpha)$ such that $\delta_U \theta_{U_0} \gamma_U = 0$ for all $\theta \in \Theta$. Note, that $\check{\mathcal{M}}(M_\omega, K, M_\alpha; \Theta)_{red}$ becomes a full subcategory of $\check{\mathcal{M}}(M_\omega, K, M_\alpha)_{red}$.

For a biextension algebra $[R'_1, \ldots, R'_s]B[R_1, \ldots, R_t]$ and a subbimodule W of $\mathrm{D}(R')\otimes_B R$, any element $\sum_{i=1}^n d_i \otimes r_i$ of W furnishes a natural transformation $\theta : \mathrm{Hom}_B(R, -) \to D(R')\otimes_B(-)$ which for X in $B-\mathrm{mod}$ sends a homomorphism $f : R \to X$ to $\sum_{i=1}^n d_i \otimes f(r_i)$.

Let us denote by Θ_W the set of all these natural transformations. Under the natural identification of $[R'_1, \dots, R'_s]B[R_1, \dots, R_t]-\mathrm{mod}$ with $\check{\mathcal{M}}(\mathrm{Hom}_B(R,-), B-\mathrm{mod}, D(R') \otimes_B (-))$ the full subcategory $[R'_1, \dots, R'_s]B[R_1, \dots, R_t]/J(W)-\mathrm{mod}$ is identified with $\check{\mathcal{M}}(\mathrm{Hom}_B(R,-), B-\mathrm{mod}, D(R') \otimes_B (-); \Theta_W)$. Obviously, it is sufficient to put into Θ_W only a generating set of the bimodule W.

2.4. Note that for a fixed k^t-module W_ω, a fixed k^s-module W_α and a fixed object X of K the set of objects U in $\check{\mathcal{M}}(M_\omega, K, M_\alpha)$ satisfying $U_\omega = W_\omega$, $U_\alpha = W_\alpha$ and $U_0 = X$ can be identified with the linear variety of pairs of matrices (G, D) over k where G has $\dim_k W_\omega$ columns and $\dim_k M_\omega(X)$ rows whereas D has $\dim_k M_\alpha(X)$ columns and $\dim_k W_\alpha$ rows. Moreover, any natural transformation $\theta : M_\omega \to M_\alpha$ furnishes a matrix T_θ with $\dim_k M_\omega(X)$ columns and $\dim_k M_\alpha(X)$ rows. Thus, the set of pairs (G, D) lying in $\check{\mathcal{M}}(M_\omega, K, M_\alpha; \Theta)$ is the closed subvariety given by all (G, D) such that $DT_\theta G = 0$ for all $\theta \in \Theta$.

In this way 1-parameter families and tameness are defined for multiple vector space categories with relations $(M_\omega, K, M_\alpha; \Theta)$ like for usual vector space categories. By adapting the proof of [Dr1, Theorem 3.3] one obtains the following result:

Lemma. *Let us consider a biextension algebra $[R'_1, \dots, R'_s]B[R_1, \dots, R_t]$, let W be a subbimodule of $\mathrm{D}(R') \otimes_B R$ and let $(\mathrm{Hom}_B(R,-), B-\mathrm{mod}, \mathrm{D}(R') \otimes_B (-); \Theta_W)$ be the associated multiple vector space category with relations. Then the algebra $[R'_1, \dots, R'_s]B[R_1, \dots, R_t]/J(W)$ is tame if and only if the following three conditions are satisfied:*
(i) The algebra B is tame.
(ii) For every $n \in \mathbb{N}$ the subset of $B-\mathrm{ind}$ given by all X such that $\dim_k X \leq n$ and $\mathrm{Hom}_B(R, X) \oplus (D(R') \otimes_B X) \neq 0$ is finite.
(iii) $(\mathrm{Hom}_B(R,-), B-\mathrm{mod}, D(R') \otimes_B (-); \Theta_W)_{red}$ is a tame multiple vector space category with relations.

2.5. Let (M_ω, K, M_α) be a multiple vector space category. We denote by $S_1, \dots, S_t$ (resp. $S'_1, \dots, S'_s$) the simple k^t- resp. k^s-modules. It is easy to see that U in $\check{\mathcal{M}}(M_\omega, K, M_\alpha)$ does not admit a summand isomorphic to some $(S_i, 0, 0, 0, 0)$ or $(0, 0, 0, 0, S'_j)$ if and only if γ_U is injective and δ_U is surjective. We denote by $\mathcal{M}(M_\omega, K, M_\alpha)$ the full subcategory defined by objects of this shape. In the case of ordinary vector space categories this coincides with the usual constructions of $\mathcal{U}(K, M)$ and $\mathcal{V}(K, M)$.

For any set of relations Θ the objects $(S_i, 0, 0, 0, 0)$ and $(0, 0, 0, 0, S'_j)$ belong to $\check{\mathcal{M}}(M_\omega, K, M_\alpha; \Theta)$. Thus we can introduce $\mathcal{M}(M_\omega, K, M_\alpha; \Theta)$ in the analogous way. Since only finitely many isoclasses of indecomposable representations are lost by passing from $\check{\mathcal{M}}(M_\omega, K, M_\alpha; \Theta)$ to $\mathcal{M}(M_\omega, K, M_\alpha; \Theta)$, the representation type remains unchanged.

3 Multipatterns

3.1. Returning to our particular situation we want to apply lemma 2.4 to the algebra $[R'_1, \dots, R'_s]B[R_1, \dots, R_t]/J(W)$ where B and the sequences $R'_1, \dots, R'_s$ and $R_1, \dots, R_t$ are as in the assumption of Theorem B and $W := \oplus_{\nu \in \mathbb{Z}} \mathrm{D}(R'_{\nu+1}) \otimes_B R_\nu$. The conditions (i) and (ii) are well-known for tame hereditary algebras and by [AS] carry over immediately to iterated tilted algebras of these. Remembering the isomorphism $\mathrm{D}(R'_i) \otimes_B (-) \cong \mathrm{D}\,\mathrm{Hom}_B(-, R'_i)$, it remains to establish the tameness of the multiple vector space category with relations $(\mathrm{Hom}_B(R, -), B\text{−mod}, \mathrm{D}\,\mathrm{Hom}_B(-, R'); \Theta_W)_{red}$. Clearly, it is sufficient to prove the tameness of $(\mathrm{Hom}_{D^b(B)}(R, -), D^b(B), \mathrm{D}\,\mathrm{Hom}_{D^b(B)}(-, R'); \Theta_W)_{red}$ which using the derived equivalence F is the same as showing the tameness of $(\mathrm{Hom}_{D^b(H)}(Z, -), D^b(H), \mathrm{D}\,\mathrm{Hom}_{D^b(H)}(-, Z'); \Theta)_{red}$ where $Z_i := FR_i$, $i = 1, \dots, t$, $Z'_j := FR'_j$, $j = 1, \dots, s$, $Z := \oplus_{i=1}^t Z_i$ and $Z' := \oplus_{j=1}^s Z'_j$. Moreover, Θ is the set of natural transformations $\mathrm{Hom}_{D^b(H)}(Z, -) \to \mathrm{D}\,\mathrm{Hom}_{D^b(H)}(-, Z')$ induced from Θ_W via F.

We denote the multiple vector space categories with relations $(\mathrm{Hom}_{D^b(H)}(Z, -), D^b(H), \mathrm{D}\,\mathrm{Hom}_{D^b(H)}(-, Z'); \Theta)_{red}$ as *multipatterns* of type $\widetilde{\mathbb{D}}_n$ since they are a natural generalization of the patterns of this type introduced in [Ri1]. For the case $\widetilde{\mathbb{A}}_n$ we obtain analogous multipatterns.

3.2. The following lemma is a special case of [Ha, Proposition 4.9]. It is one of our major tools since it allows to transform one-point extension into one-point coextension and vice versa inside the derived category.

Lemma. *For all X, Y in $D^b(H)$ there is an isomorphism* $\mathrm{Hom}_{D^b(H)}(T^{-1}\tau^- X, Y) \cong \mathrm{D}\,\mathrm{Hom}_{D^b(H)}(Y, X)$ *which is natural in both arguments.*

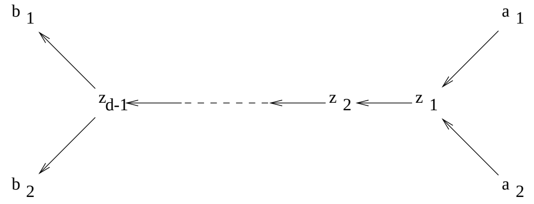
If we use the lemma to replace some functor $\mathrm{D}\,\mathrm{Hom}_{D^b(H)}(-, Z'_j)$ by $\mathrm{Hom}_{D^b(H)}(T^{-1}\tau^- Z'_j, -)$ or some $\mathrm{Hom}_{D^b(H)}(Z_i, -)$ by $\mathrm{D}\,\mathrm{Hom}_{D^b(H)}(-, T\tau Z_i)$, we will have to modify the relations Θ in the corresponding way. Nevertheless, we will denote this modified relations by Θ as well.

3.3. To understand the set of relations Θ in our situation, by the lemma above we have to understand the spaces $\mathrm{Hom}_{D^b(H)}(T^{-1}\tau^- Z'_j, Z_i) \cong \mathrm{D}(R'_j) \otimes_B R_i$ where $Z_i \in \mathcal{R}[\nu]$ and $Z'_j \in \mathcal{R}[\nu+1]$. More general, let us analyze the possible positions of Z_i and $T^{-1}\tau^- Z'_j$ in $\mathcal{R}[\nu]$.

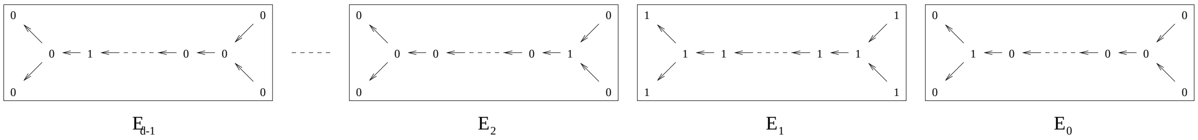
Lemma. *In case $Z_i \in \mathcal{R}[\nu]$ and $Z'_j \in \mathcal{R}[\nu+1]$ the following assertion holds:*
(a) $T^{-1}\tau^- Z'_j \ncong \tau^- Z_i$.
(b) If $\mathrm{Hom}_{D^b(H)}(T^{-1}\tau^- Z'_j, Z_i) \neq 0$, then either $T^{-1}\tau^- Z'_j \cong Z_i$ or $T^{-1}\tau^- Z'_j \cong \tau Z_i$.

Proof. For the proof of (a) we use [Ha, lemma 5.1] in order to see that the objects R'_j and $T^{-1}R'_j$ of $D^b(B)$ cannot both be B-modules. For the proof of (b) we only have to remember that $T^{-1}\tau^- Z'_j$ and Z_i are both objects of length 2 in a tube of rank $n-2$. $\square$

3.4. Remember that $H = k\vec{\Delta}$ where $\vec{\Delta}$ is a directed quiver such that the underlying graph Δ is $\widetilde{\mathbb{D}}_n$. Since reflections of $\vec{\Delta}$ induce automorphisms of $D^b(H)$, we may suppose without loss of generality that $\vec{\Delta} = \vec{\Delta}_n$ where $\vec{\Delta}_n$ is a quiver with underlying graph $\widetilde{\mathbb{D}}_n$ bearing the 'standard orientations' used in the tables in [DR].



Adapting the notation from [DR], we denote by by $E_0, \dots, E_{d-1}$, $d := n-2$, the simple regular modules in the tube $\mathcal{T}$ of rank $n-2$ of $k\vec{\Delta}_n$−mod containing a simple module. The tables in [DR] display the dimension vectors of these modules.



To fix further notation, we call $E_h^{(l)}$ the indecomposable regular module of regular length l with regular top E_h. In particular, $E_h^{(1)} = E_h$ Note, that the Auslander-Reiten translation τ_H acts on these modules by $\tau_H E_h^{(l)} = E_{h+1}^{(l)}$ where we calculate modulo $d = n-2$ in the obvious way.

By embedding H−mod into $D^b(H)$ in the usual fashion, we obtain that the indecomposable objects of length 2 in the tubes of rank $n-2$ which we are considering coincide with the objects $T^\nu E_h^{(2)} \in T^\nu\mathcal{T}$ where $\nu \in \mathbb{Z}$ and $h \in \mathbb{Z}/d$. Thus we find our objects Z_i and Z'_j among these.

3.5. For the final tameness proof in the next section we possibly will need to increase n. It will consume the rest of this section to make this precise and work it out. By Lemma 3.3(a) and the assumptions (i) and (ii) of Theorem B it is clear that for each $\nu \in \mathbb{Z}$ there is $h(\nu)$ in $\mathbb{Z}/d$ such that $E_{h(\nu)}^{(2)}$ does not appear among the Z_i and $T^{-1}\tau^- Z'_j$ lying in $\mathcal{R}[\nu]$. We want to modify the function $h : \mathbb{Z} \to \mathbb{Z}/d$. The following lemma shows how to do this.

Lemma. *Let $\mu \in \mathbb{Z}$. We may replace Z_i by $\tau^- Z_i$ for all i such that $Z_i \in \mathcal{R}[\nu]$ with $\nu \geq \mu$ and Z'_j by $\tau^- Z'_j$ for all j such that $Z'_j \in \mathcal{R}[\nu]$ with $\nu > \mu$.*

Proof. Let us define $\mathcal{H}$ as the ideal of $D^b(H)$ formed by all morphisms which factor through tubes different form the $T^\nu\mathcal{T}$. Since we are only interested in the multiple vector space category $(\mathrm{Hom}_{D^b(H)}(Z,-), D^b(H), \mathrm{D}\,\mathrm{Hom}_{D^b(H)}(-,Z'); \Theta)_{red}$, we may replace $D^b(H)$ by $D^b(H)/\mathcal{H}$, because $\mathcal{H}$ is contained in the kernel of the functor $\mathrm{Hom}_{D^b(H)}(Z,-) \oplus \mathrm{D}\,\mathrm{Hom}_{D^b(H)}(-,Z')$.

We assume $\mu = 0$ and define as $\mathcal{T}_{\leq}$ the full subcategory of $\operatorname{ind} D^b(H)$ given by all indecomposables lying in some $\mathcal{R}[\nu]$ or $\mathcal{P}[\nu]$ where $\nu \leq 0$. To obtain $\mathcal{T}_{<}$ we skip the objects from $\mathcal{R}[0]$. We define $\mathcal{T}_{\geq}$ as the complement of $\mathcal{T}_{<}$ and $\mathcal{T}_{>}$ as the complement of $\mathcal{T}_{\leq}$ in $\operatorname{ind} D^b(H)$. The intersection of $\mathcal{T}_{\leq}/\mathcal{H}$ and $\mathcal{T}_{\geq}/\mathcal{H}$ is $\mathcal{T}$. Moreover, there is no non-zero morphism from $\mathcal{T}_{<}/\mathcal{H}$ to $\mathcal{T}_{>}/\mathcal{H}$ in $\mathrm{D}^b(H)/\mathcal{H}$.

We define a full and dense endofunctor $G_{\leq}$ of $\mathcal{T}_{\leq}/\mathcal{H}$ which is the identity on $\mathcal{T}_{<}/\mathcal{H}$ and the projection on the unique maximal factor object in $\mathcal{T}$. Analogously we define a full and dense endofunctor $G_{\geq}$ of $\mathcal{T}_{\geq}/\mathcal{H}$ which is the identity on $\mathcal{T}_{>}/\mathcal{H}$ and the inclusion of the unique maximal subobject on $\mathcal{T}$. Restricted to $\mathcal{T}$ we have $G_{\geq} = \tau G_{\leq}$. Therefore we can splice together $G_{\geq}$ and $\tau G_{\leq}$ to a full and dense endofunctor G of $D^b(H)/\mathcal{H}$.

Using Lemma 3.1 we can replace all objects Z_i and Z'_j appearing in $\mathcal{T}$ by $T\tau Z_i$ (resp. $T^{-1}\tau^{-}Z'_j$). The full and dense functor G then yields an inclusion of $\check{\mathcal{M}}(\operatorname{Hom}_{D^b(H)}(GZ,-), D^b(H)/\mathcal{H}, \operatorname{D}\operatorname{Hom}_{D^b(H)}(-,GZ');\Theta)$ into $\check{\mathcal{M}}(\operatorname{Hom}_{D^b(H)}(Z,-), D^b(H)/\mathcal{H}, \operatorname{D}\operatorname{Hom}_{D^b(H)}(-,Z');\Theta)$ which preserves wildness. □

The transformations performed above on our vector space category do not interfere with the conditions imposed on the Z_i, Z'_j by the conditions (i), (ii), (iii) in Theorem B.

3.6. To increase n we embed $k\vec{\Delta}_n$ into $k\vec{\Delta}_{n+1}$. We use again the notation of [DR] and send the vertices a_i, b_i and z_i of $\vec{\Delta}_n$ simply to themselves as vertices of $\vec{\Delta}_{n+1}$. Thus the only vertex not hit is z_d. We map the arrows of $\vec{\Delta}_n$ to compositions of arrows of $\vec{\Delta}_{n+1}$ in the only possible way.

In order to distinguish the indecomposables in the $d+1$-tube of $k\vec{\Delta}_{n+1}-\mathrm{mod}$ from those in the d-tube of $k\vec{\Delta}_n-\mathrm{mod}$, we denote them by $F_h^{(l)}$ where now $h \in \mathbb{Z}/(d+1)$.

The left adjoint $L : k\vec{\Delta}_n-\mathrm{mod} \to k\vec{\Delta}_{n+1}-\mathrm{mod}$ of the restriction functor $k\vec{\Delta}_{n+1}-\mathrm{mod} \to k\vec{\Delta}_n-\mathrm{mod}$ is a tensor product by a projective right $k\vec{\Delta}_n$-module and consequently exact. The modules in $L(k\vec{\Delta}_n-\mathrm{mod})$ are precisely those $k\vec{\Delta}_{n+1}$-modules X such that in the minimal projective presentation $P_1 \longrightarrow P_0 \longrightarrow X \to 0$ the projective indecomposable $k\vec{\Delta}_{n+1}$-module attached to the vertex z_d does not appear as summand of P_0 or P_1. Since this indecomposable projective is just the projective cover of the simple module F_{d+1}, we obtain that $L(k\vec{\Delta}_n-\mathrm{mod})$ is just the perpendicular category ${}^{\perp}F_{d+1}$ which is defined as the full subcategory of $k\vec{\Delta}_{n+1}-\mathrm{mod}$ given by all X satisfying $\operatorname{Hom}_{k\vec{\Delta}_{n+1}}(X, F_{d+1}) = 0$ and $\operatorname{Ext}^1_{k\vec{\Delta}_{n+1}}(X, F_{d+1}) = 0$.

The fully faithful functor L maps the projective $k\vec{\Delta}_n$-modules to the projective $k\vec{\Delta}_{n+1}$-modules. Hence it induces a fully faithful functor $K^b(k\vec{\Delta}_n-\mathrm{proj}) \to K^b(k\vec{\Delta}_{n+1}-\mathrm{proj})$. Since L is exact, we obtain an extension of the functor $L : k\vec{\Delta}_n-\mathrm{mod} \to k\vec{\Delta}_{n+1}-\mathrm{mod}$ to a fully faithful functor $D^b(k\vec{\Delta}_n) \to D^b(k\vec{\Delta}_{n+1})$ which we denote by L as well. It is then easy to see that $L(D^b(k\vec{\Delta}_n))$ is the perpendicular category ${}^{\perp}F_{d+1}$ of F_{d+1} inside $D^b(k\vec{\Delta}_{n+1})$ which is by definition given by all X such that $\operatorname{Hom}_{D^b(k\vec{\Delta}_{n+1})}(X, T^{\nu}F_{d+1}) = 0$ for all $\nu \in \mathbb{Z}$.

Let us calculate the objects $LT^{\nu}E_h^{(2)}$. Up to shift it is sufficient to calculate the modules $LE_h^{(2)}$. Since L happens to be exact, this reduces to calculate the modules LE_h which turn out to be $LE_h = F_h$ for all $h = 1, \ldots, d-1$ but $LE_d = F_d^{(2)}$. We obtain that $LT^\nu E_h^{(2)} = T^\nu F_h^{(2)}$ for all $h = 1, \ldots, d-2$ whereas $LT^\nu E_d^{(2)} = T^\nu F_d^{(3)}$ and $LT^\nu E_{d-1}^{(2)} = T^\nu F_{d-1}^{(3)}$ for all $\nu \in \mathbb{Z}$.

Lemma. *We may suppose that the objects Z_i and $T^{-1}\tau^- Z'_j$ are among the $T^\nu E_h^{(2)}$ with $0 \leq h \leq d-3$.*

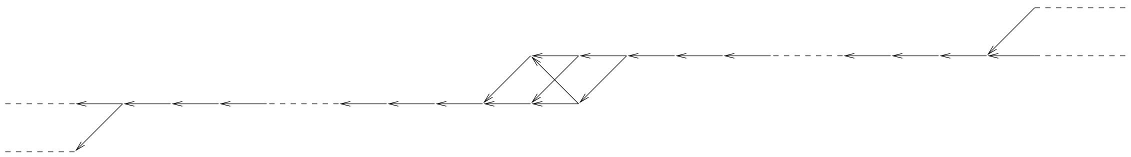
Proof. We will increase n to $n+1$ using the embedding of $D^b(k\Delta_n)$ into $D^b(k\Delta_{n+1})$ outlined above.

By the previous lemma we my assume that $E_{d-1}^{(2)}$ is not isomorphic to any of the Z_i and $T^{-1}\tau^- Z'_j$. As we have seen above that $LD^b(k\vec{\Delta}_n)$ equals ${}^\perp F_{d+1}$, we may identify $(\mathrm{Hom}(Z,-), D^b(k\vec{\Delta}_n), \mathrm{Hom}(T^{-1}\tau^- Z', -); \Theta)$ with $(\mathrm{Hom}(LZ,-), {}^\perp F_{d+1}, \mathrm{Hom}(LZ',-); \Theta)$. Using the above formulas for the $LT^\nu E_h^{(2)}$, we are almost done. The only problem appears if some of the objects Z_i or $T^{-1}\tau^- Z'_j$ is isomorphic to some $T^\nu E_d^{(2)}$ because in this case we observed that $LT^\nu E_d^{(2)} = T^\nu F_d^{(3)}$. Nevertheless, the proof is complete if we can show that the functors $\mathrm{Hom}(T^\nu F_d^{(3)}, -)$ and $\mathrm{Hom}(T^\nu F_{d+1}^{(2)}, -)$ coincide on ${}^\perp F_{d+1}$. Without loss of generality we may suppose that $\nu = 0$ and consider the canonical exact sequence $0 \to F_{d+1}^{(2)} \longrightarrow F_d^{(3)} \longrightarrow F_d \to 0$ which by [Ha] yields an exact sequence $\mathrm{Hom}(F_d, -) \longrightarrow \mathrm{Hom}(F_d^{(3)}, -) \longrightarrow \mathrm{Hom}(F_{d+1}^{(2)}, -) \longrightarrow \mathrm{Hom}(TF_d, -)$ of functors acting on $D^b(k\vec{\Delta}_{n+1})$. The outer terms vanish on the subcategory ${}^\perp F_{d+1}$ because by lemma 3.1 we know that $\mathrm{Hom}(T^\nu F_d, -) \cong \mathrm{D}\,\mathrm{Hom}(-, T^{\nu+1} F_{d+1})$. $\square$

As in the previous lemma the transformations performed above on our vector space category do not interfere with the conditions imposed on the Z_i, Z'_j by the conditions (i), (ii), (iii) in Theorem B.

4 Reduction to one algebra C

4.1. Since $\mathrm{gl.dim}\, H = 1$, by [Ha] the triangulated category $D^b(H)$ can be identified with $\hat{H}-\underline{\mathrm{mod}}$ which is the stable category of the the category of finite-dimensional modules over the repetitive algebra $\hat{H}$ (see [HW]). The repetitive algebra $\hat{H}$ is an infinite-dimensional algebra given by the following quiver endowed with all possible commutativity relations and a lot of zero-relations which we will not specify, since they will not play any role.

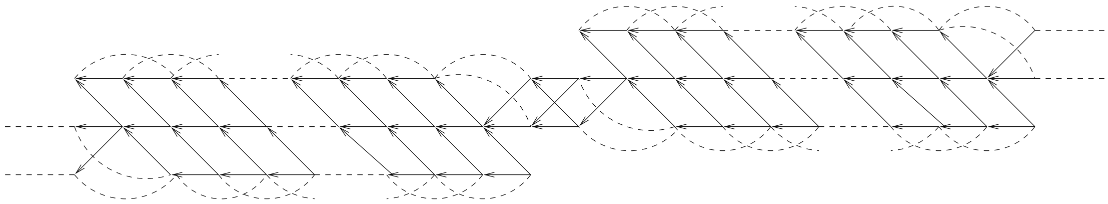


Thus we have to study $(\underline{\mathrm{Hom}}_{\hat{H}}(Z,-), \hat{H}-\underline{\mathrm{mod}}, \mathrm{D}\,\underline{\mathrm{Hom}}_{\hat{H}}(-,Z'); \Theta)$. The repetitive algebra $\hat{H}$ has the usual Nakayama shift as an automorphism which induces an automorphism ϕ_H of $\hat{H}-\underline{\mathrm{mod}}$. The induced automorphism on $\hat{H}-\underline{\mathrm{mod}}$ will be denoted by ϕ_H as well. The two shifts T and ϕ_H on $\hat{H}-\underline{\mathrm{mod}}$ are related by the formula $\phi_H = T^2\tau$.

4.2. We want to pass from $(\underline{\mathrm{Hom}}_{\hat{H}}(Z,-), \hat{H}-\underline{\mathrm{mod}}, \mathrm{D}\,\underline{\mathrm{Hom}}_{\hat{H}}(-,Z');\Theta)$ to $(\mathrm{Hom}_{\hat{H}}(Z,-), \hat{H}-\mathrm{mod}, \mathrm{D}\,\mathrm{Hom}_{\hat{H}}(-,Z');\Theta)$, but this multiple vector space category with relations in general fails to be tame. In order to arrange this, we have to modify the Z_i and Z'_j in an appropriate way. Using Lemma 3.2, we can replace each functor $\mathrm{D}\,\mathrm{Hom}(-,Z'_j)$ such that Z'_j is in some $\mathcal{R}[\nu]$ with ν odd by $\mathrm{Hom}(T^{-1}\tau^- Z'_j,-)$ where the object $T^{-1}\tau^- Z'_j$ now sits in $\mathcal{R}[\nu-1]$. Dually we replace each functor $\mathrm{Hom}(Z_i,-)$ such that Z_i is in some $\mathcal{R}[\nu]$ with ν odd by $\mathrm{D}\,\mathrm{Hom}(-,T\tau Z_i,-)$ where the object $T\tau Z_i$ sits in $\mathcal{R}[\nu+1]$. Thus we collect all objects representing or corepresenting the functors in our multiple vector space category into those $\mathcal{R}[\nu]$ such that ν is even.

Let us consider the set of representing objects in some $\mathcal{R}[\nu]$. From Lemma 3.6 we know that they do not form a complete τ-orbit but at least two objects which are subsequent under τ are missing. Applying Lemma 3.5, we can rearrange these objects such that precisely $\phi^{\nu}_H(E^{(2)}_0)$ and $\phi^{\nu}_H(E^{(2)}_1)$ are missing. We do this arrangements with the representing and corepresenting objects in all $\mathcal{R}[\nu]$. Under this condition we will prove the tameness of $(\mathrm{Hom}_{\hat{H}}(Z,-), \hat{H}-\mathrm{mod}, \mathrm{D}\,\mathrm{Hom}_{\hat{H}}(-,Z');\Theta)$.

4.3. Let us convince ourselves that for showing the tameness of the above multiple vector space category with relations it is sufficient to show the tameness of the infinite-dimensional algebra C given by the following quiver endowed with all commutativity relations and the indicated zero-relations.



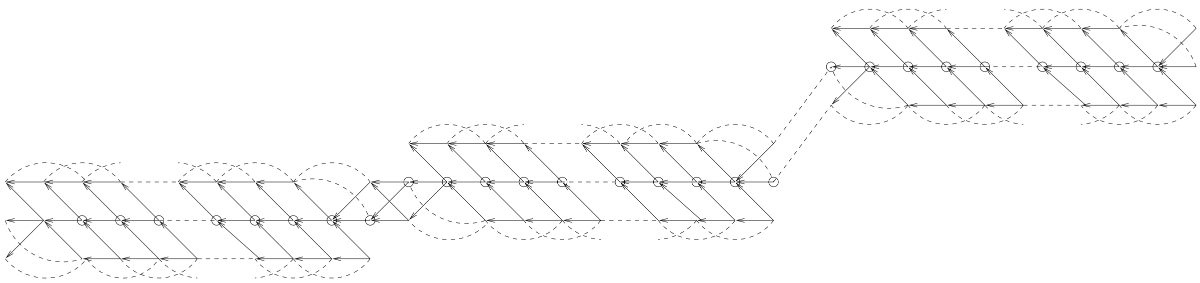
We want to establish that $\mathcal{M}(\mathrm{Hom}_{\hat{H}}(Z,-), \hat{H}-\mathrm{mod}, \mathrm{D}\,\mathrm{Hom}_{\hat{H}}(-,Z');\Theta)$ can be identified with a full subcategory of $C-\mathrm{mod}$. Then the wildness of $\mathcal{M}(\mathrm{Hom}_{\hat{H}}(Z,-), \hat{H}-\mathrm{mod}, \mathrm{D}\,\mathrm{Hom}_{\hat{H}}(-,Z');\Theta)$ would obviously imply the wildness of C.

Let us consider an object $U = (U_\omega, \gamma_U, U_0, \delta_U, U_\omega)$ of the category $\mathcal{M}(\mathrm{Hom}(Z,-), \hat{H}-\mathrm{mod}, \mathrm{D}\,\mathrm{Hom}(-,Z');\Theta)$. Thus U_ω is a t-tuple $(U^{(1)}_\omega, \dots, U^{(t)}_\omega)$ and U_α an s-tuple $(U^{(1)}_\alpha, \dots, U^{(s)}_\alpha)$ of k-spaces. Analogously the map γ_U is a t-tuple $(\gamma^{(1)}_U, \dots, \gamma^{(t)}_U)$ and δ_U is a s-tuple $(\delta^{(1)}_U, \dots, \delta^{(s)}_U)$ of k-linear maps. If Z'_j is in some $\mathcal{R}[\nu]$ such that ν is odd, then we replace the k-epimorphism $\delta^{(j)}_U : \mathrm{Hom}(T^{-1}\tau^- Z'_j, U_0) \to U^{(j)}_\alpha$ by its kernel $\bar{\delta}^{(j)}_U : \bar{U}^{(j)}_\alpha \to \mathrm{Hom}(T^{-1}\tau^- Z'_j, U_0)$. In the dual way we replace the k-monomorphism $\gamma^{(i)}_U : U^{(i)}_\omega \to \mathrm{D}\,\mathrm{Hom}(U_0, T\tau Z_i) \to U^{(j)}_\alpha$ by its cokernel $\bar{\gamma}^{(i)}_U : \mathrm{D}\,\mathrm{Hom}(U_0, T\tau Z_i) \to \bar{U}^{(i)}_\omega$ if Z_i is in some $\mathcal{R}[\nu]$ such that ν is odd.

It remains to take into account the set of relations Θ. A non-trivial relation only appears provided $\mathrm{Hom}(T^{-1}\tau^- Z'_j, Z_i) \neq 0$ for some Z_i in $\mathcal{R}[\nu]$ and Z'_j in $\mathcal{R}[\nu+1]$ or dually $\mathrm{Hom}(Z'_j, T\tau Z_i) \neq 0$ for some Z_i in $\mathcal{R}[\nu-1]$ and Z'_j in $\mathcal{R}[\nu]$. By duality we will only deal with the first case. Using Lemma 3.3.(b), we know that either $T^{-1}\tau^- Z'_j \cong Z_i$ or $T^{-1}\tau^- Z'_j \cong \tau Z_i$. The first case is excluded by condition (iii) of Theorem B. Let us choose a generator ϵ of the 1-dimensional k-space $\mathrm{Hom}(T^{-1}\tau^- Z'_j, Z_i)$. That U satisfies the relations Θ implies that the composition $\delta_U^{(j)}\,\mathrm{Hom}(\epsilon, U_0)\gamma_U^{(i)}$ equals 0. Hence there exists a k-linear map $\eta : \bar{U}_\alpha^{(j)} \to U_\omega^{(i)}$ such that $\bar{\delta}_U^{(j)}\eta = \mathrm{Hom}(\epsilon, U_0)\gamma_U^{(i)}$. In this way we transformed the object U into a representation of the quiver of C where η represents the corresponding arrow at the rim of the quiver of C. Moreover, all relations are satisfied with the possible exception of the zero relations at the upper and lower rims. To get also this we use Lemma 3.3.(a) and the conditions (i) and (ii) to see that if $\mathrm{Hom}(T^{-1}\tau^{-}Z'_j, Z_i) \neq 0$, then neither $\tau^2 Z_i$ nor $\tau^{-} Z_i$ can occur among the representing objects in $\mathcal{R}[\nu]$.

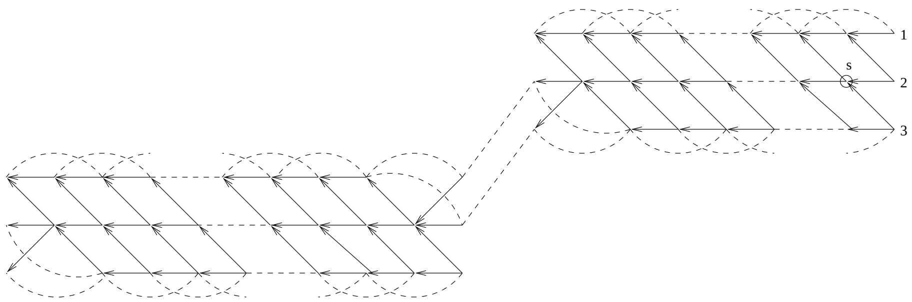
5 Tameness of C via generalized one-point extensions

5.1. Finally we are left with the problem to show the tameness of the infinite-dimensional algebra C which amounts to show the tameness of finite-dimensional convex truncations $\bar{C}$ of C given by a quiver of the following shape again endowed with all commutativity relations and the indicated zero-relations. Certain vertices are encircled which we will need below to write $\bar{C}$ as inductive generalized one-point extension.



5.2. The vertices of the quiver of $\bar{C}$ appear in vertical slices. We use the technique of generalized one-point extensions developed in [Dr3] to build up $\bar{C}$ slice by slice from left to write using the encircled vertex in the previous slice as extension point. In each step we can show that the appearing ordinary vector space category is actually tame. Clearly, various calculations corresponding to the 'nodes' of the quiver are necessary. We leave these to the reader and only will show in detail the 'general step'. But let us note the the calculations in the particular cases are very similar to those we will give.

Thus we want to show the tameness of the algebra A which is given by the following quiver endowed with all commutativity relations and the indicated zero relations. By induction we know that the algebra A' obtained by removing the vertices 1,2,3 is tame.



Let P be the indecomposable projective A-module and $e(s)$ the primitive idempotent of A associated with the encircled vertex s. We denote by K the full subcategory of $A-\mathrm{mod}$ given by all modules V satisfying $\mathrm{Hom}_A(P, \tau_A V) = 0$ and consider the vector space category $(K, \mathrm{Hom}_A(P, -))$. By [Dr1][Theorem 3.3] the algebra A is tame if and only if the following three conditions are satisfied:
(i) The factor algebra $A/Ae(s)A$ is tame.
(ii) The vector space category $(K^* \setminus P, \mathrm{Hom}_A(P, -))_{red}$ is tame.
(iii) For every $n \in \mathbb{N}$ the set of objects V in $\mathrm{ind}\, K^*$ satisfying $\dim_k V \leq n$ is finite.

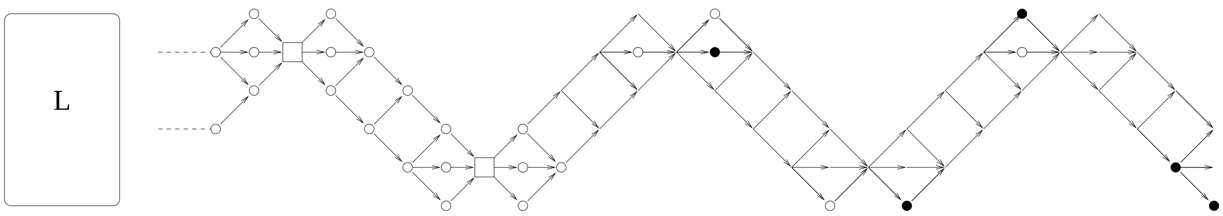
From now on we will use the notation introduced in [Dr3]. It is easy to see that A is a generalized one-point extension by s which means that $A/Ae(s)A-\mathrm{mod} = A^s-\mathrm{mod} \vee A_s-\mathrm{mod}$. The algebra A^s is by definition obtained from A by removing all predecessors of s. Hence it can be constructed from the tame algebra A' by firstly removing the vertex s and secondly gluing the vertex 3 using a splitting zero relation. Thus A^s is tame. The algebra A^s which is defined dually by removing all successors of s is obviously of finite representation type. Consequently, condition (i) is satisfied.

By [Dr3][Proposition 3.3] $K^* \setminus P$ is a disjoint union of K^+ and K^- where $(K^+, \mathrm{Hom}_A(P, -))_{red}$ can be identified with $(A^s-\mathrm{mod}, \mathrm{Hom}_{A^s}(R^+, -))_{red}$ and $(K^-, \mathrm{Hom}_A(P, -))_{red}$ can be identified with $(A_s-\mathrm{mod}, R^- \otimes_{A_s} (-))_{red}$ where R^+ is the radical of the left A-module $Ae(s)$ and R^- is the radical of the right A-module $e(s)A$.

The algebra $\overline{A}^s$ is by definition obtained form A by removing all proper predecessors of s. Hence it can be constructed from the tame algebra A' by gluing the vertex 3 using a splitting zero relation. Consequently, $\overline{A}^s$ is tame. The category K^+ is the same for A and $\overline{A}^s$. Therefore property (iii) is satisfied for $\mathrm{ind}\, K^+$. As $\mathrm{ind}\, K^-$ has only finitely many objects, property (iii) is also satisfied for K^*.

It remains to check property (ii). We calculate the preinjective component of the Auslander-Reiten quiver of A^s, the Auslander-Reiten quiver of A_s and the actions of $\mathrm{Hom}_{A^s}(R^+, -)$ resp. $R^- \otimes_{A_s} (-) \cong \mathrm{D}\,\mathrm{Hom}_{A_s}(-, R^-)$ on the vertices of these quivers. In order to get $(K^* \setminus P, \mathrm{Hom}_A(P, -))_{red}$ we have to glue $(K^+, \mathrm{Hom}_A(P, -))_{red}$ and $(A_s-\mathrm{mod}, R^- \otimes_{A_s} (-))_{red}$ using [Dr3][sections 4,5]. It turns out that the computed components contain all object in which connecting arrows start and end. Below we display the obtained quiver. The bullets correspond to indecomposable objects V in $K^* \setminus P$ with $\dim_k \mathrm{Hom}_A(P, V) = 1$ and the squares to those with $\dim_k \mathrm{Hom}_A(P, V) = 2$. The solid bullets (resp. squares) correspond to objects of K^-, the empty ones to K^+. The relations on the obtained component of the quiver of $(K^* \setminus P)/\,\mathrm{Ker}\,\mathrm{Hom}_A(P, -)$ are induced from the preinjective component of A_s. In L we comprise all indecomposable objects in $K^* \setminus P$ which do not come from this component.

We enlarge the quiver by some 'imaginary' vertices in order to see that $(K^* \setminus P)/\,\mathrm{Ker}\,\mathrm{Hom}_A(P, -)$ is a finite 'prolongation' of $A^s-\mathrm{mod}/\,\mathrm{Ker}\,\mathrm{Hom}_{A^s}(R^+, -)$.



5.3. The tameness of $((K^* \setminus P)_{red}, \mathrm{Hom}_A(P,-))$ finally follows by shifting back to the vector space category $(A^s\text{-mod}, \mathrm{Hom}_{A^s}(R^+,-))$ whose tameness is known because the algebra $\overline{A}^s$ is tame.

The shifting is done using the following lemma from [DG1]. Note, that the assumptions of the lemma are satisfied because by [Dr3][Corollary 4.3] the category $K^* \setminus P$ has almost split morphisms.

Lemma. *Let (K, M) be a faithful vector space category and L, X, Y be a partition of* $\operatorname{ind} K$ *into subspectroids. Suppose that there exists an isomorphism $\tau = (\tau_0, \tau_\omega)$: $(\operatorname{add} X, M) \to (\operatorname{add} Y, M)$, there exist objects $x_1, \ldots, x_n$ in X and non-zero morphisms $u_i : x_i \to y_i := \tau_0 x_i$ such that the following properties are satisfied:*
(a) $\dim_k M(x_i) = 1$ *and* $M(u_i) = (\tau_\omega)_{x_i}$ *for all* $i = 1, \ldots, n$.
(b) $K(X, L) = 0 = K(Y, L)$.
(c) $K(z, x) = \sum_{i=1}^n K(z, x_i)K(x_i, x)$ *for all z in L and x in X.*
(d) $K(z, y) = \sum_{i=1}^n K(z, y_i)K(y_i, y)$ *for all z in L and y in Y.*
(e) $K(z, \oplus_{i=1}^n y_i) = K(z, \oplus_{i=1}^n x_i)u$ *for all z in L where $u : \oplus_{i=1}^n x_i \to \oplus_{i=1}^n y_i$ is the map whose components are $u_{ii} = u_i$ and $u_{ij} = 0$ for $i \neq j$.*
If all these conditions are satisfied, then $(\operatorname{add}(L \cup X), M) \cong (\operatorname{add}(L \cup Y), M)$.

6 Applications and comments

6.1. In general the converses of the Theorems A and B are not true. The reason is that although we can show that the multiple vector space category obtained by embedding mod$-B$ into $D^b(B)$ is still tame, it will usually be much bigger, because only few relevant complexes may be modules.

We are in a better situation in the following special case where it is easy to construct a wild full subcategory of the multiple vector space category $(\text{mod}-B, \mathrm{Hom}_B(R,-))$ if the orthogonality condition is not satisfied.

Corollary. *Let H be a connected tame hereditary k-algebra of type $\widetilde{\mathbb{A}}_n$ or $\widetilde{\mathbb{D}}_n$. Suppose T is a tilting A-module without preinjective direct summands, $B = \mathrm{End}_H(T)$ is the associated tilted algebra and $F = \mathrm{Hom}_H(T,-)$ the corresponding tilting functor.*

If $R_1, \ldots, R_r$ is a sequence of indecomposable T-torsion H-modules of regular length 2 lying in the non-homogeneous tubes in case $\widetilde{\mathbb{A}}_n$ and in one tube of rank $n-2$ in case $\widetilde{\mathbb{D}}_n$, then the multiple one-point extension $B[F(R_1), \ldots, F(R_n)]$ is of tame representation type if and only if the modules R_i are pairwise Hom*-orthogonal.*

6.2. Using recent results from [DG2] one can see that the category of representations of the multiple vector space category without relations corresponding to Theorem A is equivalent to the category of representations of a clan. This gives another tameness proof in this special situation but fails in the general situation of Theorem B.

For the tameness of the algebra C there is an alternative proof. Namely, C degenerates to the clannish algebra obtained from C by transforming the commutativity relations in the squares at the rim of the quiver to zero relations. Nevertheless, our proof using generalized one-point extensions preserves some information about the structure of the indecomposable C-modules which we will study in a subsequent paper.

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