(NONSYMMETRIC) DIRICHLET OPERATORS ON L^1: EXISTENCE, UNIQUENESS AND ASSOCIATED MARKOV PROCESSES

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Abstract. Let L be a nonsymmetric operator of type $Lu = \sum a_{ij}\partial_i\partial_j u + \sum b_i\partial_i u$ on an arbitrary subset $U \subset \mathbb{R}^d$. We analyse L as an operator on $L^1(U,\mu)$ where μ is an invariant measure, i.e., a possibly infinite measure μ satisfying the equation $L^*\mu = 0$ (in the weak sense). We explicitely construct, under mild regularity assumptions, extensions of L generating sub–Markovian C_0–semigroups on $L^1(U,\mu)$ as well as associated diffusion processes. We give sufficient conditions on the coefficients so that there exists only one extension of L generating a C_0–semigroup and apply the results to prove uniqueness of the invariant measure μ. Our results imply in particular that if $\varphi \in H^{1,2}_{loc}(R^d, dx)$, $\varphi \neq 0$ dx–a.e., the generalized Schrödinger operator $(\Delta + 2\varphi^{-1}\nabla\varphi\cdot\nabla, C_0^\infty(R^d))$ has exactly one extension generating a C_0–semigroup if and only if the Friedrich's extension is conservative. We also give existence and uniqueness results for a corresponding class of infinite dimensional operators acting on smooth cylinder functions on a separable real Banach space.

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0. Introduction

The purpose of this paper is to study the Cauchy–Problem of second order differential operators with measurable coefficients of type

$$Lu = \sum_{i,j=1}^{d} a_{ij}\partial_i\partial_j u + \sum_{i=1}^{d} b_i\partial_i u \tag{0.1}$$

defined on $C_0^\infty(U)$, $U \subset \mathbb{R}^d$ open, on the space $L^1(U,\mu)$, where μ is an *invariant measure*, i.e., a possibly infinite measure satisfying $Lu \in L^1(U,\mu)$ for all $u \in C_0^\infty(U)$ and

$$\int Lu\, d\mu = 0 \text{ for all } u \in C_0^\infty(U)\ . \tag{0.2}$$

Here the $(a_{ij})_{1\le i,j\le d}$ are supposed to be locally strictly elliptic and $b_i \in L^2_{loc}(\mathbb{R}^d,\mu)$, $1 \le i \le d$. We do not assume that L is symmetric or that the first order part is a small perturbation of the second order part in any classical sense. Results obtained by V.I. Bogachev, N. Krylov and M. Röckner (cf. [BKR1], [BKR2] and [BR]) show that it is natural to suppose that μ is absolutely continuous w.r.t. the Lebesgue measure dx and that the density admits a representation φ^2, $\varphi \in H^{1,2}_{loc}(U)$. Moreover, we will assume that a_{ij}, $1 \le i,j \le d$, is contained locally in the weighted Sobolev space $H_0^{1,2}(U,\mu)$ (= the closure of $C_0^\infty(U)$ in $L^2(U,\mu)$ w.r.t. the norm given by $\int |\nabla u|^2 + u^2\, d\mu$).

We are mainly concerned with the following three problems:

(i) Construct and analyse extensions $\overline{L}$ of L generating C_0–semigroups $(\overline{T}_t)_{t\ge0}$ on $L^1(U,\mu)$ that are *sub–Markovian* (i.e., $0 \le f \le 1$ implies $0 \le \overline{T}_t f \le 1$). Henceforth, any extension of L generating a C_0–semigroup will be called *maximal*.

(ii) Find conditions on $(a_{ij})_{1\le i,j\le d}$, $(b_i)_{1\le i\le d}$ and μ so that there is only one maximal extension.

(iii) Construct diffusion processes with transition probabilities given by $(\overline{T}_t)_{t\ge0}$.

Concerning problem (i) we will construct in Theorem 1.5 below a maximal extension $\overline{L}$ generating a sub–Markovian C_0–semigroup $(\overline{T}_t)_{t\ge0}$ in such a way that the space $D(\overline{L})_b$ of all bounded functions in the domain $D(\overline{L})$ is contained locally in the weighted Sobolev space $H_0^{1,2}(U,\mu)$. This implies that, although $\overline{L}$ is not symmetric, this maximal extension is still associated with a bilinear form. More precisely, the following representation holds:

$$\sum_{i,j=1}^{d} \int a_{ij}\partial_i u\, \partial_j v\, d\mu + \sum_{i=1}^{d} \int (b_i^0 - b_i)\partial_i u\, v\, d\mu = -\int \overline{L}u\, v\, d\mu \tag{0.3}$$

for all $u \in D(\overline{L})_b$ and $v \in H_0^{1,2}(U,\mu)_0$ (= the space of all elements $v \in H_0^{1,2}(U,\mu)$ with compact support contained in U). Here

$$b_i^0 = \sum_{j=1}^{d} (\partial_j a_{ij} + 2a_{ij}\partial_j\varphi/\varphi)\, , 1 \le i \le d\, , \tag{0.4}$$

(cf. Theorem 1.5). Since $\overline{L}$ is a Dirichlet operator, i.e., the generator of a sub–Markovian semigroup, $D(\overline{L})_b$ is dense in $D(\overline{L})$ w.r.t. the graph norm, so that our result implies that $\overline{L}$ is completely determined by the first order object (0.3). We would like to emphasize that the construction of $(\overline{L}, D(\overline{L}))$ is purely analytic and uses only results from semigroup theory as well as Dirichlet form techniques adapted to the nonsymmetric case.

The question whether μ is $(\overline{T}_t)$–*invariant* (i.e., $\int \overline{T}_t u\, d\mu = \int u\, d\mu$ for all $u \in L^1(U,\mu)$) is of particular interest for applications. Clearly, μ is $(\overline{T}_t)$–invariant if and only if $\int \overline{L}u\, d\mu = 0$ for all $u \in D(\overline{L})$. However, it can happen, even in the case where the measure μ is finite and $U = \mathbb{R}^d$, that μ is not $(\overline{T}_t)$–invariant although $\int Lu\, d\mu = 0$ for all $u \in C_0^\infty(\mathbb{R}^d)$ (cf. Example 1.12). In the symmetric case $(\overline{T}_t)$–invariance of μ is equivalent to the conservativeness of $(\overline{T}_t)_{t\geq 0}$ and the latter has been well–studied by many authors (cf. [D], [FOT, Section 1.6], [S] and references therein). However, in the context of an L^1–framework, there are no nontrivial results on $(\overline{T}_t)$–invariance of μ in the nonsymmetric case.

Proposition 1.9 completely characterizes $(\overline{T}_t)$–invariance of μ in terms of the bilinear form (0.3). More precisely, μ is $(\overline{T}_t)$–invariant if and only if there exist $\chi_n \in H^{1,2}_{loc}(\mathbb{R}^d,\mu)$ and $\alpha > 0$ such that $(\chi_n - 1)^- \in H_0^{1,2}(\mathbb{R}^d,\mu)_0$, $\lim_{n\to\infty} \chi_n = 0$ μ–a.e. and

$$\alpha \int \chi_n\, v\, d\mu + \sum_{i,j=1}^d \int a_{ij}\partial_i \chi_n\, \partial_j v\, d\mu + \sum_{i=1}^d \int (b_i - b_i^0)\partial_i \chi_n v\, d\mu \geq 0$$

for all $v \in H_0^{1,2}(\mathbb{R}^d,\mu)_0$, $v \geq 0$. This characterization allows one in particular to apply the method of Lyapunov functions to derive sufficient conditions for $(\overline{T}_t)$–invariance of μ (cf. Proposition 1.10 (b) and (c)). These criteria are well–known in the symmetric case (cf. [D]) but have been proved in cases only where classical regularity theory to the operator L can be applied which is not the case here. Since we are working in an L^1–framework it is also possible to formulate sufficient conditions for $(\overline{T}_t)$–invariance of μ in terms of integrability assumptions on the coefficients (cf. Proposition 1.10 (a)). More precisely, the measure μ is $(\overline{T}_t)$–invariant if

$$a_{ij}, b_i - b_i^0 \in L^1(\mathbb{R}^d,\mu)\,, 1 \leq i,j \leq d\,.$$

Our next result is related to the uniqueness of maximal extensions of L (defined on $C_0^\infty(U)$) both in the symmetric and in the nonsymmetric case. Our main result in the symmetric case states that if the $(a_{ij})_{1\leq i,j\leq d}$ are locally Hölder–continuous then $(L, C_0^\infty(\mathbb{R}^d))$ is L^1–unique if and only if the Friedrich's extension (= the closure of $\int Lu\, v\, d\mu$, $u,v \in C_0^\infty(\mathbb{R}^d)$, on $L^2(U,\mu)$) is conservative (cf. Theorem 2.1). This implies in particular that the generalized Schrödinger operator $(\Delta + 2\varphi^{-1}\nabla\varphi\cdot\nabla, C_0^\infty(R^d))$ has exactly one maximal extension if and only if the Friedrich's extension (or equivalently, the associated diffusion process) is conservative. This result completes on the one hand the well–known results on Markov–uniqueness of such operators obtained by M. Röckner and T.S. Zhang (cf. [RZ]) and illustrates on the other hand the difference between the two notions.

The results on existence and uniqueness of maximal extensions of $(L, C_0^\infty(\mathbb{R}^d))$ can be applied to obtain results on uniqueness of the invariant measure μ. As an example we prove, based on a regularity result for invariant measures obtained by V.I. Bogachev, N. Krylov and M. Röckner in [BKR2] (cf. also [ABR]), in the particular case $a_{ij} = \delta_{ij}$, $b_i \in L^p_{loc}(\mathbb{R}^d, dx)$, $1 \le i, j \le d$, for some $p > d \ge 2$ and $\sum_{i=1}^d b_i(x)x_i \le M(|x|^2 \ln(|x|^2 + 1) + 1)$ for some $M \ge 0$ that there exists at most one probability measure ν satisfying $b_i \in L^1_{loc}(\mathbb{R}^d, \nu)$, $1 \le i \le d$, and $\int Lu \, d\nu = 0$ for all $u \in C_0^\infty(\mathbb{R}^d)$ (cf. Proposition 2.8). This result complements a recent result on uniqueness of the invariant measure μ obtained by S. Albeverio, V.I. Bogachev and M. Röckner in [ABR].

Our last result in the finite dimensional case is related to the existence of diffusions whose transition semigroups are given by $(\overline{T}_t)_{t \ge 0}$. Using the framework of *generalized Dirichlet forms* (cf. [St1]) one can construct, only using the explicit description (0.3) of the maximal extension $\overline{L}$, in a similar way to the construction of associated Markov processes in the classical theory of Dirichlet forms (cf. [MR]), associated Markov processes having a resolvent that is quasi–strong Feller (cf. Theorem 3.5). A standard technique in the classical theory of Dirichlet forms can then be carried over to the more general nonsymmetric case to show that such Markov processes are in fact diffusions (cf. Proposition 3.6).

All our methods we developed for the analysis of L and the construction of associated diffusions are independent of the dimension and do not use any finite dimensional specialities. This way they can help to prove new results on existence and uniqueness of Dirichlet operators in infinite dimensions. As a particular example we consider in Section 4 (nonsymmetric) operators of type $L = L^0 + \beta \cdot \nabla$, where L^0 is the generator of a gradient form of type $\int_E \langle A\nabla u, \nabla v\rangle \, d\mu$ acting on smooth cylinder functions on a real separable Banach space E. The results improve the results obtained in [St2] and at the same time the class of operators under consideration is much more general.

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1. Existence and Invariance in the finite dimensional case

a) Existence under global assumptions.

Let $U \subset \mathbb{R}^d$ be open, μ a *finite* positive measure on $\mathcal{B}(U)$ with $\mathrm{supp}(\mu) \equiv U$. Suppose that $d\mu \ll dx$ and that the density admits a representation φ^2, where $\varphi \in H^{1,2}(U)$. We denote by $\| \cdot \|_p$, $p \in [1, \infty]$, the usual norm on $L^p(U, \mu)$. If $W \subset L^p(U, \mu)$ is an arbitrary subspace, denote by W_0 the subspace of all elements $u \in W$ such that $\mathrm{supp}(|u|\mu)$ is a compact subset contained in U, and by W_b the subspace of all bounded elements in W. Finally, let $W_{0,b} = W_0 \cap W_b$.

Let $(\mathbb{D}^0, H_0^{1,2}(U, \mu))$ be the closure of $\int \langle \nabla u, \nabla v\rangle \, d\mu \, ; u, v \in C_0^\infty(U)$, on $L^2(U, \mu)$. Let $H^{1,2}_{loc}(U, \mu)$ be the space of all elements u such that $u\chi \in H_0^{1,2}(U, \mu)$ for all

$\chi \in C_0^\infty(U)$.

Let $A = (a_{ij})_{1\le i,j\le d}$ with

$$a_{ij} = a_{ji} \in H^{1,2}_{loc}(U,\mu) \text{ and } \partial_j a_{ij} \in L^2(U,\mu)\,, 1 \le i,j \le d, \tag{1.1}$$

be uniformly strictly elliptic, i.e., for some constant $\nu > 0$

$$\nu^{-1}|h|^2 \le \langle A(x)h,h\rangle \le \nu|h|^2 \text{ for all } h\in\mathbb{R}^d, x\in U\ . \tag{1.2}$$

Let

$$B \in L^2(U;\mathbb{R}^d,\mu) \tag{1.3}$$

such that

$$\int L^A u + \langle B,\nabla u\rangle\, d\mu = 0 \text{ for all } u\in C_0^\infty(U), \tag{1.4}$$

where $L^A u := \sum_{i,j=1}^d a_{ij}\partial_i\partial_j u\,, u\in C_0^\infty(U)$.

Our first result in this section is on existence of closed extensions of $L^A + B\cdot\nabla$ on $L^1(U,\mu)$ generating C_0–semigroups that are *sub–Markovian.* Recall that in the symmetric case, i.e., $B = B^0 = (b_1^0,\dots,b_d^0)$, where

$$b_i^0 = \sum_{j=1}^d (\partial_j a_{ij} + a_{ij} 2\partial_j\varphi/\varphi)\,, 1\le i\le d\,, \tag{1.5}$$

one can use bilinear form techniques to obtain such an extension. In fact, the bilinear form

$$\mathcal{E}^0(u,v) = \int \langle A\nabla u,\nabla v\rangle\, d\mu\,; u,v\in H_0^{1,2}(U,\mu)\,,$$

is a well–defined symmetric Dirichlet form on $L^2(U,\mu)$. Let $(L^0, D(L^0))$ be the associated generator and $(T_t^0)_{t\ge0}$ the associated semigroup. It is easy to see that our assumptions imply that $C_0^\infty(U)\subset D(L^0)$ and $L^0u = L^Au + \langle B^0,\nabla u\rangle$, $u\in C_0^\infty(U)$. Since T_t^0 is sub–Markovian and symmetric it can be extended uniquely to a sub–Markovian contraction $\overline{T_t^0}$ on $L^1(U,\mu)$. It is easy to see that $(\overline{T_t^0})_{t\ge0}$ is a C_0–semigroup on $L^1(U,\mu)$ and that the corresponding generator $(\overline{L^0}, D(\overline{L^0}))$ is the closure of $(L^0, D(L^0))$ on $L^1(U,\mu)$.

In the more general nonsymmetric case there is no symmetric bilinear form associated with $L^A + B\cdot\nabla$. Note that on the other hand we have that $L^Au + \langle B,\nabla u\rangle = L^0u + \langle\beta,\nabla u\rangle$, $u\in C_0^\infty(U)$, where $\beta := B - B^0$ is such that $\beta\in L^2(U;\mathbb{R}^d,\mu)$ and

$$\int \langle\beta,\nabla u\rangle\, d\mu = 0 \text{ for all } u\in C_0^\infty(U), \tag{1.6}$$

since $\int\langle\beta,\nabla u\rangle\,d\mu = \int L^Au + \langle B,\nabla u\rangle\,d\mu - \int L^0u\,d\mu = 0$ for all $u\in C_0^\infty(U)$. Hence $L^A + B\cdot\nabla$ is associated with a first order perturbation of the symmetric bilinear form $\mathcal{E}^0$. Clearly, (1.6) extends to all $u\in H_0^{1,2}(U,\mu)$, in particular,

$$\int\langle\beta,\nabla u\rangle v\, d\mu = -\int\langle\beta,\nabla v\rangle u\, d\mu \text{ for all } u,v\in H_0^{1,2}(U,\mu)_b\ .$$

Therefore, $Lu := L^0u + \langle\beta,\nabla u\rangle$, $u\in D(L^0)_b$, is an extension of $L^Au + \langle B,\nabla u\rangle$, $u\in C_0^\infty(U)$.

Proposition 1.1. *Let (1.1)–(1.4) be satisfied. Then:*

(i) The operator $Lu = L^0u + \langle\beta, \nabla u\rangle$, $u \in D(L^0)_b$, is dissipative, hence in particular closable, on $L^1(U,\mu)$. The closure $(\overline{L}, D(\overline{L}))$ generates a sub–Markovian C_0–semigroup of contractions $(\overline{T}_t)_{t\geq 0}$.

(ii) $D(\overline{L})_b \subset H_0^{1,2}(U,\mu)$ and

$$(1.7) \qquad \mathcal{E}^0(u,v) - \int \langle\beta, \nabla u\rangle v\, d\mu = -\int \overline{L}u\, v\, d\mu\,; u \in D(\overline{L})_b, v \in H_0^{1,2}(U,\mu)_b\ .$$

In particular,

$$(1.8) \qquad \mathcal{E}^0(u,u) = -\int \overline{L}u\, u\, d\mu\,; u \in D(\overline{L})_b\ .$$

Recall that a densely defined linear operator $(A, D(A))$ on a Banach space X is called *dissipative* if for any $u \in D(A)$ there exists a normalized tangent functional ℓ (i.e., an element $\ell \in X'$ (= the topological dual of X) satisfying $\|\ell\|_{X'} = \|u\|_X$ and $\ell(u) = \|u\|_X^2$) such that $\ell(Au) \leq 0$.

Note that the sub–Markovian C_0–semigroup of contractions $(\overline{T}_t)_{t\geq 0}$ on $L^1(U,\mu)$ can be restricted to a semigroup of contractions on $L^p(U,\mu)$ for all $p \in (0,\infty)$ by the Riesz–Thorin Interpolation Theorem (cf. [ReSi, Theorem IX.17]) and that the restricted semigroup is strongly continuous on $L^p(U,\mu)$. The corresponding generator $(\overline{L}_p, D(\overline{L}_p))$ is the *part of* $(\overline{L}, D(\overline{L}))$ *on* $L^p(U,\mu)$, i.e., $D(\overline{L}_p) = \{u \in D(\overline{L}) \cap L^p(U,\mu) | \overline{L}u \in L^p(U,\mu)\}$ and $\overline{L}_p u = \overline{L}u$, $u \in D(\overline{L}_p)$.

For the proof of 1.1 we need additional information on the closure $(\overline{L^0}, D(\overline{L^0}))$ of $(L^0, D(L^0))$ on $L^1(U,\mu)$:

Lemma 1.2.

(i) $D(\overline{L^0})_b \subset H_0^{1,2}(U,\mu)$.

(ii) $\lim_{t\to 0} T_t^0 u = u$ in $H_0^{1,2}(U,\mu)$ for all $u \in D(\overline{L^0})_b$.

(iii) $\mathcal{E}^0(u,v) = -\int \overline{L^0}u\, v\, d\mu$ for all $u \in D(\overline{L^0})_b$, $v \in H_0^{1,2}(U,\mu)_b$.

(iv) Let $\varphi \in C^2(\mathbb{R})$, $\varphi(0) = 0$, and $u \in D(\overline{L^0})_b$. Then $\varphi(u) \in D(\overline{L^0})$ and

$$\overline{L^0}\varphi(u) = \varphi'(u)\overline{L^0}u + \varphi''(u)\langle A\nabla u, \nabla u\rangle\ .$$

Proof. Let $u \in L(\overline{L^0})_b$. Then $T_t^0 u \in D(L^0) \subset H_0^{1,2}(U,\mu)$ if $t > 0$ and

$$\begin{aligned}\mathcal{E}^0(T_t^0u - T_s^0u, T_t^0u - T_s^0u) &= -\int L^0(T_t^0u - T_s^0u)(T_t^0u - T_s^0u)\, d\mu \\ &= -\int (\overline{T_t^0}\overline{L^0}u - \overline{T_s^0}\overline{L^0}u)(T_t^0u - T_s^0u)\, d\mu \\ &\leq 2\|u\|_\infty \|\overline{T_t^0}\overline{L^0}u - \overline{T_s^0}\overline{L^0}u\|_1\,, s,t > 0\ .\end{aligned}$$

Therefore $(T_t^0 u)_{t>0}$ is an $H_0^{1,2}(U,\mu)$–Cauchy sequence, which implies that $u \in H_0^{1,2}(U,\mu)$ and $\lim_{t\to 0} T_t^0 u = u$ in $H_0^{1,2}(U,\mu)$. Hence (i) and (ii) are proved.

Let $v \in H_0^{1,2}(U,\mu)_b$. Then

$$\begin{aligned}\mathcal{E}^0(u,v) &= \lim_{t\to 0} \mathcal{E}^0(T_t^0 u, v) = \lim_{t\to 0} -(L^0 T_t^0 u, v)_{L^2(U,\mu)} \\ &= \lim_{t\to 0} - \int \overline{T_t^0 L^0} u\, v\, d\mu = - \int \overline{L^0} u\, v\, d\mu\end{aligned}$$

which proves (iii).

(iv) Since $u \in H_0^{1,2}(U,\mu)_b$ we obtain that $\varphi(u) \in H_0^{1,2}(U,\mu)_b$ and $\partial_i \varphi(u) = \varphi'(u)\partial_i u$. Hence for all $v \in H_0^{1,2}(U,\mu)_b$ by (iii)

$$\begin{aligned}\mathcal{E}^0(\varphi(u),v) &= \int \langle A\nabla u, \nabla v\rangle \varphi'(u)\, d\mu \\ &= \int \langle A\nabla u, \nabla(v\varphi'(u))\rangle\, d\mu - \int \langle A\nabla u, \nabla u\rangle \varphi''(u) v\, d\mu \\ &= - \int (\varphi'(u)\overline{L^0}u + \varphi''(u)\langle A\nabla u, \nabla u\rangle) v\, d\mu\ .\end{aligned}$$

Since $\varphi'(u)\overline{L^0}u + \varphi''(u)\langle A\nabla u, \nabla u\rangle \in L^1(\mathbb{R}^d,\mu)$ the assertion now follows from [BH, I.4.2.2.1]. $\square$

Proof (of 1.1). (i) **Step 1:** Let $u \in D(L^0)_b$ then $\int Lu 1_{\{u>1\}}\, d\mu \le 0$.

To show this let $\psi_\varepsilon \in C^2(\mathbb{R})$, $\varepsilon > 0$, be such that $\psi_\varepsilon(t) = 0$ if $t \le 1$, $0 \le \psi_\varepsilon' \le 1$, $\psi_\varepsilon'(t) = 1$ if $t \ge 1+\varepsilon$ and $\psi_\varepsilon'' \ge 0$. Then $\psi_\varepsilon(u) \in D(\overline{L^0})_b$ by 1.2, $\psi_\varepsilon(u) \ge 0$ and thus

$$\begin{aligned}(1.9) \qquad 0 &\ge \int \overline{L^0}\psi_\varepsilon(u)\, d\mu = \int L^0 u\, \psi_\varepsilon'(u)\, d\mu + \int \psi_\varepsilon''(u)\langle A\nabla u, \nabla u\rangle\, d\mu \\ &\ge \int L^0 u\, \psi_\varepsilon'(u)\, d\mu\ .\end{aligned}$$

Since $\lim_{\varepsilon\to 0} \psi_\varepsilon'(u) = 1_{\{u>1\}}$ and $\|\psi_\varepsilon'(u)\|_\infty \le 1$ it follows from Lebesgue's Theorem that

$$\int L^0 u\, 1_{\{u>1\}}\, d\mu = \lim_{\varepsilon\to 0} \int L^0 u\, \psi_\varepsilon'(u)\, d\mu \le 0\ .$$

Similarly,

$$\begin{aligned}\int \langle \beta, \nabla u\rangle 1_{\{u>1\}}\, d\mu &= \lim_{\varepsilon\to 0} \int \langle \beta, \nabla u\rangle \psi_\varepsilon'(u) d\mu \\ &= \lim_{\varepsilon\to 0} \int \langle \beta, \nabla \psi_\varepsilon(u)\rangle\, d\mu = 0\end{aligned}$$

by (1.5). Hence $\int Lu\, 1_{\{u>1\}}\, d\mu \le 0$ and Step 1 is proved.

Note that Step 1 implies in particular $n\int Lu\,1_{\{nu>1\}}\,d\mu \le 0$ for all n, hence $\int Lu1_{\{u>0\}}\,d\mu \le 0$ and consequently,

$$\int Lu(1_{\{u>0\}} - 1_{\{u<0\}})\,d\mu \le 0 \ .$$

Since $\|u\|_1(1_{\{u>0\}} - 1_{\{u<0\}}) \in L^\infty(U,\mu) = (L^1(U,\mu))'$ is a normalized tangent functional to u we obtain that $(L, D(L^0)_b)$ is dissipative.

It follows from [ReSi, Theorem X.48] that the closure $(\overline{L}, D(\overline{L}))$ generates a C_0–semigroup of contractions on $L^1(U,\mu)$ if and only if $(1-L)(D(L^0)_b) \subset L^1(U,\mu)$ dense which will be proved in the next step.

Step 2: $(1-L)(D(L^0)_b) \subset L^1(U,\mu)$ dense.

Let $h \in L^\infty(U,\mu)$ be such that $\int(1-L)u\,h\,d\mu = 0$ for all $u \in D(L^0)_b$. Then $u \mapsto \int(1-L^0)u\,h\,d\mu = \int\langle\beta,\nabla u\rangle h\,d\mu$, $u \in D(L^0)_b$, is continuous w.r.t. the norm on $H_0^{1,2}(U,\mu)$ which implies the existence of some element $v \in H_0^{1,2}(U,\mu)$ such that $\mathcal{E}_1^0(u,v) = \int(1-L^0)u\,h\,d\mu$ for all $u \in D(L^0)_b$. It follows that $\int(1-L^0)u(h-v)\,d\mu = 0$ for all $u \in D(L^0)_b$. Since $(L^0, D(L^0))$ is a Dirichlet operator we obtain that $(1-L)(D(L^0)_b) \subset L^1(U,\mu)$ dense and consequently, $h = v$. In particular, $h \in H_0^{1,2}(U,\mu)$ and

$$\begin{aligned}\mathcal{E}_1^0(h,h) &= \lim_{t\to 0}\mathcal{E}_1^0(T_t^0 h, h) = \lim_{t\to 0}\int(1-L^0)T_t^0 h\,h\,d\mu \\ &= \lim_{t\to 0}\int\langle B,\nabla T_t^0 h\rangle h\,d\mu = \int\langle B,\nabla h\rangle h\,d\mu = 0\end{aligned}$$

by (1.5) and therefore $h = 0$.

It follows that the closure $(\overline{L}, D(\overline{L}))$ generates a C_0–semigroup of contractions $(\overline{T}_t)_{t\ge 0}$.

Step 3: $(\overline{T}_t)_{t\ge 0}$ is sub–Markovian.

Let $(\overline{G}_\alpha)_{\alpha>0}$ be the associated resolvent, i.e., $\overline{G}_\alpha := (\alpha - \overline{L})^{-1}$. We will show that $(\overline{G}_\alpha)_{\alpha>0}$ is sub–Markovian. Since $\overline{T}_t u = \lim_{\alpha\to\infty}\exp(t\alpha(\alpha\overline{G}_\alpha - 1))u$ for all $u \in L^1(U,\mu)$ (cf. [Pa, 1.3.5]) we then obtain that $(\overline{T}_t)_{t\ge 0}$ is sub–Markovian too.

To this end let $u \in D(\overline{L})$ and $u_n \in D(\overline{L^0})_b$ such that $\lim_{n\to\infty}\|u_n - u\|_1 + \|Lu_n - \overline{L}u_n\|_1 = 0$ and $\lim_{n\to\infty} u_n = u$ μ–a.e. Let ψ_ε, $\varepsilon > 0$, be as in Step 1. Then $\int Lu_n\psi_\varepsilon'(u_n)\,d\mu \le 0$ for all n by (1.9) and thus $\int\overline{L}u\,\psi_\varepsilon'(u)\,d\mu \le 0$. Taking the limit $\varepsilon \to 0$ we conclude that

$$\int\overline{L}u\,1_{\{u>1\}}\,d\mu \le 0 \ .$$

Let $f \in L^1(U,\mu)$ and $u := \alpha\overline{G}_\alpha f \in D(\overline{L})$. If $f \le 1$ then

$$\begin{aligned}\alpha\int u1_{\{u>1\}}\,d\mu &\le \int(\alpha u - \overline{L}u)1_{\{u>1\}}\,d\mu \\ &= \alpha\int f1_{\{u>1\}}\,d\mu \le \alpha\int 1_{\{u>1\}}\,d\mu \ .\end{aligned}$$

Consequently, $\alpha\int(u-1)1_{\{u>1\}}\,d\mu \leq 0$ which implies that $u \leq 1$. If $f \geq 0$ then $-nf \leq 1$, hence $-nu \leq 1$ for all n, i.e., $u \geq 0$. Hence $(\overline{G}_\alpha)_{\alpha>0}$ is sub–Markovian and (i) is proved.

(ii) **Step 1:** $D(\overline{L^0})_b \subset D(\overline{L})$ and $\overline{L}u = \overline{L^0}u + \langle \beta, \nabla u\rangle$, $u \in D(\overline{L^0})_b$.

Let $u \in D(\overline{L^0})_b$. Then $T_t^0 u \in D(L^0)_b \subset D(\overline{L})$ and $\overline{L}T_t^0 u = L^0 T_t^0 u + \langle \beta, \nabla T_t^0 u\rangle = \overline{T^0}_t \overline{L^0} u + \langle \beta, \nabla T_t^0 u\rangle$. Since $\lim_{t\to 0} T_t^0 u = u$ in $H_0^{1,2}(U,\mu)$ by 1.2 it follows that $\lim_{t\to 0} \overline{L}T_t^0 u = \overline{L^0}u + \langle\beta, \nabla u\rangle$ in $L^1(U,\mu)$. Hence $u \in D(\overline{L})$ and $\overline{L}u = \overline{L^0}u + \langle \beta, \nabla u\rangle$ by closedness of $(\overline{L}, D(\overline{L}))$.

Step 2: Let $u \in D(\overline{L})_b$, $u_n \in D(L^0)_b$ be as in Step 3 of (i) and $\|u\|_\infty < M_1 < M_2$. Then

$$\lim_{n\to\infty} \int_{\{M_1 \leq |u_n| \leq M_2\}} \langle A\nabla u_n, \nabla u_n\rangle\, d\mu = 0\ . \tag{1.10}$$

Indeed, let $\varphi \in C^1(\mathbb{R})$ be such that $\varphi'(t) := (t-M_1)^+ \wedge (M_2 - M_1)$, $t \in \mathbb{R}$, and $\varphi(0) = 0$. Then by (1.6)

$$\begin{aligned}\int_{\{M_1 \leq u_n \leq M_2\}} \langle A\nabla u_n, \nabla u_n\rangle\, d\mu &= \int \langle A\nabla u_n, \nabla \varphi'(u_n)\rangle\, d\mu \\ &= -\int L^0 u_n \varphi'(u_n)\, d\mu - \int \langle \beta, \nabla\varphi(u_n)\rangle\, d\mu \\ &= -\int L u_n \varphi'(u_n)\, d\mu \to -\int \overline{L}u\varphi'(u)\, d\mu = 0\,,\end{aligned}$$

if $n \to \infty$, since $\varphi'(u) \equiv 0$. Similarly, $\lim_{n\to\infty}\int_{\{-M_2 \leq u_n \leq -M_1\}} \langle A\nabla u_n, \nabla u_n\rangle\, d\mu = 0$.

Step 3: Let $(u_n)_{n\geq 1}$ be as in Step 2 and $\psi \in C_b^2(\mathbb{R})$ be such that $\psi(t) = t$ if $|t| \leq \|u\|_\infty + 1$ and $\psi(t) = 0$ if $|t| \geq \|u\|_\infty + 2$. Then $\lim_{n\to\infty}\psi(u_n) = u$ μ–a.e. and in $L^p(U,\mu)$ for all $p \in [1,\infty)$, $\psi(u_n) \in D(\overline{L^0})_b \subset D(\overline{L})$ and $\lim_{n\to\infty} \overline{L}\psi(u_n) = \lim_{n\to\infty} \psi'(u_n) L u_n + \psi''(u_n)\langle A\nabla u_n, \nabla u_n\rangle = \overline{L}u$ in $L^1(U,\mu)$ by 1.2, Step 1 and (1.10). Consequently,

$$\begin{aligned}&\mathcal{E}^0(\psi(u_n) - \psi(u_m), \psi(u_n) - \psi(u_m)) \\ &\qquad = -\int \overline{L}(\psi(u_n) - \psi(u_m))(\psi(u_n) - \psi(u_m))\, d\mu \\ &\qquad \leq 2\|\psi\|_\infty \|\overline{L}\psi(u_n) - \overline{L}\psi(u_m)\|_1 \to 0\,; n, m \to \infty\,.\end{aligned}$$

Hence $u \in H_0^{1,2}(U,\mu)$ and $\lim_{n\to\infty}\psi(u_n) = u$ in $H_0^{1,2}(U,\mu)$ since $H_0^{1,2}(U,\mu)$ is complete. If $v \in H_0^{1,2}(U,\mu)_b$ then

$$\begin{aligned}\mathcal{E}^0(u,v) - \int \langle\beta, \nabla u\rangle v\, d\mu &= \lim_{n\to\infty} \mathcal{E}^0(\psi(u_n), v) - \int \langle \beta, \nabla\psi(u_n)\rangle v\, d\mu \\ &= -\lim_{n\to\infty} \int \overline{L}\psi(u_n) v\, d\mu = -\int \overline{L}uv\, d\mu\ .\end{aligned}$$

Hence (1.7) is proved. Clearly, (1.8) follows from (1.7) by taking $v = u$ and using (1.6). This completes the proof of 1.1. $\square$

Remark 1.3. Since $-\beta$ satisfies the same assumptions as β the closure $(\overline{L}', D(\overline{L}'))$ of $L^0u - \langle\beta, \nabla u\rangle$, $u \in D(L^0)_b$, on $L^1(U,\mu)$ generates a sub–Markovian C_0–semigroup of contractions $(\overline{T}'_t)_{t\geq 0}$, $D(\overline{L}')_b \subset H_0^{1,2}(U,\mu)$ and

$$\mathcal{E}^0(u,v) + \int \langle\beta, \nabla u\rangle v\, d\mu = -\int \overline{L}' u\, v\, d\mu\,; u \in D(\overline{L}')_b, v \in H_0^{1,2}(U,\mu)_b\ .$$

If $(L', D(L'))$ is the part of $(\overline{L}', D(\overline{L}'))$ on $L^2(U,\mu)$ and $(L, D(L))$ is the part of $(\overline{L}, D(\overline{L}))$ on $L^2(U,\mu)$ then

$$\begin{aligned}(Lu,v)_{L^2(U,\mu)} &= -\mathcal{E}^0(u,v) + \int \langle\beta, \nabla u\rangle v\, d\mu \\ &= -\mathcal{E}^0(v,u) - \int \langle\beta, \nabla v\rangle u\, d\mu \\ &= (L'v, u)_{L^2(U,\mu)}\end{aligned} \tag{1.11}$$

for all $u \in D(L)_b$, $v \in D(L')_b$. Since $(L, D(L))$ (resp. $(L', D(L'))$) is the generator of a sub–Markovian C_0–semigroup it follows that $D(L)_b \subset D(L)$ (resp. $D(L')_b \subset D(L')$) dense w.r.t. the graph norm and hence (1.11) extends to all $u \in D(L)$, $v \in D(L')$ which implies that L and L' are adjoint operators on $L^2(U,\mu)$.

Remark 1.4. It is a remarkable fact that the well–known correspondences between Dirichlet operators and sub–Markovian semigroups of contractions on L^2–spaces (cf. [MR, I.4.4]) do have analogues on L^1. More precisely, let $(A, D(A))$ be the generator of a C_0–semigroup of contractions $(S_t)_{t\geq 0}$ on $L^1(X,m)$. Then it is easy to see that the following statements are equivalent:

(i) $\int Au\, 1_{\{u>1\}}\, dm \leq 0$ for all $u \in D(A)$.

(ii) $(S_t)_{t\geq 0}$ is sub–Markovian.

Similarly, the following statements are equivalent:

(i') $\int Au\, 1_{\{u>0\}}\, dm \leq 0$ for all $u \in D(A)$.

(ii') $(S_t)_{t\geq 0}$ is positivity preserving (i.e., $S_t f \geq 0$ if $f \geq 0$).

Also note that for a linear operator $(A, D(A))$ on $L^1(X,m)$ we always have that (i) implies (i') and (i') implies that $(A, D(A))$ is dissipative.

b) Existence under local assumptions.

1.1 can be generalized in various directions. First of all, $(\mathcal{E}^0, H_0^{1,2}(U,\mu))$ can be replaced by any other sectorial Dirichlet form $(\mathcal{E}, H_0^{1,2}(U,\mu))$ in the sense of [MR] such that the $\mathcal{E}_1^{\frac{1}{2}}$–norm is equivalent to the norm on $H_0^{1,2}(U,\mu)$. A generalization to the infinite dimensional case will be considered in Section 4. Another possible generalization in the finite dimensional case is obtained by "localizing" the assumptions on μ, A and B.

More precisely, suppose from now on that the density φ^2 of μ is such that $\varphi \in H^{1,2}_{loc}(U)$. In particular, the measure μ can be infinite. We replace assumption (1.1) by

$$a_{ij} = a_{ji} \in H^{1,2}_{loc}(U,\mu)\ , 1 \le i,j \le d\ . \tag{1.1'}$$

Instead of (1.2) we suppose that for all V relatively compact in U there exist $\nu_V > 0$ such that

$$\nu_V^{-1}|h|^2 \le \langle A(x)h,h\rangle \le \nu_V |h|^2 \text{ for all } h \in \mathbb{R}^d, x \in V\ . \tag{1.2'}$$

By the results of [MR, Subsection II.2b] the bilinear form

$$\mathcal{E}^0(u,v) = \int \langle A\nabla u, \nabla v\rangle\, d\mu\,; u,v \in C_0^\infty(U)\,,$$

is closable on $L^2(U,\mu)$, and we denote its closure by $(\mathcal{E}^0, D(\mathcal{E}^0))$, the associated generator by $(L^0, D(L^0))$ and the corresponding semigroup by $(T_t^0)_{t\ge 0}$. Similar to Subsection a) $C_0^\infty(U) \subset D(L^0)$ and $L^0u = L^Au + \langle B^0, \nabla u\rangle$, $u \in C_0^\infty(U)$. Finally we replace (1.3) by

$$B \in L^2_{loc}(U;\mathbb{R}^d,\mu) \tag{1.3'}$$

i.e., $\int_V |B|^2\, d\mu < \infty$ for all V relatively compact in U, and we suppose that (1.4) holds. Let $B^0 = (b_1^0, \dots, b_d^0)$ be as in (1.5). Then similar to Subsection a) $L^Au + \langle B, \nabla u\rangle = L^0u + \langle \beta, \nabla u\rangle$, $u \in C_0^\infty(U)$, where $\beta := B - B^0 \in L^2_{loc}(U;\mathbb{R}^d,\mu)$ again satisfies $\int \langle \beta, \nabla u\rangle\, d\mu = 0$ for all $u \in C_0^\infty(U)$. Therefore, $Lu := L^0u + \langle \beta, \nabla u\rangle$, $u \in D(L^0)_{0,b}$, is an extension of $L^Au + \langle B, \nabla u\rangle$, $u \in C_0^\infty(U)$.

For all open subsets V relatively compact in U the bilinear form $\int \langle A\nabla u, \nabla v\rangle\, d\mu$, $u,v \in H_0^{1,2}(V,\mu)$, is closed. Let $(L^{0,V}, D(L^{0,V}))$ be the corresponding generator. Since $\int \langle \beta, \nabla u\rangle\, d\mu = 0$ for all $u \in H_0^{1,2}(V,\mu)$ the closure $(\overline{L}^V, D(\overline{L}^V))$ of $L^Vu := L^{0,V}u + \langle \beta, \nabla u\rangle$, $u \in D(L^{0,V})_b$, on $L^1(V,\mu)$ generates a sub–Markovian C_0–semigroup of contractions by 1.1. Let $(\overline{G}_\alpha^V)_{\alpha>0}$ be the corresponding resolvent. If we define

$$\overline{G}_\alpha^V f := \overline{G}_\alpha^V (f 1_V)\,; f \in L^1(U,\mu)\,, \alpha > 0\,,$$

then $\overline{G}_\alpha^V$, $\alpha > 0$, can be extended to a sub–Markovian contraction on $L^1(U,\mu)$.

Theorem 1.5. *Let (1.1')–(1.3') and (1.4) be satisfied. Then there exists a (closed) extension $(\overline{L}, D(\overline{L}))$ of $Lu := L^0u + \langle \beta, \nabla u\rangle$, $u \in D(L^0)_{0,b}$, on $L^1(U,\mu)$ satisfying the following properties:*

(a) $(\overline{L}, D(\overline{L}))$ generates a sub–Markovian C_0–semigroup of contractions $(\overline{T}_t)_{t\ge 0}$.

(b) If $(U_n)_{n\ge 1}$ is an increasing sequence of open subsets relatively compact in U such that $U = \bigcup_{n\ge 1} U_n$ then $\lim_{n\to\infty} \overline{G}_\alpha^{U_n} f = (\alpha - \overline{L})^{-1} f$ in $L^1(U,\mu)$ for all $f \in L^1(U,\mu)$ and $\alpha > 0$.

(c) $D(\overline{L})_b \subset D(\mathcal{E}^0)$ *and*

$$\mathcal{E}^0(u,v) - \int \langle \beta, \nabla u\rangle v\, d\mu = -\int \overline{L}u\, v\, d\mu\,;\, u \in D(\overline{L})_b, v \in H_0^{1,2}(U,\mu)_{0,b}\ .$$

Moreover,

$$\mathcal{E}^0(u,u) \le -\int \overline{L}u\, u\, d\mu\,;\, u \in D(\overline{L})_b\ .$$

The proof of 1.5 is based on the following lemma.

Lemma 1.6. *Let V_1, V_2 be relatively compact in U and $V_1 \subset V_2$. Let $u \in L^1(U,\mu)$, $u \ge 0$, and $\alpha > 0$. Then $\overline{G}_\alpha^{V_1} u \le \overline{G}_\alpha^{V_2} u$.*

Proof. Clearly, we may assume that u is bounded. Let $w_\alpha := \overline{G}_\alpha^{V_1} u - \overline{G}_\alpha^{V_2} u$. Then $w_\alpha \in H_0^{1,2}(V_2,\mu)$ but also $w_\alpha^+ \in H_0^{1,2}(V_1,\mu)$ since $w_\alpha^+ \le \alpha \overline{G}_\alpha^{V_1} u$ and $\alpha\overline{G}_\alpha^{V_1} u \in H_0^{1,2}(V_1,\mu)$. Note that $\int \langle \beta, \nabla w_\alpha\rangle w_\alpha^+\, d\mu = \int \langle \beta, \nabla w_\alpha^+\rangle w_\alpha^+\, d\mu = 0$ and $\mathcal{E}^0(w_\alpha^+, w_\alpha^-) \le 0$ since $(\mathcal{E}^0, H_0^{1,2}(V_2,\mu))$ is a Dirichlet form. Hence by (1.7)

$$\begin{aligned}\mathcal{E}_\alpha^0(w_\alpha^+, w_\alpha^+) &\le \mathcal{E}_\alpha^0(w_\alpha, w_\alpha^+) - \int \langle \beta, \nabla w_\alpha\rangle w_\alpha^+\, d\mu \\ &= \int (\alpha - \overline{L}^{V_1})\overline{G}_\alpha^{V_1} u\, w_\alpha^+\, d\mu - \int (\alpha - \overline{L}^{V_2})\overline{G}_\alpha^{V_2} u\, w_\alpha^+\, d\mu = 0\ .\end{aligned}$$

Consequently, $w_\alpha^+ = 0$, i.e., $\overline{G}_\alpha^{V_1} u \le \overline{G}_\alpha^{V_2} u$. $\square$

Proof of 1.5. Let $(V_n)_{n\ge 1}$ be an increasing sequence of open subsets relatively compact in U such that $\overline{V_n} \subset V_{n+1}^\circ$, $n \ge 1$, and $U = \bigcup_{n\ge 1} V_n$. Let $f \in L^1(U,\mu)$, $f \ge 0$. Then $\lim_{n\to\infty} \overline{G}_\alpha^{V_n} f =: \overline{G}_\alpha f$ exists μ–a.e. by 1.6. We will show below that $(\overline{G}_\alpha)_{\alpha>0}$ is a sub–Markovian C_0–resolvent of contractions on $L^1(U,\mu)$. We will then show that the corresponding generator satisfies properties (a)–(c) as stated in the Theorem.

Since $\int \alpha \overline{G}_\alpha^{V_n} f\, d\mu \le \int f 1_{V_n}\, d\mu$ it follows that $\alpha\overline{G}_\alpha f \in L^1(U,\mu)$, $\lim_{n\to\infty} \alpha\overline{G}_\alpha^{V_n} f = \alpha \overline{G}_\alpha f$ in $L^1(U,\mu)$ and

$$\int \alpha \overline{G}_\alpha f\, d\mu \le \int f\, d\mu\ . \tag{1.12}$$

For arbitrary $f \in L^1(U,\mu)$ let $\alpha\overline{G}_\alpha f := \alpha\overline{G}_\alpha f^+ - \alpha\overline{G}_\alpha f^-$. Then $\int |\alpha \overline{G}_\alpha f|\, d\mu \le \int \alpha\overline{G}_\alpha f^+ + \alpha\overline{G}_\alpha f^-\, d\mu \le \int |f|\, d\mu$ by (1.12). Hence $\alpha\overline{G}_\alpha$ is a contraction on $L^1(U,\mu)$. Clearly, $\alpha\overline{G}_\alpha$ is sub–Markovian. The family $(\overline{G}_\alpha)_{\alpha>0}$ satisfies the resolvent equation since $(\overline{G}_\alpha^{V_n})_{\alpha>0}$ satisfies the resolvent equation for all n, $\lim_{n\to\infty} \|\overline{G}_\alpha^{V_n}\overline{G}_\beta f - \overline{G}_\alpha^{V_n}\overline{G}_\beta^{V_n} f\|_1 \le \lim_{n\to\infty} \frac{1}{\alpha}\|\overline{G}_\beta f - \overline{G}_\beta^{V_n} f\|_1 = 0$ if $\alpha, \beta > 0$ and thus

$$\begin{aligned}(\beta-\alpha)\overline{G}_\alpha\overline{G}_\beta f &= \lim_{n\to\infty} (\beta-\alpha)\overline{G}_\alpha^{V_n}\overline{G}_\beta f = \lim_{n\to\infty}(\beta-\alpha)\overline{G}_\alpha^{V_n}\overline{G}_\beta^{V_n} f \\ &= \lim_{n\to\infty} \overline{G}_\alpha^{V_n} f - \overline{G}_\beta^{V_n} f = \overline{G}_\alpha f - \overline{G}_\beta f\end{aligned}$$

for all $\alpha, \beta > 0$.

Let $f \in L^1(U,\mu)_b$. Then by (1.8)

$$\mathcal{E}^0_\alpha(\overline{G}^{V_n}_\alpha f, \overline{G}^{V_n}_\alpha f) = \int f 1_{V_n} \overline{G}^{V_n}_\alpha f \, d\mu \leq \frac{1}{\alpha} \|f\|_\infty \|f\|_1 \ .$$

Consequently, $\sup_{n\geq 1} \mathcal{E}^0_\alpha(\overline{G}^{V_n}_\alpha f, \overline{G}^{V_n}_\alpha f) < +\infty$, hence $\overline{G}^{V_n}_\alpha f \in D(\mathcal{E}^0)$ by Banach–Alaoglu, $\lim_{n\to\infty} \overline{G}^{V_n}_\alpha f = \overline{G}_\alpha f$ weakly in $D(\mathcal{E}^0)$ and

$$\mathcal{E}^0_\alpha(\overline{G}_\alpha f, \overline{G}_\alpha f) \leq \liminf_{n\to\infty} \mathcal{E}^0_\alpha(\overline{G}^{V_n}_\alpha f, \overline{G}^{V_n}_\alpha f) = \int f \overline{G}_\alpha f \, d\mu \ . \tag{1.13}$$

If $v \in H^{1,2}_0(U,\mu)_{0,b}$ then $v \in H^{1,2}_0(V_n,\mu)_b$ for big n and hence by (1.7)

(1.14)
$$\begin{aligned}\mathcal{E}^0_\alpha(\overline{G}_\alpha f, v) - \int \langle \beta, \nabla \overline{G}_\alpha f\rangle v \, d\mu &= \lim_{n\to\infty} \mathcal{E}^0_\alpha(\overline{G}^{V_n}_\alpha f, v) - \int \langle \beta, \nabla \overline{G}^{V_n}_\alpha f\rangle v \, d\mu \\ &= \int f v \, d\mu \ .\end{aligned}$$

To see the strong continuity of $(\overline{G}_\alpha)_{\alpha>0}$ note that $u = \overline{G}^{V_n}_\alpha (\alpha - L^{V_n})u = \overline{G}^{V_n}_\alpha(\alpha - L)u$ for $u \in D(L^0)_{0,b}$ and for big n. Hence

$$u = \overline{G}_\alpha(\alpha - L)u \ . \tag{1.15}$$

In particular, $\|\alpha \overline{G}_\alpha u - u\|_1 = \|\alpha\overline{G}_\alpha u - \overline{G}_\alpha(\alpha - L)u\|_1 = \|\overline{G}_\alpha L u\|_1 \leq \frac{1}{\alpha}\|Lu\|_1 \to 0$, $\alpha \to \infty$, for all $u \in C^\infty_0(U)$ and the strong continuity then follows by a 3–ε–argument.

Let $(\overline{L}, D(\overline{L}))$ be the generator of $(\overline{G}_\alpha)_{\alpha>0}$. Then $(\overline{L}, D(\overline{L}))$ extends $(L, D(L^0)_{0,b})$ by (1.15). By the Hille–Yosida Theorem $(\overline{L}, D(\overline{L}))$ generates a C_0–semigroup of contractions $(\overline{T}_t)_{t\geq 0}$. Since $\overline{T}_t u = \lim_{\alpha\to\infty} \exp(t\alpha(\alpha\overline{G}_\alpha - 1))u$ for all $u \in L^1(U,\mu)$ (cf. [Pa, 1.3.5]) we obtain that $(\overline{T}_t)_{t\geq 0}$ is sub–Markovian.

We will show next that $(\overline{L}, D(\overline{L}))$ satisfies property (b). To this end let $(U_n)_{n\geq 1}$ be an increasing sequence of open subsets relatively compact in U such that $U = \bigcup_{n\geq 1} U_n$. Let $f \in L^1(U,\mu)$, $f \geq 0$. If $n \geq 1$ then by compactness of $\overline{V}_n$ there exist m such that $V_n \subset U_m$ and therefore $\overline{G}^{V_n} f \leq \overline{G}^{U_m} f$ by 1.6. Hence $\overline{G} f \leq \lim_{n\to\infty} \overline{G}^{U_n} f$. Similarly, $\lim_{n\to\infty} \overline{G}^{U_n} f \leq \overline{G} f$ hence (b) is satisfied.

Finally we will prove that $(\overline{L}, D(\overline{L}))$ satisfies property (c). Let $u \in D(\overline{L})_b$. Then $\alpha\overline{G}_\alpha u \in D(\mathcal{E}^0)$ and by (1.13)

$$\begin{aligned}\mathcal{E}^0(\alpha\overline{G}_\alpha u, \alpha\overline{G}_\alpha u) &\leq -\int \alpha \overline{L}\overline{G}_\alpha u \, \alpha\overline{G}_\alpha u \, d\mu \\ &= -\int \alpha\overline{G}_\alpha \overline{L} u \, \alpha \overline{G}_\alpha u \, d\mu \\ &\leq \|\overline{L}u\|_1 \|u\|_\infty \ .\end{aligned}$$

Consequently, $\sup_{\alpha>0}\mathcal{E}^0(\alpha\overline{G}_\alpha u, \alpha\overline{G}_\alpha u) < \infty$, hence $u \in D(\mathcal{E}^0)$, $\lim_{\alpha\to\infty}\alpha\overline{G}_\alpha u = u$ weakly in $D(\mathcal{E}^0)$ by Banach–Alaoglu and

$$\begin{aligned}\mathcal{E}^0(u,u) &\le \liminf_{\alpha\to\infty}\mathcal{E}^0(\alpha\overline{G}_\alpha u, \alpha\overline{G}_\alpha u) \le \liminf_{\alpha\to\infty} -\int \alpha\overline{G}_\alpha\overline{L}u\,\alpha\overline{G}_\alpha u\,d\mu \\ &= -\int \overline{L}u\,u\,d\mu\ .\end{aligned}$$

If $v \in D(\mathcal{E}^0)_{0,b}$ then by (1.14)

$$\begin{aligned}\mathcal{E}^0(u,v) - \int\langle\beta,\nabla u\rangle v\,d\mu &= \lim_{\alpha\to\infty}\mathcal{E}^0(\alpha\overline{G}_\alpha u, v) - \int\langle\beta,\nabla\alpha\overline{G}_\alpha u\rangle v\,d\mu \\ &= \lim_{\alpha\to\infty} -\int \alpha\overline{G}_\alpha\overline{L}u\,v\,d\mu = -\int\overline{L}u\,v\,d\mu\ .\end{aligned}$$

This completes the proof of 1.5. □

Remark 1.7. (i) Clearly, $(\overline{L}, D(\overline{L}))$ is uniquely determined by properties (a) and (b) in 1.5.

(ii) Similar to $(\overline{L}, D(\overline{L}))$ we can construct a closed extension $(\overline{L}', D(\overline{L}'))$ of $L^0u - \langle\beta,\nabla u\rangle$, $u \in D(L^0)_{0,b}$, that generates a sub–Markovian C_0–semigroup of contractions $(\overline{T}'_t)_{t\ge0}$. Since for all V relatively compact in U by (1.7) and (1.6)

$$\int\overline{G}^V_\alpha u\,v\,d\mu = \int u\overline{G}^{V,\prime}_\alpha v\,d\mu \text{ for all } u,v \in L^1(U,\mu)_b\ , \tag{1.16}$$

where $(\overline{G}^{V,\prime}_\alpha)_{\alpha>0}$ is the resolvent of the closure $(\overline{L}^{V,\prime}, D(\overline{L}^{V,\prime}))$ of $L^{0,V}u - \langle\beta,\nabla u\rangle$, $u \in D(L^{0,V})_b$, it follows that

$$\int\overline{G}_\alpha u\,v\,d\mu = \int u\,\overline{G}'_\alpha v\,d\mu \text{ for all } u,v \in L^1(U,\mu)_b\ , \tag{1.17}$$

where $\overline{G}'_\alpha = (\alpha - \overline{L}')^{-1}$.

(iii) Similar to the case of symmetric Dirichlet operators that admit a carré du champ (cf. [BH, I.4]) $D(\overline{L})_b$ is an algebra.

Proof: Let $u \in D(\overline{L})_b$. Clearly it is enough to show that $u^2 \in D(\overline{L})_b$. To this end it suffices to prove that if $g := 2u\overline{L}u + 2\langle A\nabla u,\nabla u\rangle$ then

$$\int\overline{L}'v\,u^2\,d\mu = \int gv\,d\mu \text{ for all } v = \overline{G}'_1 h\,, h \in L^1(U,\mu)_b\,, \tag{1.18}$$

since then $\int\overline{G}_1(u^2-g)h\,d\mu = \int(u^2-g)\overline{G}'_1 h\,d\mu = \int u^2(\overline{G}'_1 h - \overline{L}'\overline{G}'_1 h)\,d\mu = \int u^2\,h\,d\mu$ for all $h \in L^1(U,\mu)_b$. Consequently, $u^2 = \overline{G}_1(u^2-g) \in D(\overline{L})_b$.

For the proof of (1.18) fix $v = \overline{G}_1' h$, $h \in L^1(U,\mu)_b$, and suppose first that $u = \overline{G}_1 f$ for some $f \in L^1(U,\mu)_b$. Let $u_n := \overline{G}_1^{V_n} f$ and $v_n = \overline{G}_1^{V_n,\prime} h$, where $(V_n)_{n\geq 1}$ is as in the proof of 1.5. Then by 1.1 and 1.5

$$
\begin{aligned}
\int \overline{L}^{V_n,\prime} v_n\, u u_n \, d\mu &= -\mathcal{E}^0(v_n, u u_n) - \int \langle \beta, \nabla v_n\rangle u u_n \, d\mu \\
&= -\mathcal{E}^0(v_n u_n, u) - \int \langle A\nabla v_n, \nabla u_n\rangle u \, d\mu + \int \langle A\nabla u_n, \nabla u\rangle v_n \, d\mu \\
&\quad + \int \langle \beta, \nabla u\rangle v_n u_n \, d\mu + \int \langle \beta, \nabla u_n\rangle v_n u \, d\mu \\
&= \int \overline{L} u\, v_n u_n \, d\mu + \int \overline{L}^{V_n} u_n\, v_n u \, d\mu + \int \langle A\nabla u_n, \nabla(v_n u)\rangle \, d\mu \\
&\quad - \int \langle A\nabla v_n, \nabla u_n\rangle u \, d\mu + \int \langle A\nabla u_n, \nabla u\rangle v_n \, d\mu \\
&= \int \overline{L} u\, v_n u_n \, d\mu + \int \overline{L}^{V_n} u_n\, v_n u \, d\mu + 2\int \langle A\nabla u_n, \nabla u\rangle v_n \, d\mu \; .
\end{aligned}
$$

Note that $\lim_{n\to\infty} \int \langle A\nabla u_n, \nabla u\rangle v_n \, d\mu = \int \langle A\nabla u, \nabla u\rangle v \, d\mu$ since $\lim_{n\to\infty} u_n = u$ weakly in $D(\mathcal{E}^0)$ and $\lim_{n\to\infty} \langle A\nabla u, \nabla u\rangle v_n^2 = \langle A\nabla u, \nabla u\rangle v^2$ (strongly) in $L^1(U,\mu)$. Hence

$$
\begin{aligned}
\int \overline{L}' v\, u^2 \, d\mu &= \lim_{n\to\infty} \int \overline{L}^{V_n,\prime} v_n\, u u_n \, d\mu \\
&= \lim_{n\to\infty} \int \overline{L} u\, v_n u_n \, d\mu + \int \overline{L}^{V_n} u_n\, v_n u \, d\mu + 2\int \langle A\nabla u_n, \nabla u\rangle v_n \, d\mu \\
&= \int g v \, d\mu \; .
\end{aligned}
$$

Finally, if $u \in D(\overline{L})_b$ arbitrary, let $g_\alpha := 2(\alpha\overline{G}_\alpha u)\overline{L}(\alpha\overline{G}_\alpha u) + 2\langle A\nabla\alpha\overline{G}_\alpha u, \nabla\alpha\overline{G}_\alpha u\rangle$, $\alpha > 0$. Note that by 1.5 (c)

$$
\begin{aligned}
\mathcal{E}^0(\alpha\overline{G}_\alpha u - u, \alpha\overline{G}_\alpha u - u) &\leq -\int \overline{L}(\alpha\overline{G}_\alpha u - u)(\alpha\overline{G}_\alpha u - u)\, d\mu \\
&\leq 2\|u\|_\infty \|\alpha\overline{G}_\alpha\overline{L}u - \overline{L}u\|_1 \to 0
\end{aligned}
$$

if $\alpha \to \infty$ which implies that $\lim_{\alpha\to\infty} \alpha\overline{G}_\alpha u = u$ in $D(\mathcal{E}^0)$ and thus $\lim_{\alpha\to\infty} g_\alpha = g$ in $L^1(U,\mu)$. Since $\alpha u + (1-\alpha)\overline{G}_\alpha u \in L^1(U,\mu)_b$ and $\overline{G}_1(\alpha u + (1-\alpha)\overline{G}_\alpha u) = \alpha\overline{G}_\alpha u$ by the resolvent equation it follows from what we have just proved that

$$
\int \overline{L}' v (\alpha\overline{G}_\alpha u)^2 \, d\mu = \int g_\alpha v \, d\mu
$$

for all $\alpha > 0$ and thus, taking the limit $\alpha \to \infty$,

$$
\int \overline{L}' v\, u^2 \, d\mu = \int g v \, d\mu
$$

and (1.18) is shown. $\square$

c) Invariance.

Throughout this subsection let μ, A and B be as in Subsection b) but we suppose from now on that $U = \mathbb{R}^d$. Let $(\overline{L}, D(\overline{L}))$ be the closed extension of $Lu = L^0 u + \langle \beta, \nabla u\rangle$, $u \in D(L^0)_{0,b}$, satisfying properties (a)–(c) in 1.5 and denote by $(\overline{T}_t)_{t\geq 0}$ the associated semigroup. We say that the measure μ is $(\overline{T}_t)$*-invariant* if

$$\int \overline{T}_t u \, d\mu = \int u \, d\mu \text{ for all } u \in L^1(\mathbb{R}^d, \mu) \ . \tag{1.19}$$

Clearly, (1.19) is equivalent to the fact that $\int \overline{L}u \, d\mu = 0$ holds for all $u \in D(\overline{L})$ (or more generally for all $u \in D$ where $D \subset D(\overline{L})$ dense w.r.t. the graph norm). Note that although it is true that $\int \overline{L}u \, d\mu = 0$ for all $u \in D(L^0)_{0,b}$ the measure μ is not $(\overline{T}_t)$–invariant in general (cf. 1.12 below).

Definition 1.8. Let $p \in [1,\infty)$ and (A, D) be a densely defined operator on $L^p(X, m)$. We say that (A, D) is L^p*–unique*, if there is only one extension of (A, D) on $L^p(X, m)$ that generates a C_0–semigroup.

It follows from [Na, Theorem A–II, 1.33] that if (A, D) is L^p–unique and $(\overline{A}, \overline{D})$ the unique extension of (A, D) generating a C_0–semigroup it follows that $D \subset \overline{D}$ dense w.r.t. the graph norm. Equivalently, (A, D) is L^p–unique if and only if $(\alpha - A)(D) \subset L^p(X, m)$ dense for some $\alpha > 0$.

Proposition 1.9. *The following statements are equivalent:*

(i) There exist $\chi_n \in H^{1,2}_{loc}(\mathbb{R}^d, \mu)$ *and* $\alpha > 0$ *such that* $(\chi_n - 1)^- \in H^{1,2}_0(\mathbb{R}^d, \mu)_{0,b}$*,* $\lim_{n\to\infty} \chi_n = 0$ *μ–a.e. and*

$$\mathcal{E}^0_\alpha(\chi_n, v) + \int \langle \beta, \nabla \chi_n\rangle v \, d\mu \geq 0 \textit{ for all } v \in H^{1,2}_0(\mathbb{R}^d, \mu)_{0,b}, v \geq 0 \ . \tag{1.20}$$

(ii) $(L, D(L^0)_{0,b})$ *is* L^1*–unique.*

(iii) μ *is* $(\overline{T}_t)$*–invariant.*

Proof. *(i)* $\Rightarrow$ *(ii)*: It is sufficient to show that if $h \in L^\infty(\mathbb{R}^d, \mu)$ is such that $\int (\alpha - L)u \, h \, d\mu = 0$ for all $u \in D(L^0)_{0,b}$ it follows that $h = 0$.

To this end let $\chi \in C_0^\infty(\mathbb{R}^d)$. If $u \in D(L^0)_b$ it is easy to see that $\chi u \in D(L^0)_{0,b}$ and $L^0(\chi u) = \chi L^0 u + 2\langle A\nabla\chi, \nabla u\rangle + uL^0\chi$. Hence

$$\begin{aligned}\int (\alpha - L^0)u(\chi h) \, d\mu &= \int (\alpha - L^0)(u\chi)h \, d\mu + 2\int \langle A\nabla u, \nabla\chi\rangle h \, d\mu \\ &\quad + \int uL^0\chi \, h \, d\mu \\ &= \int \langle \beta, \nabla(u\chi)\rangle h \, d\mu + 2\int \langle A\nabla u, \nabla\chi\rangle h \, d\mu + \int uL^0\chi \, h \, d\mu \ .\end{aligned} \tag{1.21}$$

Since $|\beta| \in L^2_{loc}(\mathbb{R}^d, \mu)$ we obtain that $u \mapsto \int (\alpha - L^0)u(\chi h)\, d\mu$, $u \in D(L^0)_b$, is continuous w.r.t. the norm on $D(\mathcal{E}^0)$. Hence there exists some element $v \in D(\mathcal{E}^0)$ such that $\mathcal{E}^0_\alpha(u,v) = \int (\alpha - L^0)u(\chi h)\, d\mu$ and consequently, $\int (\alpha - L^0)u(v - \chi h)\, d\mu = 0$ for all $u \in D(L^0)_b$. Hence $v = \chi h$, in particular $\chi h \in D(\mathcal{E}^0)$ and (1.21) implies that

$$\mathcal{E}^0_\alpha(u, \chi h) = \int \langle \beta, \nabla(\chi u)\rangle h\, d\mu + 2\int \langle A\nabla u, \nabla \chi\rangle h\, d\mu + \int L^0 \chi\, uh\, d\mu \tag{1.22}$$

for all $u \in D(L^0)_b$ and subsequently for all $u \in D(\mathcal{E}^0)$. From (1.22) it follows that

$$\mathcal{E}^0_\alpha(u, h) - \int \langle \beta, \nabla u\rangle h\, d\mu = 0 \text{ for all } u \in H_0^{1,2}(\mathbb{R}^d, \mu)_0 \ .$$

Let $v_n := \|h\|_\infty \chi_n - h$. Then $v_n^- \in H_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$ and

$$0 \le \mathcal{E}^0_\alpha(v_n, v_n^-) - \int \langle \beta, \nabla v_n^-\rangle v_n\, d\mu \le -\int (v_n^-)^2\, d\mu \ ,$$

since $\int \langle \beta, \nabla v_n^-\rangle v_n\, d\mu = \int \langle \beta, \nabla v_n^-\rangle v_n^-\, d\mu = 0$. Thus $v_n^- = 0$, i.e., $h \le \|h\|_\infty \chi_n$. Similarly, $-h \le \|h\|_\infty \chi_n$, hence $|h| \le \|h\|_\infty \chi_n$. Since $\lim_{n\to\infty} \chi_n = 0$ μ–a.e. it follows that $h = 0$.

(ii) $\Rightarrow$ *(iii)*: Since $\int \overline{L}u\, d\mu = 0$ for all $u \in D(L^0)_{0,b}$ we obtain that $\int \overline{L}u\, d\mu = 0$ for all $u \in D(\overline{L})$ and thus

$$\int \overline{T}_t u\, d\mu = \int u\, d\mu + \int_0^t \int \overline{LT}_s u\, d\mu\, ds = \int u\, d\mu$$

for all $u \in D(\overline{L})$. Since $D(\overline{L}) \subset L^1(\mathbb{R}^d, \mu)$ dense we obtain that μ is $(\overline{T}_t)$–invariant.

(iii) $\Rightarrow$ *(i)*: Let $V_n := B_n(0)$, $n \ge 1$. By 1.1 the closure of $L^{0,V_n}u - \langle \beta, \nabla u\rangle$, $u \in D(L^{0,V_n})_b$, on $L^1(V_n, \mu)$ generates a sub–Markovian C_0–semigroup. Let $(\overline{G}^{V_n,\prime}_\alpha)_{\alpha > 0}$ be the corresponding resolvent and $\chi_n := 1 - \overline{G}^{V_n,\prime}_1(1_{V_n})$, $n \ge 1$. Clearly, $\chi_n \in H^{1,2}_{loc}(\mathbb{R}^d, \mu)$ and $(\chi_n - 1)^- \in H_0^{1,2}(\mathbb{R}^d, \mu)_{0,b}$.

Fix $n \ge 1$ and let $w_\beta := \beta \overline{G}'_{\beta+1}\overline{G}^{V_n,\prime}_1(1_{V_n})$, $\beta > 0$. Since $w_\beta \ge \beta \overline{G}^{V_n,\prime}_{\beta+1}\overline{G}^{V_n,\prime}_1(1_{V_n})$ and $\beta \overline{G}^{V_n,\prime}_{\beta+1}\overline{G}^{V_n,\prime}_1(1_{V_n}) = \overline{G}^{V_n,\prime}_1(1_{V_n}) - \overline{G}^{V_n,\prime}_{\beta+1}(1_{V_n}) \ge \overline{G}^{V_n,\prime}_1(1_{V_n}) - 1/(\beta+1)$ by the resolvent equation it follows that

$$w_\beta \ge \overline{G}^{V_n,\prime}_1(1_{V_n}) - 1/(\beta+1)\,, \beta > 0 \ . \tag{1.23}$$

Note that by 1.5

$$\begin{aligned}
\mathcal{E}^0_1(w_\beta, w_\beta) &\le \beta(\overline{G}^{V_n,\prime}_1(1_{V_n}) - w_\beta, w_\beta)_{L^2(\mathbb{R}^d,\mu)} \\
&\le \beta(\overline{G}^{V_n,\prime}_1(1_{V_n}) - w_\beta, \overline{G}^{V_n,\prime}_1(1_{V_n}))_{L^2(\mathbb{R}^d,\mu)} \\
&= \mathcal{E}^0_1(w_\beta, \overline{G}^{V_n,\prime}_1(1_{V_n})) + \int \langle \beta, \nabla w_\beta\rangle \overline{G}^{V_n,\prime}_1(1_{V_n})\, d\mu \\
&\le \mathcal{E}^0_1(w_\beta, w_\beta)^{\frac12}(\mathcal{E}^0_1(\overline{G}^{V_n,\prime}_1(1_{V_n}), \overline{G}^{V_n,\prime}_1(1_{V_n}))^{\frac12} + \sqrt{\nu_{V_n}}\||\beta|1_{V_n}\|_2) \ .
\end{aligned}$$

Consequently, $\lim_{\beta\to\infty} w_\beta = \overline{G}_1^{V_n,\prime}(1_{V_n})$ weakly in $D(\mathcal{E}^0)$ and now (1.23) implies for $u \in H_0^{1,2}(\mathbb{R}^d,\mu)_{0,b}$, $u \geq 0$,

$$\begin{aligned}\mathcal{E}_1^0(\chi_n,u) + \int \langle \beta, \nabla\chi_n\rangle u\,d\mu &= \lim_{\beta\to\infty} \int u\,d\mu - \mathcal{E}_1^0(w_\beta,u) - \int \langle\beta,\nabla w_\beta\rangle u\,d\mu \\ &= \lim_{\beta\to\infty}\int u\,d\mu - \beta\int (\overline{G}_1^{V_n,\prime}(1_{V_n}) - w_\beta)u\,d\mu \geq 0\ .\end{aligned}$$

Finally note that $(\chi_n)_{n\geq 1}$ is decreasing by 1.6 and therefore $\chi_\infty := \lim_{n\to\infty}\chi_n$ exists μ–a.e. If $g \in L^1(\mathbb{R}^d,\mu)_b$ then by (1.16)

$$\begin{aligned}\int g\chi_\infty\,d\mu = \lim_{n\to\infty}\int g\chi_n\,d\mu &= \lim_{n\to\infty}\int g\,d\mu - \int g\overline{G}_1^{V_n,\prime}(1_{V_n})\,d\mu \\ &= \lim_{n\to\infty}\int g\,d\mu - \int \overline{G}_1^{V_n} g\, 1_{V_n}\,d\mu \\ &= \int g\,d\mu - \int \overline{G}_1 g\,d\mu\ .\end{aligned}$$

Since μ is $(\overline{T}_t)$–invariant it follows that $\int g\chi_\infty\,d\mu = 0$ for all $g \in L^1(\mathbb{R}^d,\mu)_b$ and thus $\chi_\infty = 0$ which implies (i). $\square$

Remark. The proof of $(iii) \Rightarrow (i)$ in 1.9 shows that if μ is $(\overline{T}_t)$–invariant then there exists for *all* $\alpha > 0$ a sequence $(\chi_n)_{n\geq 1} \subset H_{loc}^{1,2}(\mathbb{R}^d,\mu)$ such that $(\chi_n - 1)^- \in H_0^{1,2}(\mathbb{R}^d,\mu)_{0,b}$, $\lim_{n\to\infty}\chi_n = 0$ μ–a.e. and

$$\mathcal{E}_\alpha^0(\chi_n,v) + \int\langle\beta,\nabla\chi_n\rangle v\,d\mu \geq 0 \text{ for all } v \in H_0^{1,2}(\mathbb{R}^d,\mu)_{0,b}, v\geq 0\ .$$

Indeed, it suffices to take $\chi_n := 1 - \alpha\overline{G}_\alpha^{V_n,\prime}(1_{V_n})$, $n \geq 1$.

Finally let us give some sufficient conditions on μ, A and B that imply $(\overline{T}_t)$–invariance of μ. Clearly, $(\overline{T}_t)$–invariance of μ is equivalent to the conservativeness of the dual semigroup $(\overline{T}_t^{\prime})_{t\geq 0}$ of $(\overline{T}_t)_{t\geq 0}$ acting on $L^\infty(\mathbb{R}^d,\mu)$. Recall that $(\overline{T}_t^{\prime})_{t\geq 0}$ is called *conservative*, if $\overline{T}_t^{\prime}1 = 1$ for some (hence all) $t > 0$. Since in the *symmetric* case (i.e., $B = B^0$) $\overline{T}^{\prime}_{t|L^1(\mathbb{R}^d,\mu)_b}$ coincides with $\overline{T}_{t|L^1(\mathbb{R}^d,\mu)_b}$ we obtain that both notions coincide in this particular case. But conservativeness in the symmetric case has been well–studied by many authors. We refer to [D], [FOT, Section 1.6], [S] and references therein.

Proposition 1.10. *Each of the following conditions (a), (b) and (c) imply that there exist $\chi_n \in H_{loc}^{1,2}(\mathbb{R}^d,\mu)$ and $\alpha > 0$ as in 1.9 (i).*

(a) $a_{ij}, b_i - b_i^0 \in L^1(\mathbb{R}^d,\mu)$, $1 \leq i,j \leq d$.

(b) There exist $u \in C^2(\mathbb{R}^d)$ and $\alpha > 0$ such that $\lim_{|x|\to\infty} u(x) = +\infty$ and $L^A u + \langle 2B^0 - B, \nabla u\rangle \leq \alpha u$.

(c) $-2\langle A(x)x,x\rangle/(|x|^2+1)+trace(A(x))+\langle(2B^0-B)(x),x\rangle \leq M(|x|^2\ln(|x|^2+1)+1)$ *for some* $M \geq 0$.

Proof. (a) By 1.9 it is sufficient to show that $(L, D(L^0)_{0,b})$ is L^1–unique. But if $h \in L^\infty(\mathbb{R}^d,\mu)$ is such that $\int (1-L)u\,h\,d\mu = 0$ for all $u \in D(L^0)_{0,b}$ we have seen in the proof of the implication $(i) \Rightarrow (ii)$ in 1.9 that $h \in H^{1,2}_{loc}(\mathbb{R}^d,\mu)$ and

$$\mathcal{E}^0_1(u,h) - \int \langle \beta, \nabla u\rangle h\,d\mu = 0 \text{ for all } u \in H^{1,2}_0(\mathbb{R}^d,\mu)_0 \ . \tag{1.24}$$

Let $\psi_n \in C_0^\infty(\mathbb{R}^d)$ be such that $1_{B_n(0)} \leq \psi_n \leq 1_{B_{2n}(0)}$ and $\||\nabla\psi_n|\|_\infty \leq c/n$. Then (1.24) implies that

$$\begin{aligned}\int \psi_n^2 h^2\,d\mu + \mathcal{E}^0(\psi_n h, \psi_n h) &= \mathcal{E}^0_1(\psi_n^2 h, h) + \int \langle A\nabla\psi_n, \nabla\psi_n\rangle h^2\,d\mu \\ &\quad - \int \langle \beta, \nabla\psi_n^2 h\rangle h\,d\mu + \int \langle \beta, \nabla\psi_n\rangle \psi_n h^2\,d\mu \\ &\leq \frac{c^2}{n^2}\|h\|_\infty^2 \Big(\sum_{i,j=1}^d \int |a_{ij}|\,d\mu\Big) + \sqrt{\frac{c}{n}}\|h\|_\infty^2\Big(\sum_{i=1}^d \int |b_i - b_i^0|\,d\mu\Big)\end{aligned}$$

and thus $\int h^2\,d\mu = \lim_{n\to\infty}\int \psi_n^2 h^2\,d\mu = 0$.

(b) Let $\chi_n := \frac{u}{n}$. Then $\chi_n \in H^{1,2}_{loc}(\mathbb{R}^d,\mu)$, $(\chi_n - 1)^-$ is bounded and has compact support, $\lim_{n\to\infty}\chi_n = 0$ and

$$\mathcal{E}^0_\alpha(\chi_n, v) + \int \langle \beta, \nabla\chi_n\rangle v\,d\mu \geq 0 \text{ for all } v \in H^{1,2}_0(\mathbb{R}^d,\mu)_0, v \geq 0 \ .$$

Finally (c) implies (b) since we can take $u(x) = \ln(|x|^2+1) + r$ for r sufficiently big. □

Remark 1.11. Suppose that there exist a bounded, nonnegative and nonzero function $u \in C^2(\mathbb{R}^d)$ and $\alpha > 0$ such that $L^A u + \langle 2B^0 - B, \nabla u\rangle \geq \alpha u$. Then μ is not $(\overline{T}_t)$–invariant.

Proof: We may suppose that $u \leq 1$. If μ would be $(\overline{T}_t)$–invariant it would follow that there exist $\chi_n \in H^{1,2}_{loc}(\mathbb{R}^d,\mu)$, $n \geq 1$, such that $(\chi_n - 1)^- \in H^{1,2}_0(\mathbb{R}^d,\mu)_{0,b}$, $\lim_{n\to\infty}\chi_n = 0$ μ–a.e. and $\mathcal{E}^0_\alpha(\chi_n,v) + \int\langle\beta,\nabla\chi_n\rangle v\,d\mu \geq 0$ for all $v \in H^{1,2}_0(\mathbb{R}^d,\mu)_{0,b}$, $v \geq 0$ (cf. the Remark following 1.9). Let $v_n := (\chi_n - u)$. Then $v_n^- \in H^{1,2}_0(\mathbb{R}^d,\mu)_{0,b}$ and

$$0 \leq \mathcal{E}^0_\alpha(v_n, v_n^-) - \int \langle\beta, \nabla v_n^-\rangle v_n\,d\mu \leq -\int (v_n^-)^2\,d\mu \ ,$$

since $\int\langle\beta,\nabla v_n^-\rangle v_n\,d\mu = \int\langle\beta,\nabla v_n^-\rangle v_n^-\,d\mu = 0$. Thus $v_n^- = 0$, i.e., $u \leq \chi_n$. Since $\lim_{n\to\infty}\chi_n = 0$ μ–a.e. and $u \geq 0$ it follows that $u = 0$ which is a contradiction to our assumption $u \neq 0$. □

Example 1.12. Let $\mu := e^{-x^2}\,dx$, $B(x) = -2x - 6e^{x^2}$, $Lu := u'' + B \cdot u'$, $u \in C_0^\infty(\mathbb{R})$, $(\overline{L}, D(\overline{L}))$ be the maximal extension having properties (a)–(c) in 1.5 and $(\overline{T}_t)_{t\geq 0}$ be the associated semigroup. Let $h(x) := \int_{-\infty}^x e^{-t^2}\,dt$, $x \in \mathbb{R}$. Then $h'' + (2B^0 - B)h' \geq h$. It follows from 1.11 that μ is not $(\overline{T}_t)$–invariant.

2. Uniqueness in the finite dimensional case

Let $U = \mathbb{R}^d$. Throughout this section we assume the local assumptions (1.1')–(1.3') of Subsection 1b) and (1.4) on μ, A and B. Let $(L^0, D(L^0))$ be the generator corresponding to the closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ of

$$\int \langle A\nabla u, \nabla v\rangle \, d\mu \,;\, u, v \in C_0^\infty(\mathbb{R}^d)\,,$$

$(T_t^0)_{t\geq 0}$ be the associated semigroup, $B^0 = (b_1^0, \ldots, b_d^0)$ be as in (1.5) and $\beta = B - B^0$ so that $Lu := L^0 u + \langle \beta, \nabla u\rangle$, $u \in D(L^0)_{0,b}$, is an extension of $L^A u + \langle B, \nabla u\rangle$, $u \in C_0^\infty(\mathbb{R}^d)$. In 1.5 we constructed a closed extension $(\overline{L}, D(\overline{L}))$ of $(L, C_0^\infty(\mathbb{R}^d))$ generating a sub–Markovian C_0–semigroup $(\overline{T}_t)_{t\geq 0}$ on $L^1(\mathbb{R}^d, \mu)$. In this section we will study the question whether $(\overline{L}, D(\overline{L}))$ is the *only* extension generating a C_0–semigroup on $L^1(\mathbb{R}^d, \mu)$, i.e., whether $(L, C_0^\infty(\mathbb{R}^d))$ is L^1–unique. We will consider the symmetric and the nonsymmetric case separately (cf. 2.1 and 2.4 below).

Consider the following additional assumption on A: Suppose that for all compact V there exist $L_V \geq 0$ and $\alpha_V \in (0,1)$ such that

$$|a_{ij}(x) - a_{ij}(y)| \leq L_V |x - y|^{\alpha_V} \text{ for all } x, y \in V\,. \tag{2.1}$$

Theorem 2.1. *Let $(a_{ij})_{1\leq i,j\leq d}$ satisfy (2.1). Then $(L^0, C_0^\infty(\mathbb{R}^d))$ is L^1–unique if and only if $(\mathcal{E}^0, D(\mathcal{E}^0))$ is conservative.*

Remark 2.2. Note that 2.1 implies in particular that the generalized Schrödinger operator $S_\varphi u := \Delta u + 2\varphi^{-1}\langle\nabla\varphi, \nabla u\rangle$, $u \in C_0^\infty(\mathbb{R}^d)$, is L^1–unique if and only if the Friedrich's extension (or equivalently, the associated diffusion process) is conservative (which is in particular the case if the measure μ is *finite*). Hence $(S_\varphi, C_0^\infty(\mathbb{R}^d))$ is *Markov–unique* in the sense that there is exactly one self–adjoint extension on $L^2(\mathbb{R}^d, \mu)$ which generates a sub–Markovian semigroup. On the other hand, it has been shown by M. Röckner and T.S. Zhang in [RZ] that $(S_\varphi, C_0^\infty(\mathbb{R}^d))$ is Markov–unique (in the sense described above) for *all* $\varphi \in H_{loc}^{1,2}(\mathbb{R}^d)$, $\varphi \neq 0$ dx–a.e.

For the proof of 2.1 we first state a lemma.

Lemma 2.3. *Let $(a_{ij})_{1\leq i,j\leq d}$ satisfy (2.1). Let $h \in L^\infty(\mathbb{R}^d, \mu)$ be such that $\int (1 - L^0)u\,h\,d\mu = 0$ for all $u \in C_0^\infty(\mathbb{R}^d)$. Then $h \in H_{loc}^{1,2}(\mathbb{R}^d, \mu)$ and $\mathcal{E}_1^0(u, h) = 0$ for all $u \in H_0^{1,2}(\mathbb{R}^d, \mu)_0$.*

Proof. First note that $C_0^2(\mathbb{R}^d) \subset D(L^0)_{0,b}$ and that $\int (1 - L^0)u\,h\,d\mu = 0$ for all $u \in C_0^2(\mathbb{R}^d)$. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ and $r > 0$ be such that $\mathrm{supp}(\chi) \subset B_r(0)$. We have to show that $\chi h \in H_0^{1,2}(\mathbb{R}^d, \mu)$. Let $L \geq 0$ and $\alpha \in (0,1)$ be such that $|a_{ij}(x) - a_{ij}(y)| \leq L|x-y|^\alpha$ for all $x, y \in B_r(0)$ and define

$$\overline{a}_{ij}(x) := a_{ij}((\frac{r}{|x|} \wedge 1)x) \ , x \in \mathbb{R}^d \ .$$

Then $\overline{a}_{ij}(x) := a_{ij}(x)$ for all $x \in B_r(0)$ and $|\overline{a}_{ij}(x) - \overline{a}_{ij}(y)| \leq 2L|x-y|^\alpha$ for all $x, y \in \mathbb{R}^d$. Let $L^{\overline{A}} = \sum_{i,j=1}^d \overline{a}_{ij}\partial_i\partial_j$. By [K, 4.3.1 and 4.3.2] there exists for all $f \in C_0^\infty(\mathbb{R}^d)$ and $\alpha > 0$ a unique function $\overline{R}_\alpha f \in C_b^2(\mathbb{R}^d)$ satisfying $\alpha\overline{R}_\alpha f - L^{\overline{A}}\overline{R}_\alpha f = f$ and $\|\alpha\overline{R}_\alpha f\|_\infty \leq \|f\|_\infty$. Moreover, $\alpha\overline{R}_\alpha f \geq 0$ if $f \geq 0$ by [K, 2.9.2].

Since $C_0^\infty(\mathbb{R}^d) \subset C_\infty(\mathbb{R}^d)$ dense (where $C_\infty(\mathbb{R}^d)$ is the space of all bounded continuous functions vanishing at infinity) we obtain that $f \mapsto \alpha\overline{R}_\alpha f$, $f \in C_0^\infty(\mathbb{R}^d)$, can be uniquely extended to a positive linear map $\alpha\overline{R}_\alpha : C_\infty(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$, which is a contraction on $C_b(\mathbb{R}^d)$, hence a sub–Markovian kernel on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, which will be denoted again by $\alpha\overline{R}_\alpha$.

Let $f_n \in C_0^\infty(\mathbb{R}^d)$, $n \geq 1$, such that $\|f_n\|_\infty \leq \|h\|_\infty$ and $\tilde{h} := \lim_{n\to\infty} f_n$ is a μ–version of h. Then $\lim_{n\to\infty} \alpha\overline{R}_\alpha f_n(x) = \alpha\overline{R}_\alpha\tilde{h}(x)$ for all $x \in \mathbb{R}^d$ by Lebesgue's Theorem and $\|\alpha\overline{R}_\alpha\tilde{h}\|_\infty \leq \|h\|_\infty$. Then

$$\begin{aligned}
\mathcal{E}^0(\chi\alpha\overline{R}_\alpha f_n, \chi\alpha\overline{R}_\alpha f_n) &= -\int L^0(\chi\alpha\overline{R}_\alpha f_n)\chi\alpha\overline{R}_\alpha f_n \, d\mu \\
&= -\int \chi L^A\chi\,(\alpha\overline{R}_\alpha f_n)^2 \, d\mu - 2\int \langle A\nabla\chi, \nabla\alpha\overline{R}_\alpha f_n\rangle\chi\alpha\overline{R}_\alpha f_n \, d\mu \\
&\qquad - \int L^{\overline{A}}(\alpha\overline{R}_\alpha f_n)\,\chi^2\,\alpha\overline{R}_\alpha f_n \, d\mu - \int \langle B^0, \nabla(\chi\alpha\overline{R}_\alpha f_n)\rangle\chi\alpha\overline{R}_\alpha f_n \, d\mu \\
&= -\int \chi L^A\chi\,(\alpha\overline{R}_\alpha f_n)^2 \, d\mu - 2\int \langle A\nabla\chi, \nabla(\chi\alpha\overline{R}_\alpha f_n)\rangle\alpha\overline{R}_\alpha f_n \, d\mu \\
&\qquad + 2\int \langle A\nabla\chi, \nabla\chi\rangle(\alpha\overline{R}_\alpha f_n)^2 \, d\mu - \alpha\int (\alpha\overline{R}_\alpha f_n - f_n)\chi^2\,\alpha\overline{R}_\alpha f_n \, d\mu \\
&\qquad - \int \langle B^0, \nabla(\chi\alpha\overline{R}_\alpha f_n)\rangle\chi\,\alpha\overline{R}_\alpha f_n \, d\mu \ .
\end{aligned} \tag{2.2}$$

Hence $\mathcal{E}^0(\chi\alpha\overline{R}_\alpha f_n, \chi\alpha\overline{R}_\alpha f_n) \leq c\,\mathcal{E}^0(\chi\alpha\overline{R}_\alpha f_n, \chi\alpha\overline{R}_\alpha f_n)^{1/2} + M$ for some positive constants c and M independent of n. Consequently, $\sup_{n\geq 1} \mathcal{E}^0(\chi\alpha\overline{R}_\alpha f_n, \chi\alpha\overline{R}_\alpha f_n) < +\infty$, hence $\chi\alpha\overline{R}_\alpha\tilde{h} \in D(\mathcal{E}^0)$ and $\lim_{n\to\infty} \chi\alpha\overline{R}_\alpha f_n = \chi\alpha\overline{R}_\alpha\tilde{h}$ weakly in $D(\mathcal{E}^0)$.

Note that

$$
\begin{aligned}
-\alpha\int(\alpha\overline{R}_\alpha\tilde{h}-\tilde{h})\alpha\overline{R}_\alpha\tilde{h}\chi^2\,d\mu &\le -\alpha\int(\alpha\overline{R}_\alpha\tilde{h}-\tilde{h})\tilde{h}\chi^2\,d\mu\\
&= \lim_{n\to\infty} -\alpha\int(\alpha\overline{R}_\alpha f_n - f_n)\tilde{h}\chi^2\,d\mu\\
&= \lim_{n\to\infty} -\int L^A(\alpha\overline{R}_\alpha f_n)\tilde{h}\chi^2\,d\mu\\
&= \lim_{n\to\infty} -\int L^A(\chi^2\alpha\overline{R}_\alpha f_n)\tilde{h}\,d\mu + 4\int\langle A\nabla\chi,\nabla\alpha\overline{R}_\alpha f_n\rangle\chi\tilde{h}\,d\mu\\
&\qquad + \int L^A(\chi^2)\alpha\overline{R}_\alpha f_n\,\tilde{h}\,d\mu\\
&= \lim_{n\to\infty} -\int \chi^2\alpha\overline{R}_\alpha f_n\,\tilde{h}\,d\mu + \int\langle B^0,\nabla(\chi^2\alpha\overline{R}_\alpha f_n)\rangle\tilde{h}\,d\mu\\
&\qquad + 4\int\langle A\nabla\chi,\nabla\alpha\overline{R}_\alpha f_n\rangle\chi\tilde{h}\,d\mu + \int L^A(\chi^2)\alpha\overline{R}_\alpha f_n\,\tilde{h}\,d\mu\\
&= -\int\chi^2(\alpha\overline{R}_\alpha\tilde{h})\,\tilde{h}\,d\mu + \int\langle B^0,\nabla(\chi\alpha\overline{R}_\alpha\tilde{h})\rangle\chi\tilde{h}\,d\mu\\
&\qquad + \int\langle B^0,\nabla\chi\rangle\chi(\alpha\overline{R}_\alpha\tilde{h})\,\tilde{h}\,d\mu + 4\int\langle A\nabla\chi,\nabla(\chi\alpha\overline{R}_\alpha\tilde{h})\rangle\tilde{h}\,d\mu\\
&\qquad - 4\int\langle A\nabla\chi,\nabla\chi\rangle(\alpha\overline{R}_\alpha\tilde{h})\,\tilde{h}\,d\mu + \int L^A(\chi^2)(\alpha\overline{R}_\alpha\tilde{h})\,\tilde{h}\,d\mu\\
&\le c\,\mathcal{E}^0(\chi\alpha\overline{R}_\alpha\tilde{h},\chi\alpha\overline{R}_\alpha\tilde{h})^{1/2} + M
\end{aligned}
\tag{2.3}
$$

for some positive constants c and M independent of α. Combining (2.2) and (2.3) we obtain that

$$
\begin{aligned}
\mathcal{E}^0(\chi\alpha\overline{R}_\alpha\tilde{h},\chi\alpha\overline{R}_\alpha\tilde{h}) &\le \liminf_{n\to\infty}\mathcal{E}^0(\chi\alpha\overline{R}_\alpha\tilde{f}_n,\chi\alpha\overline{R}_\alpha\tilde{f}_n)\\
&\le -\int\chi L^A\chi(\alpha\overline{R}_\alpha\tilde{h})^2\,d\mu - 2\int\langle A\nabla\chi,\nabla(\chi\alpha\overline{R}_\alpha\tilde{h})\rangle\alpha\overline{R}_\alpha\tilde{h}\,d\mu\\
&\quad + 2\int\langle A\nabla\chi,\nabla\chi\rangle(\alpha\overline{R}_\alpha\tilde{h})^2\,d\mu - \alpha\int(\alpha\overline{R}_\alpha\tilde{h}-\tilde{h})\chi^2\,\alpha\overline{R}_\alpha\tilde{h}\,d\mu\\
&\quad - \int\langle B^0,\nabla(\chi\alpha\overline{R}_\alpha\tilde{h})\rangle\chi\alpha\overline{R}_\alpha\tilde{h}\,d\mu\\
&\le \tilde{c}\,\mathcal{E}^0(\chi\alpha\overline{R}_\alpha\tilde{h},\chi\alpha\overline{R}_\alpha\tilde{h})^{1/2} + \tilde{M}
\end{aligned}
$$

for some positive constants $\tilde{c}$ and $\tilde{M}$ independent of α. Hence $(\chi\alpha\overline{R}_\alpha\tilde{h})_{\alpha>0}$ is bounded in $D(\mathcal{E}^0)$.

If $u\in D(\mathcal{E}^0)$ is the limit of some weakly convergent subsequence $(\chi\alpha_k R_{\alpha_k}\tilde{h})_{k\ge1}$

with $\lim_{k\to\infty} \alpha_k = +\infty$ it follows for all $v \in C_0^\infty(\mathbb{R}^d)$ that

$$\begin{aligned}
\int (u - \chi\tilde{h}) v \, d\mu &= \lim_{k\to\infty} \int \chi(\alpha_k \overline{R}_{\alpha_k} \tilde{h} - \tilde{h}) v \, d\mu \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \int \chi(\alpha_k \overline{R}_{\alpha_k} f_n - f_n) v \, d\mu \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \int \chi L^A(\overline{R}_{\alpha_k} f_n) v \, d\mu \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \int \overline{R}_{\alpha_k} f_n \, L^0(\chi v) \, d\mu - \int \langle B^0, \nabla \overline{R}_{\alpha_k} f_n \rangle \chi v \, d\mu \\
&= \lim_{k\to\infty} \int \overline{R}_{\alpha_k} \tilde{h} \, L^0(\chi v) \, d\mu - \int \langle B^0, \nabla(\chi \overline{R}_{\alpha_k} \tilde{h}) \rangle v \, d\mu \\
&\qquad + \int \langle B^0, \nabla \chi \rangle \overline{R}_{\alpha_k} \tilde{h} \, v \, d\mu \\
&\le \lim_{k\to\infty} \frac{1}{\alpha_k} (\|h\|_\infty \|L^0(\chi v)\|_1 + \sqrt{\nu} \||B^0| v\|_2 \mathcal{E}^0(\chi \alpha_k \overline{R}_{\alpha_k} \tilde{h}, \chi \alpha_k \overline{R}_{\alpha_k} \tilde{h})^{1/2} \\
&\qquad + \sqrt{\nu} \|h\|_\infty \||B^0| v\|_2 \mathcal{E}^0(\chi, \chi)^{1/2}) = 0 \; .
\end{aligned}$$

Consequently, $\chi\tilde{h}$ is a μ–version of u. In particular, $\chi h \in H_0^{1,2}(\mathbb{R}^d, \mu)$.

Let $u \in H_0^{1,2}(\mathbb{R}^d, \mu)$ with compact support, $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi \equiv 1$ on $\mathrm{supp}(|u|\mu)$ and $u_n \in C_0^\infty(\mathbb{R}^d)$, $n \ge 1$, such that $\lim_{n\to\infty} u_n = u$ in $H_0^{1,2}(\mathbb{R}^d, \mu)$. Then

$$\begin{aligned}
\int \langle A\nabla u, \nabla h \rangle \, d\mu + \int u h \, d\mu &= \lim_{n\to\infty} \int \langle A\nabla(\chi u_n), \nabla h \rangle \, d\mu + \int \chi u_n h \, d\mu \\
&= \lim_{n\to\infty} \int (1 - L^0) u_n \, \chi h \, d\mu = 0 \; . \quad \square
\end{aligned}$$

Proof of 2.1. Clearly, if $(L^0, C_0^\infty(\mathbb{R}^d))$ is L^1–unique it follows that $(L^0, D(L^0)_{0,b})$ is L^1–unique too. Hence μ is $(\overline{T_t^0})$–invariant by 1.9. Since $(T_t^0)_{t\ge 0}$ is symmetric it follows that $T_t^0 1 = 1$, i.e., $(\mathcal{E}^0, D(\mathcal{E}^0))$ is conservative.

Conversely, let $h \in L^\infty(\mathbb{R}^d, \mu)$ be such that $\int (1 - L^0) u \, h \, d\mu = 0$ for all $u \in C_0^\infty(\mathbb{R}^d)$. Then $h \in H_{loc}^{1,2}(\mathbb{R}^d, \mu)$ and $\mathcal{E}_1^0(u, h) = 0$ for all $u \in H_0^{1,2}(\mathbb{R}^d, \mu)_0$ by 2.3. In particular,

$$\int (1 - L^0) u \, h \, d\mu = \mathcal{E}_1^0(u, h) = 0 \text{ for all } u \in D(L^0)_0 \; . \tag{2.4}$$

Since $(T_t^0)_{t\ge 0}$ is symmetric and $T_t^0 1 = 1$ it follows that μ is $(\overline{T_t^0})$–invariant. Hence $(L^0, D(L^0)_{0,b})$ is L^1–unique by 1.9 and (2.4) now implies that $h = 0$. Hence $(L^0, C_0^\infty(\mathbb{R}^d))$ is L^1–unique too. $\square$

Our next result is on uniqueness in the nonsymmetric case.

Proposition 2.4. *Let $(L^0, C_0^\infty(\mathbb{R}^d))$ be L^1–unique. Suppose that there exist $\chi_n \in H^{1,2}_{loc}(\mathbb{R}^d, \mu)$, $n \geq 1$, and $\alpha > 0$ as in 1.9 (i). Then $(L, C_0^\infty(\mathbb{R}^d))$ is L^1–unique too.*

Proof. $(L, D(L^0)_{0,b})$ is L^1–unique by 1.9. Hence it is enough to prove that for $u \in D(L^0)_{0,b}$ there exist $u_n \in C_0^\infty(\mathbb{R}^d)$ such that $\lim_{n\to\infty} \|u_n - u\|_1 + \|Lu_n - Lu\|_1 = 0$.

To this end let $u \in D(L^0)_{0,b}$, $u \neq 0$. Since $(L^0, C_0^\infty(\mathbb{R}^d))$ is L^1–unique there exist $u_n \in C_0^\infty(\mathbb{R}^d)$, $n \geq 1$, such that $\lim_{n\to\infty} \|u_n - u\|_1 + \|L^0 u_n - L^0 u\|_1 = 0$ and $\lim_{n\to\infty} u_n = u$ μ–a.e. Similar to the proof of (1.10) (cf. Step 2 in the proof of 1.1) it can be shown that

$$\lim_{n\to\infty} \int_{\{M_1 \leq |u_n| \leq M_2\}} \langle A\nabla u_n, \nabla u_n\rangle \, d\mu = 0 \tag{2.5}$$

for all $\|u\|_\infty < M_1 < M_2$.

Let $\psi \in C_b^\infty(\mathbb{R})$ be such that $\psi(t) = t$ if $|t| \leq \|u\|_\infty + 1$ and $\psi(t) = 0$ if $|t| \geq \|u\|_\infty + 2$. Then $\psi(u_n) \in C_0^\infty(\mathbb{R}^d)$, $n \geq 1$, and $\lim_{n\to\infty} L^0\psi(u_n) =$
$\lim_{n\to\infty} \psi'(u_n) L^0 u_n + \psi''(u_n)\langle A\nabla u_n, \nabla u_n\rangle = L^0 u$ in $L^1(\mathbb{R}^d, \mu)$ by (2.5). Consequently,

$$\begin{aligned}\mathcal{E}^0(\psi(u_n) - u, \psi(u_n) - u) &= -\int L^0(\psi(u_n) - u)(\psi(u_n) - u)\, d\mu \\ &\leq 2\|\psi\|_\infty \|L^0\psi(u_n) - L^0 u\|_1 \to 0\,, n \to \infty\,,\end{aligned}$$

i.e., $\lim_{n\to\infty} \psi(u_n) = u$ in $D(\mathcal{E}^0)$.

Finally let $\chi \in C_0^\infty(\mathbb{R}^d)$ such that $\chi \equiv 1$ on $\operatorname{supp}(|u|\mu)$. Then $\chi\psi(u_n) \in C_0^\infty(\mathbb{R}^d)$ for all n and it is easy to see that $\lim_{n\to\infty} \|\chi\psi(u_n) - u\|_1 + \|L(\chi\psi(u_n)) - Lu\|_1 = 0$ which implies the assertion. $\square$

The uniqueness results can be applied to derive results on the uniqueness of related martingale problems. According to [AR] we make the following definition:

Definition 2.5. A right process $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in\mathbb{R}^d_\Delta})$ with state space $\mathbb{R}^d$ and natural filtration $(\mathcal{F}_t)_{t\geq 0}$ is said to solve the martingale problem for $(L, C_0^\infty(\mathbb{R}^d))$ if for all $u \in C_0^\infty(\mathbb{R}^d)$

(i) $\int_0^t Lu(X_s)\, ds$, $t \geq 0$, is (P_μ–a.s.) independent of the μ–version for Lu.

(ii) $u(X_t) - u(X_0) - \int_0^t Lu(X_s)\, ds$, $t \geq 0$, is an $(\mathcal{F}_t)$–martingale under $P_\mu = \int P_x \, d\mu(x)$.

Proposition 2.6. *Assume that $(L^0, C_0^\infty(\mathbb{R}^d))$ is L^1–unique and that there exist $\chi_n \in H^{1,2}_{loc}(\mathbb{R}^d, \mu)$ and $\alpha > 0$ as in 1.9 (i). Let $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in\mathbb{R}^d_\Delta})$ be a right process that solves the martingale problem for $(L, C_0^\infty(\mathbb{R}^d))$ such that μ is an subinvariant measure for $\mathbb{M}$. Then $E_x[f(X_t)]$ is a μ–version of $\overline{T}_t f$ for all $f \in \mathcal{B}_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, \mu)$ and μ is an invariant measure for $\mathbb{M}$.*

Proof. Let $(p_t)_{t\geq 0}$ be the transition semigroup of $\mathbb{M}$. Since μ is subinvariant for $\mathbb{M}$, i.e., $\int p_t f\, d\mu \leq \int f\, d\mu$ for all $f \in \mathcal{B}_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, \mu)$, $f \geq 0$, it follows that $(p_t)_{t\geq 0}$ induces a semigroup of (sub–Markovian) contractions $(S_t)_{t\geq 0}$ on $L^1(\mathbb{R}^d, \mu)$. Using [MR, II.4.3] and the fact that $\mathbb{M}$ is a right process it is easy to see that $(S_t)_{t\geq 0}$ is strongly continuous. Let $(A, D(A))$ be the corresponding generator.

If $u \in C_0^\infty(\mathbb{R}^d)$ and $v \in \mathcal{B}_b(\mathbb{R}^d)$ then

$$\begin{aligned}\int (p_t u - u) v\, d\mu &= E_\mu\left[(u(X_t) - u(X_0))v(X_0)\right] \\ &= E_\mu\left[\left(\int_0^t Lu(X_s)\, ds\right)v(X_0)\right] \\ &= \int \left(\int_0^t p_s Lu\, ds\right) v\, d\mu\ ,\end{aligned}$$

hence $S_t u - u = \int_0^t S_s Lu\, ds$ in $L^1(\mathbb{R}^d, \mu)$. It follows from the strong continuity of $(S_t)_{t\geq 0}$ that $u \in D(A)$ and $Au = Lu$. Since $(L, C_0^\infty(\mathbb{R}^d))$ is L^1–unique by 2.4 we obtain that $A = \overline{L}$, hence $(S_t)_{t\geq 0} = (\overline{T}_t)_{t\geq 0}$ which implies the first assertion. The second assertion follows from 1.9. $\square$

Remark 2.7. In Section 3 we will construct a diffusion process $\mathbb{M}$ associated with $(\overline{L}, D(\overline{L}))$ in the sense that its transition probabilities are given by $(\overline{T}_t)_{t\geq 0}$. It is easy to see that $\mathbb{M}$ is a solution of the martingale problem for $(L, C_0^\infty(\mathbb{R}^d))$ in the sense of 2.5.

Application to uniqueness of invariant measures.

We want to demonstrate how the results of Section 1 and 2 can be applied to obtain results on uniqueness of the invariant measure μ. For simplicity suppose that $a_{ij} = \delta_{ij}\ , 1 \leq i, j \leq d$.

Proposition 2.8. *Let $d \geq 2$ and $B \in L^p_{loc}(\mathbb{R}^d; \mathbb{R}^d, dx)$ for some $p > d$. Suppose that there exists $M \geq 0$ such that*

$$\langle B(x), x\rangle \leq M(\ln(|x|^2 + 1) + 1) \text{ for all } x \in \mathbb{R}^d\ . \tag{2.6}$$

Then there exists at most one probability measure μ satisfying

$$B \in L^1_{loc}(\mathbb{R}^d; \mathbb{R}^d, \mu) \text{ and } \int \Delta u + \langle B, \nabla u\rangle\, d\mu = 0 \text{ for all } u \in C_0^\infty(\mathbb{R}^d)\ . \tag{2.7}$$

Proof. Let μ_1, μ_2 be two probability measures satisfying (2.7) and let $\mu := \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$. Clearly, μ satsifies (2.7) again. By [ABR, Theorem 2.5] $d\mu \ll dx$ and for the density ρ we have that $\rho \in H^{1,p}_{loc}(\mathbb{R}^d)$. Moreover, ρ admits a strictly positive continuous modification and thus $\varphi := \sqrt{\rho} \in H^{1,2}_{loc}(\mathbb{R}^d)$. Let $B^0 = (b^0_1, \ldots, b^0_d)$, $b^0_i = 2\varphi^{-1}\partial_i\varphi$, $1 \leq i \leq d$.

By 1.5 there exist closed extensions $(\overline{L}, D(\overline{L}))$ of $Lu := \Delta u + \langle B, \nabla u\rangle$, $u \in C_0^\infty(\mathbb{R}^d)$, and $(\overline{L}', D(\overline{L}'))$ of $L'u := \Delta u + \langle 2B - B^0, \nabla u\rangle$, $u \in C_0^\infty(\mathbb{R}^d)$, on $L^1(\mathbb{R}^d, \mu)$ generating sub–Markovian C_0–semigroups $(\overline{T}_t)_{t\geq 0}$ and $(\overline{T}'_t)_{t\geq 0}$. It follows from (1.17) (cf. 1.7) that $\int \overline{T}_t u\, v\, d\mu = \int u \overline{T}'_t u\, d\mu$ for all $u, v \in L^\infty(\mathbb{R}^d, \mu)$. Note that (2.6) implies the existence of some function $u \in C^2(\mathbb{R}^d)$, $u \geq 0$, and some $\alpha > 0$ such that $\lim_{|x|\to+\infty} u(x) = +\infty$ and $Lu \leq \alpha u$. Hence μ is $(\overline{T}'_t)$–invariant by 1.10. Consequently, $\int \overline{T}' 1 - 1\, d\mu = 0$, i.e., $\overline{T}'_t 1 = 1$, and thus $\int \overline{T}_t u\, d\mu = \int u \overline{T}'_t 1\, d\mu = \int u\, d\mu$ for all $u \in L^\infty(\mathbb{R}^d, \mu)$. Thus μ is $(\overline{T}_t)$–invariant too.

Let $(L^0, D(L^0))$ be the generator of the closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ of $\int \langle \nabla u, \nabla v\rangle\, d\mu$, $u, v \in C_0^\infty(\mathbb{R}^d)$, on $L^2(\mathbb{R}^d, \mu)$. $(L^0, C_0^\infty(\mathbb{R}^d))$ is L^1–unique by 2.1 since μ is finite. Hence $(L, C_0^\infty(\mathbb{R}^d))$ is L^1–unique too by 1.9 and 2.4. Note that $d\mu_1 \ll d\mu$, $h := \frac{d\mu_1}{d\mu} \in L^\infty(\mathbb{R}^d, \mu)$ and

$$\int Lu\, h\, d\mu = \int Lu\, d\mu_1 = 0 \text{ for all } u \in C_0^\infty(\mathbb{R}^d)) \ . \tag{2.8}$$

Then (2.8) extends to all $u \in D(\overline{L})$ which implies that $h \in D(\overline{L}')_b$ and $\overline{L}' h = 0$. By 1.5 $h \in D(\mathcal{E}^0)$ and $\mathcal{E}^0(h, h) \leq -\int \overline{L}' h\, h\, d\mu = 0$.

Since the density $\frac{d\mu}{dx}$ admits a strictly positive continuous modification it follows that $D(\mathcal{E}^0) \subset H^{1,2}_{loc}(\mathbb{R}^d)$ and $\mathcal{E}^0(h, h) = 0$ implies that $\partial_i h = 0$, $1 \leq i \leq d$. Consequently, $h \equiv c_0$ for some constant $c_0 \in \mathbb{R}$. Since $1 = \int d\mu_1 = \int h\, d\mu = c_0 \int d\mu = c_0$ it follows that $\mu_1 = \mu$ and finally $\mu_2 = 2\mu - \mu_1 = \mu_1$. $\square$

3. Associated Markov processes in the finite dimensional case

Let $U \subset \mathbb{R}^d$ be open. Throughout this section we assume the local assumptions (1.1')–(1.3') of Subsection 1b) and (1.4) on μ, A and B. Let $(L^0, D(L^0))$ be the generator corresponding to the closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ of

$$\int \langle A\nabla u, \nabla v\rangle\, d\mu\, ; u, v \in C_0^\infty(U)\,,$$

$(T^0_t)_{t\geq 0}$ be the associated semigroup, $B^0 = (b^0_1, \dots, b^0_d)$ be as in (1.5) and $\beta = B - B^0$ so that $Lu := L^0 u + \langle \beta, \nabla u\rangle$, $u \in D(L^0)_{0,b}$, is an extension of $L^A u + \langle B, \nabla u\rangle$, $u \in C_0^\infty(U)$. Let $(\overline{L}, D(\overline{L}))$ be the closed extension of $(L, D(L^0)_{0,b})$ satisfying (a)–(c) in 1.5, $(\overline{T}_t)_{t\geq 0}$ be the associated semigroup on $L^1(U, \mu)$ and $(\overline{G}_\alpha)_{\alpha>0}$ be the associated resolvent. Since $(\overline{T}_t)_{t\geq 0}$ is a sub–Markovian semigroup of contractions it determines uniquely a semigroup of contractions on $L^2(U, \mu)$ by the Riesz–Thorin Interpolation Theorem. Clearly, $(T_t)_{t\geq 0}$ is strongly continuous again. Let $(L, D(L))$ be the generator and $(G_\alpha)_{\alpha>0}$ be the associated resolvent. Note that T_t (resp. G_α) coincides with $\overline{T}_t$ (resp. $\overline{G}_\alpha$) on $L^1(U, \mu) \cap L^2(U, \mu)$.

Lemma 3.1. *Let $f \in D(L)$. Then $f \in D(\mathcal{E}^0)$ and $\mathcal{E}^0(f, f) \leq -\int Lf f\, d\mu$.*

Proof. Let $g_n \in L^1(U, \mu)_b$ be such that $\lim_{n\to\infty} \|g_n - (1-L)f\|_2 = 0$. Then $G_1 g_n \in D(\overline{L})_b \subset D(\mathcal{E}^0)$ and $\mathcal{E}^0_1(G_1 g_n - G_1 g_m, G_1 g_n - G_1 g_m) \leq \int (g_n - g_m)(G_1 g_n - G_1 g_m)\, d\mu$.

Since $\lim_{n\to\infty}\|G_1 g_n - f\|_2 = 0$ it follows that $(G_1 g_n)_{n\ge 1}$ is an $\mathcal{E}^0$–Cauchy–sequence, hence $f \in D(\mathcal{E}^0)$ and $\mathcal{E}^0(f,f) \le -\int Lff\,d\mu$. $\square$

It is well–known that the general theory of Dirichlet forms can be used to construct a diffusion $\mathbb{M}^0 = (\Omega^0, \mathcal{F}^0, (X_t^0)_{t\ge 0}, (P_x^0)_{x\in U_\Delta})$ with life time ζ that is associated with $(\mathcal{E}^0, H_0^{1,2}(U,\mu))$ in the sense that $E.\left[f(X_t^0)\right]$ is an $\mathcal{E}^0$–quasi continuous (= $\mathcal{E}^0$–q.c.) μ–version of $T_t^0 f$ for all $f \in \mathcal{B}_b(U) \cap L^2(U,\mu)$, $t > 0$ (cf. [MR] or [FOT]). $\mathcal{E}^0$–quasi continuity of $p_t^0 f$ means that there exists an increasing sequence $(F_k)_{k\ge 1}$ of closed subsets of U such that $\bigcup_{k\ge 1} D(\mathcal{E}^0)_{F_k} \subset D(\mathcal{E}^0)$ dense (where we set

$$D(\mathcal{E}^0)_{F_k} = \{v \in D(\mathcal{E}^0) | v = 0 \text{ on } F_k^c\}\ )$$

and $p_t^0 f_{|F_k}$ is continuous for all k. Such an increasing sequence of closed subsets $(F_k)_{k\ge 1}$ is called an $\mathcal{E}^0$–nest and it is well–known in the theory of Dirichlet forms that $(F_k)_{k\ge 1}$ is an $\mathcal{E}^0$–nest if and only if $P_\mu^0\left[\lim_{k\to\infty}\sigma_{U\setminus F_k} < +\zeta\right] = 0$. Here $\sigma_{U\setminus F_k} = \inf\{t > 0 | X_t^0 \in U \setminus F_k\}$ denotes the first hitting time.

The construction of $\mathbb{M}^0$ is possible because the domain of $\mathcal{E}^0$ contains enough continuous functions with compact support in U, since $C_0^\infty(U) \subset D(\mathcal{E}^0)$ dense. This property implies quasi–regularity of $(\mathcal{E}^0, D(\mathcal{E}^0))$ and thus the existence of $\mathbb{M}^0$ (cf. [MR, IV.3.5 and V.1.5]).

Since L is neither symmetric nor sectorial we cannot use the same framework to construct a diffusion process associated with L. However, a closer look to the construction of associated stochastic processes in the framework of Dirichlet forms shows that the assumption on the symmetry of $(L^0, D(L^0))$ can be removed and only information on the domain $D(L^0)$ and the resolvent $(\alpha - L^0)^{-1}$, $\alpha > 0$, is used to construct $\mathbb{M}^0$. This observation is used in the theory of *generalized* Dirichlet forms to construct stochastic processes associated with non–symmetric operators (cf. [St1]).

First note that $(L, D(L))$ is associated with a *generalized* Dirichlet form by [St1, I.4.9 (ii)]. The *explicit construction* of $\overline{L}$ (hence L too) in 1.5 provides enough information on the domain $D(L)$ and the resolvent $(G_\alpha)_{\alpha>0}$ to apply the fundamental existence result in the theory of *generalized* Dirichlet forms (cf. [St1, IV.2.2]) to obtain a μ–tight special standard process $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t\ge 0}, (P_x)_{x\in U_\Delta})$ with life time ζ that is associated with $(L, D(L))$ in the sense that $E.\left[\int e^{-\alpha t} f(X_t)\,dt\right]$ is an $\mathcal{E}^0$–q.c. μ–version of $G_\alpha f$ for all $f \in \mathcal{B}_b(U) \cap L^2(U,\mu)$, $\alpha > 0$ (cf. 3.5 below) and we will show in 3.6 below that $\mathbb{M}$ is a diffusion in the sense that $P_x\left[t \to X_t \text{ is continuous on } [0,\zeta)\right] = 1$ $\mathcal{E}^0$–q.e.

Analytic potential theory related to L.

In order to apply [St1, IV.2.2] we have to prove quasi–regularity of L, which is defined in the framework of *generalized* Dirichlet forms in a similar way as quasi–regularity in the framework of (sectorial) Dirichlet forms (for details we refer to [St1]). To this end we first introduce some notions that are well–known in the classical framework.

For an element $f \in L^2(U,\mu)$ let $\mathcal{L}_f = \{g \in L^2(U,\mu) | g \geq f\}$. An element $f \in L^2(U,\mu)$ is called 1–*excessive* if $\beta G_{\beta+1} f \leq f$ for all $\beta \geq 0$. If $f \in L^2(U,\mu)$ and $V \subset U$, V open are such that $\mathcal{L}_{f1_V} \cap D(L) \neq \emptyset$ there exists a 1–excessive element $f_V \in \mathcal{L}_{f1_V}$ such that $f_V \leq u$ for all $u \in \mathcal{L}_{f1_V}$, u 1–excessive (cf. [St1, III.1.7]). f_V is called the 1–*reduced element* f_V of f on V. Note that $f_V = f$ on V if f itself is 1–excessive since then $f \wedge f_V$ is 1–excessive and $f \wedge f_V \geq f$ on V which implies that $f_V \leq f \wedge f_V$, hence $f_V = f$ on V. Moreover, $\|f_V\|_\infty \leq \|f\|_\infty$.

Lemma 3.2. *Let $f \in D(\overline{L})_b$ and $V \subset U$ open such that $\mathcal{L}_{f1_V} \cap D(L) \neq \emptyset$. Then:*

(i) $f_V \in D(\mathcal{E}^0)_b$.

(ii) $\mathcal{E}_1^0(f_V, g) - \int \langle \beta, \nabla f_V \rangle g \, d\mu = 0$ *for all* $g \in D(\mathcal{E}^0)_{0,b}$ *with* $g = 0$ *on* V.

(iii) $\mathcal{E}_1^0(f_V, f_V) \leq 6 \|(1-\overline{L})f\|_1 \|f\|_\infty$.

Proof. By assumption f_V exists. Let $f_V^\alpha \in D(L)_b \subset D(\mathcal{E}^0)_b$, $\alpha > 0$, be the uniquely determined element in $D(L)$ with $(1-L)f_V^\alpha = \alpha(f_V^\alpha - f1_V)^-$ (cf. [St1, III.1.6]). Then $0 \leq f_V^\alpha \leq f_V$, $\alpha > 0$, and $\lim_{\alpha\to\infty} f_V^\alpha = f_V$ in $L^2(U,\mu)$ (cf. [St1, III.1.7]). Since $w_\alpha = f_V^\alpha - f \in L^2(U,\mu)_b$ it follows from 3.1 that $\beta G_\beta w_\alpha \in D(\mathcal{E}^0)$ and

$$\begin{aligned}\mathcal{E}_1^0(\beta G_\beta w_\alpha, \beta G_\beta w_\alpha) &\leq \int (1-L)\beta G_\beta w_\alpha \, \beta G_\beta w_\alpha \, d\mu \\ &= \int (\beta G_\beta (1-L) f_V^\alpha - \beta \overline{G}_\beta (1-\overline{L}) f) \beta G_\beta w_\alpha \, d\mu \\ &\leq \|(1-L)f_V^\alpha\|_2 \|w_\alpha\|_2 + \|(1-\overline{L})f\|_1 \|w_\alpha\|_\infty\end{aligned}$$

for all $\beta > 0$. Hence $(\beta G_\beta w_\alpha)_{\beta>0}$ is bounded in $D(\mathcal{E}^0)$, $\lim_{\beta\to\infty} \beta G_\beta w_\alpha = w_\alpha$ weakly in $D(\mathcal{E}^0)$ and thus

$$\begin{aligned}\mathcal{E}_1^0(w_\alpha, w_\alpha) &\leq \int ((1-L)f_V^\alpha - (1-\overline{L})f) w_\alpha \, d\mu \\ &= \alpha \int (f_V^\alpha - f1_V)^- (f_V^\alpha - f) \, d\mu \\ &\quad - \int (1-\overline{L}) f (f_V^\alpha - f) \, d\mu \\ &\leq 2 \|(1-\overline{L})f\|_1 \|f\|_\infty \; .\end{aligned}$$

It follows that $\sup_{\alpha>0} \mathcal{E}_1^0(f_V^\alpha - f, f_V^\alpha - f) < +\infty$, therefore $f_V \in D(\mathcal{E}^0)$ and $\lim_{\alpha\to\infty} f_V^\alpha = f_V$ weakly in $D(\mathcal{E}^0)$ by Banach–Alaoglu. Let $g \in D(\mathcal{E}^0)_{0,b}$ be such that $g = 0$ on V. Then by 1.5

$$\begin{aligned}\mathcal{E}_1^0(f_V, g) - \int \langle \beta, \nabla f_V \rangle g \, d\mu &= \lim_{\alpha\to\infty} \mathcal{E}_1^0(f_V^\alpha, g) - \int \langle \beta, \nabla f_V^\alpha \rangle g \, d\mu \\ &= \lim_{\alpha\to\infty} \int (1-\overline{L}) f_V^\alpha \, g \, d\mu \\ &= \lim_{\alpha\to\infty} \alpha \int (f_V^\alpha - f1_V)^- g \, d\mu = 0 \; .\end{aligned}$$

Moreover,

$$
\begin{aligned}
\mathcal{E}_1^0(f_V, f_V) &\le \liminf_{\alpha\to\infty} \mathcal{E}_1^0(f_V^\alpha, f_V^\alpha) \\
&\le \liminf_{\alpha\to\infty} 2\mathcal{E}_1^0(f_V^\alpha - f, f_V^\alpha - f) + 2\mathcal{E}_1^0(f,f) \\
&\le \liminf_{\alpha\to\infty} -2\int (1-\overline{L})f(f_V^\alpha - f)\,d\mu + 2\int (1-\overline{L})ff\,d\mu \\
&\le 6\|(1-\overline{L})f\|_1 \|f\|_\infty\ . \quad \square
\end{aligned}
$$

Definition 3.3. An increasing sequence of closed subsets $(F_k)_{k\ge1}$ is called an L–*nest* if $f_{F_k^c} \to 0$ in $L^2(U,\mu)$ for all $f \in D(L)$, f 1–excessive.

Lemma 3.4. *An increasing sequence of closed subsets $(F_k)_{k\ge1}$ is an L–nest if and only if it is an $\mathcal{E}^0$–nest.*

Proof. First suppose that $(F_k)_{k\ge1}$ is an L–nest. Let $u = G_1 f$, $f \in L^1(U,\mu)_b$ and $v := G_1(f^+)$, $w := G_1(f^-)$. Then $v_{F_k^c}$, $w_{F_k^c}$ exist for all k and $v_{F_k^c} \to 0$ (resp. $w_{F_k^c} \to 0$) in $L^2(U,\mu)$ and weakly in $D(\mathcal{E}^0)$ by 3.2 (iii). Hence $u_k := u + w_{F_k^c} - v_{F_k^c} \to u$ weakly in $D(\mathcal{E}^0)$. Since $u_k \in D(\mathcal{E}^0)_{F_k,b}$ it follows that $\bigcup_{k\ge1} D(\mathcal{E}^0)_{F_k,b} \subset G_1(L^1(U,\mu)_b)$ dense w.r.t. the norm on $D(\mathcal{E}^0)$. But $G_1(L^1(U,\mu)_b) \subset D(\mathcal{E}^0)$ dense since $G_1(L^1(U,\mu)_b) \subset G_1(L^2(U,\mu))$ dense w.r.t. the norm on $D(\mathcal{E}^0)$ (cf. the proof of 3.1) and $C_0^\infty(U) \subset G_1(L^2(U,\mu))$.

Conversely suppose that $\bigcup_{k\ge1} D(\mathcal{E}^0)_{F_k}$ is dense in $D(\mathcal{E}^0)$. Let $(V_n)_{n\ge1}$ be as in the proof of 1.5 an increasing sequence of open subsets relatively compact in U such that $U = \bigcup_{n\ge1} V_n$ and $(\overline{G}_\alpha^{V_n})_{\alpha>0}$ be the resolvent corresponding to the closure of $L^0 u + \langle \beta, \nabla u\rangle$, $u \in D(L^{0,V_n})_b$. Let $0 < \varphi \le 1$, $\varphi \in L^1(U,\mu)$, $h := G_1\varphi$ and $h_k := (G_1\varphi)_{F_k^c}$. Since $(h_k)_{k\ge1}$ is decreasing, $h_\infty := \lim_{k\to\infty} h_k$ exists in $L^2(U,\mu)$. By 3.2 (iii) $h_\infty \in D(\mathcal{E}^0)$ and $\lim_{k\to\infty} h_k = h_\infty$ weakly in $D(\mathcal{E}^0)$.

Fix $u \in C_0^\infty(U)$ and $v \in D(\mathcal{E}^0)_{F_k}$, $k \ge 1$. Then $uv \in D(\mathcal{E}^0)_{F_k}$ and thus by 3.2 (ii)

$$
\mathcal{E}_1^0(h_l, uv) - \int \langle \beta, \nabla h_l\rangle uv\,d\mu = 0 \text{ for all } l \ge k\ .
$$

It follows that

$$
\mathcal{E}_1^0(h_\infty, uv) - \int \langle \beta, \nabla h_\infty\rangle uv\,d\mu = 0 \tag{3.1}
$$

for all $v \in \bigcup_{k\ge1} D(\mathcal{E}^0)_{F_k,b}$ and subsequently for all $v \in D(\mathcal{E}^0)_b$. Clearly, (3.1) for all $v \in D(\mathcal{E}^0)$ implies $\mathcal{E}_1^0(h_\infty, u) - \int \langle \beta, \nabla h_\infty\rangle u\,d\mu = 0$.

Fix n and let $w_n := G_1\varphi - G_1^{V_n}\varphi - h_\infty$. Then

$$
\mathcal{E}_1^0(w_n, u) - \int \langle \beta, \nabla w_n\rangle u\,d\mu = 0 \tag{3.2}
$$

by (3.1), 1.1 and 1.5 for all $u \in C_0^\infty(V_n)$ and thus for all $u \in H_0^{1,2}(V_n,\mu)_b$. Note that $w_n^- = (G_1^{V_n}\varphi + (h_\infty - G_1\varphi))1_{\{w_n \le 0\}} \le G_1^{V_n}\varphi$. Since $G_1^{V_n}\varphi \in H_0^{1,2}(V_n,\mu)_b$ it follows that $w_n^- \in H_0^{1,2}(V_n,\mu)_b$ too and (3.2) implies that $\mathcal{E}_1^0(w_n,w_n^-) = 0$. Thus $w_n^- = 0$, i.e., $h_\infty \le G_1\varphi - G_1^{V_n}\varphi$. Since $\lim_{n\to\infty} G_1\varphi - G_1^{V_n}\varphi = 0$ we obtain that $h_\infty \le 0$. Since on the other hand $h_\infty \ge 0$ we conclude that $h_\infty = 0$. It follows from [St1, III.2.10] that $(F_k)_{k\ge 1}$ is an L–nest. $\square$

Using the last lemma it is now easy to see that $(L, D(L))$ is quasi–regular (in terms of the framework of generalized Dirichlet forms (cf. [St1, IV.1.7])).

Theorem 3.5. *There exists a μ–tight special standard process $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t\ge 0}, (P_x)_{x\in U_\Delta})$ with life time ζ that is associated with $(L, D(L))$ in the sense that $E_\cdot\left[\int e^{-\alpha t} f(X_t)\,dt\right]$ is an $\mathcal{E}^0$–q.c. μ–version of $G_\alpha f$ for all $f \in \mathcal{B}_b(U) \cap L^2(U,\mu)$, $\alpha > 0$.*

Proof. Since $(\mathcal{E}^0, D(\mathcal{E}^0))$ is quasi–regular it is now easy to see by 3.4 that $(L, D(L))$ is quasi–regular too (now in terms of the framework of *generalized* Dirichlet forms (cf. [St1, IV.1.7])). In order to apply the existence theorem in the theory of generalized Dirichlet forms (cf. [St1, IV.2.2]) it remains to show the existence of a linear subspace $\mathcal{Y} \subset L^\infty(U,\mu)$ such that $\mathcal{Y}\cap D(L) \subset D(L)$ dense, $\lim_{\alpha\to\infty}(\alpha G_\alpha f - f)_U = 0$ in $L^2(U,\mu)$ for all $f \in \mathcal{Y}$ and $f \wedge \alpha \in \overline{\mathcal{Y}}$ (= the closure of $\mathcal{Y}$ in $L^\infty(U,\mu)$), if $f \in \overline{\mathcal{Y}}$ and $\alpha \ge 0$.

Let $\mathcal{Y} := D(\overline{L})_b$. Since $\mathcal{Y}$ is an algebra by 1.7 (ii) it follows that $f \wedge \alpha \in \overline{\mathcal{Y}}$ if $f \in \overline{\mathcal{Y}}$ and $\alpha \ge 0$. Clearly, $\mathcal{Y} \cap D(L) = D(L)_b \subset D(L)$ dense. Let $f \in \mathcal{Y}$. Then by 3.2 (iii) and the strong continuity of $(\overline{G}_\alpha)_{\alpha>0}$ on $L^1(U,\mu)$

$$\begin{aligned}\mathcal{E}_1^0((\alpha G_\alpha f - f)_U,(\alpha G_\alpha f - f)_U) &\le 6\|(1-\overline{L})(\alpha G_\alpha f - f)\|_1 \|\alpha G_\alpha f - f\|_\infty \\ &\le 12\|\alpha \overline{G}_\alpha(1-\overline{L})f - (1-\overline{L})f\|_1 \|f\|_\infty \to 0\,, \alpha\to\infty\ .\end{aligned}$$

Hence the theorem is proved. $\square$

Proposition 3.6. *Let $\mathbb{M}$ be as in 3.5. Then*

$$P_x\,[t \to X_t \textit{ is continuous on } [0,\zeta)] = 1 \qquad \mathcal{E}^0\text{–q.e.}$$

For the proof of 3.6 we will need one lemma. Let $\mathbb{M}$ be as in 3.5. Denote by $(p_t)_{t\ge 0}$ (resp. $(R_\alpha)_{\alpha>0}$) the corresponding transition semigroup (resp. resolvent).

For $f \in \mathcal{B}_b(U)$ and $V \subset U$, V open, let

$$H^V f(x) = E_x\left[e^{-\sigma_V} f(X_{\sigma_V})\right]\ .$$

Lemma 3.7. *Let $x \in U$, $r > 0$, such that $\overline{B_r(x)} \subset U$ and $V := U \setminus \overline{B_r(x)}$. Let $f \in C_0^\infty(U)$, $f \ge 0$. Then $H^V f$ is an $\mathcal{E}^0$–q.c. μ–version of some element in $D(\mathcal{E}^0)$ for which*

$$\mathcal{E}_1^0(H^V f, g) - \int \langle \beta, \nabla H^V f\rangle g\, d\mu = 0 \tag{3.3}$$

for all $g \in D(\mathcal{E}^0)_b$ *with* $g = 0$ *on* V.

Proof. Let $f_\alpha := \alpha R_{\alpha+1} f$, $\alpha > 0$. Then $\lim_{\alpha\to\infty} f_\alpha = f$ pointwise everywhere but also in $D(\mathcal{E}^0)$ since by 3.1 and the strong continuity of $(G_\alpha)_{\alpha>0}$ on $L^2(U,\mu)$ $\lim_{\alpha\to\infty} \mathcal{E}_1^0(f_\alpha - f, f_\alpha - f) = \lim_{\alpha\to\infty}(\alpha R_{\alpha+1}(1-L)f - (1-L)f, f_\alpha - f)_{L^2(U,\mu)} = 0$. Since $f_\alpha = \alpha R_1 f - R_1 f_\alpha$ we have that $H^V f_\alpha = \alpha H^V R_1 f - H^V R_1 f_\alpha$. By [St1, IV.3.4] $\alpha H^V R_1 f$ (resp. $H^V R_1 f_\alpha$) is a μ–version of $(\alpha G_1 f)_V$ (resp. $(G_1 f_\alpha)_V$). By 3.2 $H^V f_\alpha \in D(\mathcal{E}^0)_b$ and

$$\mathcal{E}_1^0(H^V f_\alpha, g) - \int \langle \beta, \nabla H^V f_\alpha\rangle g \, d\mu = 0 \tag{3.4}$$

for all $g \in D(\mathcal{E}^0)_b$ with $g = 0$ on V. Since $H^V f_\alpha - f_\alpha \in D(\mathcal{E}^0)_b$ and $H^V f_\alpha - f_\alpha = 0$ on V we obtain in particular that

$$\mathcal{E}_1^0(H^V f_\alpha, H^V f_\alpha - f_\alpha) - \int \langle \beta, \nabla H^V f_\alpha\rangle H^V f_\alpha - f_\alpha \, d\mu = 0 \ ,$$

hence

$$\begin{aligned}\mathcal{E}_1^0(H^V f_\alpha, H^V f_\alpha) &\le \mathcal{E}_1^0(H^V f_\alpha, f_\alpha) \\ &\quad + 2\|f\|_\infty \nu_{\overline{B_r(x)}}^{1/2} \||\beta| 1_{\overline{B_r(x)}}\|_2 \mathcal{E}^0(H^V f_\alpha, H^V f_\alpha)^{1/2} \ ,\end{aligned}$$

and consequently,

$$\sup_{\alpha>0} \mathcal{E}_1^0(H^V f_\alpha, H^V f_\alpha) < +\infty \ .$$

Since $\lim_{\alpha\to\infty} H^V f_\alpha = H^V f$ pointwise everywhere we obtain that $H^V f \in D(\mathcal{E}^0)_b$ and $\lim_{\alpha\to\infty} H^V f_\alpha = H^V f$ weakly in $D(\mathcal{E}^0)$ by Banach–Alaoglu. Therefore by (3.4)

$$\mathcal{E}_1^0(H^V f, g) - \int \langle \beta, \nabla H^V f\rangle g \, d\mu = 0$$

for all $g \in D(\mathcal{E}^0)_b$ with $g = 0$ on V.

To prove that $H^V f$ is $\mathcal{E}^0$–q.c. note that $\lim_{\alpha\to\infty} \alpha R_{\alpha+1} H^V f = H^V f$ pointwise everywhere since $\lim_{t\to 0} p_t H^V f = H^V f$ pointwise everywhere by the right continuity of $t \mapsto X_t$ and the continuity of f. Moreover, by 3.1 and 1.5

$$\begin{aligned}\mathcal{E}_1^0(\alpha R_{\alpha+1} H^V f, \alpha R_{\alpha+1} H^V f) &\le \alpha(H^V f - \alpha R_{\alpha+1} H^V f, \alpha R_{\alpha+1} H^V f)_{L^2(U,\mu)} \\ &\le \alpha(H^V f - \alpha R_{\alpha+1} H^V f, H^V f)_{L^2(U,\mu)} \\ &= \mathcal{E}_1^0(\alpha R_{\alpha+1} H^V f, H^V f) - \int \langle \beta, \nabla \alpha R_{\alpha+1} H^V f\rangle H^V f \, d\mu \, ,\end{aligned} \tag{3.5}$$

which implies that $\sup_{\alpha>0} \mathcal{E}_1^0(\alpha R_{\alpha+1} H^V f, \alpha R_{\alpha+1} H^V f) < +\infty$, hence $\lim_{\alpha\to\infty} \alpha R_{\alpha+1} H^V f = H^V f$ weakly in $D(\mathcal{E}^0)$ again by Banach–Alaoglu. Note that by (3.5)

$$\begin{aligned}\limsup_{\alpha\to\infty} \mathcal{E}_1^0(\alpha R_{\alpha+1} H^V f, \alpha R_{\alpha+1} H^V f) &\le \limsup_{\alpha\to\infty} \mathcal{E}_1^0(\alpha R_{\alpha+1} H^V f, H^V f) \\ &\qquad - \int \langle \beta, \nabla \alpha R_{\alpha+1} H^V f\rangle H^V f \, d\mu \\ &= \mathcal{E}_1^0(H^V f, H^V f) \ ,\end{aligned}$$

hence $\lim_{\alpha\to\infty} \mathcal{E}_1^0(\alpha R_{\alpha+1}H^V f, \alpha R_{\alpha+1}H^V f) = \mathcal{E}_1^0(H^V f, H^V f)$ and thus $\lim_{\alpha\to\infty} \alpha R_{\alpha+1}H^V f = H^V f$ strongly in $D(\mathcal{E}^0)$. Since $\alpha R_{\alpha+1}H^V f$ is $\mathcal{E}^0$–q.c. for all $\alpha > 0$ by 3.5 it follows by [MR, III.3.5] that for some subsequence $\alpha_n \to \infty$ $\lim_{n\to\infty} \alpha_n R_{\alpha_n+1}H^V f = H^V f$ $\mathcal{E}^0$–quasi uniformly, i.e., uniformly on F_k, $k \geq 1$, for some $\mathcal{E}^0$–nest $(F_k)_{k\geq 1}$. In particular, $H^V f$ is $\mathcal{E}^0$–q.c. □

Proof (of 3.6). Let $x \in U$ and $r > 0$ such that $\overline{B_r(x)} \subset U$ and $V := U \setminus \overline{B_r(x)}$. Let $u \in C_0^\infty(U)$ be such that $u = 0$ on $\overline{B_r(x)}$. Since $u = H^V u$ on V, $\nabla u = 0$ on $B_r(x)$, $\mu(\partial \overline{B_r(x)}) = 0$ and therefore by (3.3)

$$\begin{aligned}\mathcal{E}_1^0(u - H^V u, u - H^V u) &= \mathcal{E}_1^0(u - H^V u, u - H^V u)\\ &\qquad - \int \langle \beta, \nabla(u - H^V u)\rangle u - H^V u \, d\mu\\ &= \mathcal{E}_1^0(u, u - H^V u) - \int \langle \beta, \nabla u\rangle u - H^V u \, d\mu = 0 \ ,\end{aligned}$$

it follows that $u = H^V u$, in particular $H^V u = 0$ $\mathcal{E}^0$–q.e. on $\overline{B_r(x)}$. Let $u_n \in C_0^\infty(U)$, $n \geq 1$, $u_n \geq 0$, $u_n = 0$ on $\overline{B_r(x)}$, such that $\sup_{n\geq 1} u_n > 0$ on V. Then $H^V u_n = 0$ $\mathcal{E}^0$–q.e. on $\overline{B_r(x)}$ for all n implies that $P_x\left[X_{\sigma_V} \in V\right] = 0$ $\mathcal{E}^0$–q.e. on $\overline{B_r(x)}$.

Now, let $\mathcal{U} = \{U \setminus \overline{B_r(x)} | x \in \mathbb{Q}^d \cap U, r \in \mathbb{Q}$ such that $\overline{B_r(x)} \subset U\}$. Then there exists some $\mathcal{E}^0$–exceptional set N such that $P_x\left[X_{\sigma_V} \in V\right] = 0$ for all $x \in U \setminus N$, $V \in \mathcal{U}$. Let $(F_k)_{k\geq 1}$ be an $\mathcal{E}^0$–nest such that $N \subset \bigcap_{k\geq 1} U \setminus F_k$ and $\Omega_0 := \{\lim_{k\to\infty} \sigma_{U\setminus F_k} \geq \zeta\}$. Let

$$\Omega_d := \{\omega | X_{t-}(\omega) \neq X_t(\omega) \text{ for some } t \in (0, \zeta(\omega))\} \ .$$

Then

$$\Omega_d \cap \Omega_0 \subset \bigcup_{V\in\mathcal{U}} \bigcup_{\substack{s\in\\ \mathbb{Q}\cap(0,\infty)}} \{\omega | X_s(\omega) \in V^c \setminus N, X_{\sigma_V}(\theta_s \omega) \in V\} \ .$$

Since

$$P_x\left[X_s \in V^c \setminus N, X_{\sigma_V} \circ \theta_s \in V\right] = P_x\left[P_{X_s}\left[X_{\sigma_V} \in V\right], X_s \in V^c \setminus N\right] = 0$$

for all $x \in U$, $V \in \mathcal{U}$, it follows that $P_x\left[\Omega_d \cap \Omega_0\right] = 0$ for all $x \in U$, and thus $P_x\left[\Omega_d\right] = 0$ $\mathcal{E}^0$–q.e., since $P_x\left[\Omega_0\right] = 1$ $\mathcal{E}^0$–q.e. by [St1, IV.3.10]. □

4. The infinite dimensional case

All the results of Sections 1 and 3 have been obtained in such a way that they do not use any finite dimensional specialities such as the Lebesgue measure. Hence it is not surprising that they can be carried over to an infinite dimensional setting. To this end let us first introduce our framework. Let E be a separable real Banach space, H be a separable real Hilbert space such that $H \subset E$ densely and continuously. Identifying H with its topological dual H' we obtain that $E' \subset H \subset E$ densely and continuously. Let

$$\mathcal{F}C_b^\infty := \{f(\ell_1, \ldots, \ell_m) \mid m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), \ell_1, \ldots, \ell_m \in E'\}$$

be the set of finitely based smooth functions. For $u \in \mathcal{F}C_b^\infty$ and $k \in E$ let

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z+sk)|_{s=0}\,, z \in E\ ,$$

be the Gâteaux derivative of u in direction k. It follows that for $u = f(\ell_1, \ldots, \ell_m) \in \mathcal{F}C_b^\infty$ and $k \in H$ we have that

$$\frac{\partial u}{\partial k}(z) = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(\ell_1(z), \ldots, \ell_m(z))\langle \ell_i, k\rangle_H\,, z \in E\ .$$

Consequently, $k \mapsto \frac{\partial u}{\partial k}(z)$ is continuous on H and we can define the H–gradient $\nabla u(z)$ of u by

$$\langle \nabla u(z), k\rangle_H = \frac{\partial u}{\partial k}(z)\ .$$

Let $L_{sym}(H)$ denote the linear space of all symmetric bounded operators on H. Let $A : E \mapsto L_{sym}(H)$ be measurable such that for some positive constant ν we have that

$$\nu^{-1}|h|_H^2 \le \langle A(z)h, h\rangle_H \text{ for all } z \in E, h \in H\ . \tag{4.1}$$

Let μ be a *finite* positive measure on $\mathcal{B}(E)$ with $\mathrm{supp}\,(\mu) \equiv E$ and assume that

$$\int \|A(z)\|_{L(H)}\, d\mu(z) < \infty\ , \tag{4.2}$$

where $\|\cdot\|_{L(H)}$ is the usual operator norm. Then the bilinear form

$$\mathcal{E}^0(u,v) := \int \langle A(z)\nabla u(z), \nabla v(z)\rangle_H\, d\mu(z)\ ; u, v \in \mathcal{F}C_b^\infty\ ,$$

is well–defined. We assume that $(\mathcal{E}^0, \mathcal{F}C_b^\infty)$ is closable in $L^2(E,\mu)$. Then the closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a (symmetric) Dirichlet form. Let $(L^0, D(L^0))$ be the associated generator, $(T_t^0)_{t\ge 0}$ the associated semigroup and $(G_\alpha^0)_{\alpha>0}$ the associated resolvent. Similar to the finite dimensional case denote by $(\overline{T}_t^0)_{t\ge 0}$ the unique extension of $(T_t^0)_{t\ge 0}$ on $L^1(E,\mu)$ and by $(\overline{L}^0, D(\overline{L}^0))$ the corresponding generator which extends $(L^0, D(L^0))$.

Suppose that $\beta \in L^2(E; H, \mu)$ is such that

$$\int \langle \beta, \nabla u\rangle_H\, d\mu = 0 \text{ for all } u \in \mathcal{F}C_b^\infty\ . \tag{4.3}$$

Define the linear operator $(L, \mathcal{F}C_b^\infty \cap D(L^0))$ on $L^2(E,\mu)$ by

$$Lu := L^0 u + \langle \beta, \nabla u\rangle_H\ .$$

Note that by (4.3) $(L, \mathcal{F}C_b^\infty \cap D(L^0))$ is negative definite and that $\mathcal{F}C_b^\infty \cap D(L^0)$ may consist of constant functions only. Since $\mathcal{F}C_b^\infty$ is dense in $D(\mathcal{E}^0)$ it is easy to see that (4.3) implies that $\int \langle \beta, \nabla u\rangle_H\, d\mu = 0$ for all $u \in D(\mathcal{E}^0)$ and thus $\int \langle \beta, \nabla u\rangle_H v\, d\mu = -\int \langle \beta, \nabla v\rangle_H u\, d\mu$ for all $u, v \in D(\mathcal{E}^0)_b$.

The existence result.

Proposition 4.1. *Let (4.1)–(4.3) be satisfied. Then:*

(i) The operator

$$Lu := L^0 u + \langle \beta, \nabla u \rangle_H \ , u \in D(L^0)_b \ ,$$

is dissipative, hence in particular closable, on $L^1(E, \mu)$. The closure $(\overline{L}, D(\overline{L}))$ generates a Markovian C_0–semigroup of contractions $(\overline{T}_t)_{t \geq 0}$.

(ii) $D(\overline{L})_b \subset D(\mathcal{E}^0)$ and

$$\mathcal{E}^0(u, v) - \int \langle \beta, \nabla u \rangle_H v \, d\mu = - \int \overline{L} u \, v \, d\mu \, ; u \in D(\overline{L})_b, v \in D(\mathcal{E}^0)_b \ . \tag{4.4}$$

In particular,

$$\mathcal{E}^0(u, u) = - \int \overline{L} u \, u \, d\mu \, ; u \in D(\overline{L})_b \ . \tag{4.5}$$

Proof. (i) Similar to the proof of Step 1 in 1.1 it can be shown that

$$\int Lu \, 1_{\{u > 1\}} \, d\mu \leq 0 \ \text{ for all } u \in D(L^0)_b \ . \tag{4.6}$$

Similar to the proof of the corresponding statement in 1.1 it follows that $(L, D(L^0)_b)$ is dissipative, hence in particular closable. Exactly in the same way as in the proof of Step 2 in 1.1 (i) it is shown that $(1 - L)(D(L^0)_b) \subset L^1(E, \mu)$ dense, so that the closure $(\overline{L}, D(\overline{L}))$ generates a C_0–semigroup of contractions $(\overline{T}_t)_{t \geq 0}$.

To see that $(\overline{T}_t)_{t \geq 0}$ is Markovian note that similar to the proof of 1.1 the inequality (4.6) extends to all $u \in D(\overline{L})$ which implies that the associated resolvent $(\overline{G}_\alpha)_{\alpha > 0}$, $\overline{G}_\alpha := (\alpha - L)^{-1}$, $\alpha > 0$, is sub–Markovian. Since for all $u \in L^1(E, \mu)$ $\overline{T}_t u = \lim_{\alpha \to \infty} \exp(t\alpha(\alpha \overline{G}_\alpha - 1))u$ we then obtain that $(\overline{T}_t)_{t \geq 0}$ is sub–Markovian too. Note that $1 \in D(L^0)_b \subset D(\overline{L})$ and $\overline{L}1 = 0$, hence $\overline{T}_t 1 = 1$, $t \geq 0$, which implies that $(\overline{T}_t)_{t \geq 0}$ is in fact Markovian.

(ii) Similar to Step 1 in the proof of 1.1 (ii) one can show that $D(\overline{L}^0)_b \subset D(\overline{L})$ and $\overline{L}u = \overline{L}^0 u + \langle \beta, \nabla u \rangle_H$, $u \in D(\overline{L}^0)$.

Let $u \in D(\overline{L})_b$ and $u_n \in D(L^0)$, $n \geq 1$, such that $\lim_{n \to \infty} \|u_n - u\|_1 + \|\overline{L}u_n - \overline{L}u\|_1 = 0$ and $\lim_{n \to \infty} u_n = u$ μ–a.e. Let $\psi \in C_b^2(\mathbb{R})$ be such that $\psi(t) = t$ if $|t| \leq \|u\|_\infty + 1$ and $\psi(t) = 0$ if $|t| \geq \|u\|_\infty + 2$. Since $\psi(u_n) \in D(\overline{L})$ it follows similar to the proof of the corresponding statements in 1.1 (ii) that $u \in D(\mathcal{E}^0)$, $\lim_{n \to \infty} \psi(u_n) = u$ in $D(\mathcal{E}^0)$ and $\lim_{n \to \infty} \overline{L}\psi(u_n) = \overline{L}u$ in $L^1(E, \mu)$. If $v \in D(\mathcal{E}^0)_b$ we obtain that

$$\begin{aligned} \mathcal{E}^0(u, v) - \int \langle \beta, \nabla u \rangle v \, d\mu &= \lim_{n \to \infty} \mathcal{E}^0(\psi(u_n), v) - \int \langle \beta, \nabla \psi(u_n) \rangle v \, d\mu \\ &= - \lim_{n \to \infty} \int \overline{L}\psi(u_n) \, v \, d\mu = - \int \overline{L}u \, v \, d\mu \ . \quad \square \end{aligned}$$

Remark 4.2. We may also consider the closure $(\overline{L}', D(\overline{L}'))$ of $L^0u - \langle \beta, \nabla u\rangle_H$, $u \in D(L^0)$, on $L^1(E,\mu)$. By 4.1 the closure generates a Markovian C_0–semigroup of contractions $(\overline{T}'_t)_{t\geq 0}$, $D(\overline{L}')_b \subset D(\mathcal{E}^0)$ and

$$\mathcal{E}^0(u,v) + \int \langle B, \nabla u\rangle_H v \, d\mu = -\int \overline{L}' u \, v \, d\mu \,; u \in D(\overline{L}')_b, v \in D(\mathcal{E}^0)_b, \tag{4.7}$$

by (4.4). If $(L', D(L'))$ is the part of $(\overline{L}', D(\overline{L}'))$ on $L^2(E,\mu)$ and $(L, D(L))$ the part of $(\overline{L}, D(\overline{L}))$ on $L^2(E,\mu)$ it follows similar to the finite dimensional case (cf. Remark 1.3) that L and L' are adjoint on $L^2(E,\mu)$.

The uniqueness result.

Proposition 4.3. *Suppose that $\mathcal{F}C_b^\infty \subset D(L^0)$. If $(L^0, \mathcal{F}C_b^\infty)$ is L^1–unique then $(L, \mathcal{F}C_b^\infty)$ is L^1–unique too.*

Proof. The proof is similar to that of 2.4. In fact let $u \in D(L^0)_b$ and $u_n \in \mathcal{F}C_b^\infty$, $n \geq 1$, be such that $\lim_{n\to\infty} \|u_n - u\|_1 + \|L^0 u_n - L^0 u\|_1 = 0$ and $\lim_{n\to\infty} u_n = u$ μ–a.e. If $\psi \in C_b^\infty(\mathbb{R})$ is such that $\psi(t) = t$ if $|t| \leq \|u\|_\infty + 1$ and $\psi(t) = 0$ if $|t| \geq \|u\|_\infty + 2$ then $\psi(u_n) \in \mathcal{F}C_b^\infty$, $n \geq 1$, and similar to the corresponding statements in 2.4 it follows that $\lim_{n\to\infty} L^0\psi(u_n) = L^0 u$ in $L^1(E,\mu)$ and $\lim_{n\to\infty} \psi(u_n) = u$ in $D(\mathcal{E}^0)$. Therefore $\lim_{n\to\infty} L\psi(u_n) = Lu$ in $L^1(E,\mu)$. Since $(L, D(L^0)_b)$ is L^1–unique (cf. the proof of 4.1 (i)) this implies the assertion. □

Associated Markov processes.

Similar to the finite dimensional case the theory of *generalized* Dirichlet forms can be used to construct stochastic processes associated with $(L, D(L))$, where $(L, D(L))$ is the part of $(\overline{L}, D(\overline{L}))$ on $L^2(E,\mu)$. Let $(G_\alpha)_{\alpha>0}$ be the associated resolvent. By [St1, I.4.9 (ii)] $(L, D(L))$ is associated with a *generalized* Dirichlet form. Similar to Section 3 an element $u \in L^2(E,\mu)$ is called 1–*excessive* if $\beta G_{\beta+1} u \leq u$ for all $\beta \geq 0$. Recall that, if $f \in L^2(E,\mu)$ and $V \subset E$ open are such that $\mathcal{L}_{f1_V} \cap D(L) \neq \emptyset$, the 1–reduced element of f on V exists and is denoted by f_V. Note that $\mathcal{L}_f \cap D(L) \neq \emptyset$ for all $f \in L^\infty(E,\mu)$ since $1 \in D(L)$.

The proof of the following lemma is similar to the proof of 3.2.

Lemma 4.4. *Let $f \in D(\overline{L})_b$ and $V \subset E$ open. Then:*

(i) $f_V \in D(\mathcal{E}^0)_b$.

(ii) $\mathcal{E}_1^0(f_V, g) - \int \langle \beta, \nabla f_V\rangle g \, d\mu = 0$ *for all* $g \in D(\mathcal{E}^0)_b$ *with* $g = 0$ *on* V.

(iii) $\mathcal{E}_1^0(f_V, f_V) \leq 6\|(1-\overline{L})f\|_1 \|f\|_\infty$.

The proof of the following lemma is also similar to the proof of 3.3.

Lemma 4.5. *An increasing sequence of closed subsets $(F_k)_{k\geq 1}$ is an L–nest if and only if it is an $\mathcal{E}^0$–nest.*

Proof. If $(F_k)_{k\geq 1}$ is an L–nest it can be shown similar to the proof of the corresponding statement in 3.4 that $(F_k)_{k\geq 1}$ is an $\mathcal{E}^0$–nest. Conversely, suppose that $(F_k)_{k\geq 1}$ is an $\mathcal{E}^0$–nest, let $0<\varphi\leq 1$, $h:=G_1\varphi$ and $h_k:=(G_1\varphi)_{F_k^c}$. Similar to the proof of the corresponding statements in 3.4 it can be shown that $(h_k)_{k\geq 1}$ is decreasing, $h_\infty:=\lim_{k\to\infty}h_k\in D(\mathcal{E}^0)$ and

$$\mathcal{E}_1^0(h_\infty,u)-\int\langle\beta,\nabla h_\infty\rangle u\,d\mu=0$$

for all $u\in\bigcup D(\mathcal{E}^0)_{F_k,b}$, hence all $u\in D(\mathcal{E}^0)_b$. In particular, $\mathcal{E}_1^0(h_\infty,h_\infty)=$ $=\mathcal{E}_1^0(h_\infty,h_\infty)-\int\langle\beta,\nabla h_\infty\rangle h_\infty\,d\mu=0$, i.e., $h_\infty=0$. It follows from [St1, III.2.10] that $(F_k)_{k\geq 1}$ is an L–nest. $\square$

Theorem 4.6. *There exists a μ–tight special standard process $\mathbb{M}=(\Omega,\mathcal{F},(X_t)_{t\geq 0},(P_x)_{x\in E_\Delta})$ with life time ζ that is associated with $(L,D(L))$ in the sense that $E.\left[\int e^{-\alpha t}f(X_t)\,dt\right]$ is an $\mathcal{E}^0$–q.c. μ–version of $G_\alpha f$ for all $f\in\mathcal{B}_b(E)\cap L^2(E,\mu)$, $\alpha>0$. In particular, $P_x\,[\zeta=+\infty]=1$ $\mathcal{E}^0$–q.e.*

Proof. $(\mathcal{E}^0,D(\mathcal{E}^0))$ is quasi–regular since $\mathcal{F}C_b^\infty\subset D(\mathcal{E}^0)$ dense and by (4.2) there exists an $\mathcal{E}^0$–nest of compact sets (cf. [RSch, Subsection 4a)]). By 4.5 it is now easy to see that $(L,D(L))$ is quasi–regular too (now in terms of the framework of *generalized* Dirichlet forms (cf. [St1, IV.1.7])). Similar to 3.5 it suffices now to show the existence of a linear subspace $\mathcal{Y}\subset L^\infty(E,\mu)$ such that $\mathcal{Y}\cap D(L)\subset D(L)$ dense, $\lim_{\alpha\to\infty}(\alpha G_\alpha f-f)_E=0$ in $L^2(E,\mu)$ for all $f\in\mathcal{Y}$ and $f\wedge\alpha\in\overline{\mathcal{Y}}$ (= the closure of $\mathcal{Y}$ in $L^\infty(E,\mu)$) if $f\in\overline{\mathcal{Y}}$ and $\alpha\geq 0$. Let $\mathcal{Y}:=D(\overline{L})_b$. Clearly, $\mathcal{Y}\cap D(L)=D(L)_b$ is dense in $D(L)$. Similar to the proof of the corresponding statements in 3.5 it can be shown that $\lim_{\alpha\to\infty}(\alpha G_\alpha f-f)_E=0$ in $L^2(E,\mu)$ for all $f\in\mathcal{Y}$ (using 4.4 (iii)) and $u\wedge\alpha\in\overline{\mathcal{Y}}$ if $u\in\overline{\mathcal{Y}}$ and $\alpha\geq 0$. Now, [St1, IV.2.2] applies and we obtain the existence of $\mathbb{M}$. Since $\alpha\int_0^\infty e^{-\alpha t}P_x\,[X_t\in E]\,dt$ is an $\mathcal{E}^0$–q.c. μ–version of $\alpha G_\alpha 1=1$, it follows that $P_x\,[X_t\in E]=1$ $\mathcal{E}^0$–q.e. for all t, hence $P_x\,[\zeta=+\infty]=1$ $\mathcal{E}^0$–q.e. $\square$

Proposition 4.7. *Let $\mathbb{M}$ be as in 4.6. Then*

$$P_x\,[t\to X_t \text{ is continuous on } [0,\zeta)]=1 \qquad \mathcal{E}^0\text{–q.e.}$$

Proof. Similar to the proof of 3.7 it can be shown that if $f\in\mathcal{F}C_b^\infty$ and $U\subset E$ open then $H^Uf(x)=E_x\,[e^{-\sigma_U}f(X_{\sigma_U})]$ is an $\mathcal{E}^0$–q.c. μ–version of some element in $D(\mathcal{E}^0)_b$ for which

$$\mathcal{E}_1^0(H^Uf,g)-\int\langle\beta,\nabla H^Uf\rangle_H g\,d\mu=0 \tag{4.8}$$

for all $g\in D(\mathcal{E}^0)_b$ with $g=0$ on U.

Let $u\in\mathcal{F}C_b^\infty$, $u\geq 0$, and $U:=\{u>0\}$. Then $\operatorname{supp}((u-\varepsilon)^+)\subset U$ if $\varepsilon>0$. Since $u=H^Uu$ on U and $\nabla u=0$ μ–a.e. on $E\setminus\overline{U}$ (cf. [MR, V.1.12]) it follows that

$$\begin{aligned}&\mathcal{E}_1^0(u,u-H^Uu)-\int\langle\beta,\nabla u\rangle_H u-H^Uu\,d\mu\\&\quad=\lim_{\varepsilon\to 0}\mathcal{E}_1^0((u-\varepsilon)^+,u-H^Uu)-\int\langle\beta,\nabla(u-\varepsilon)^+\rangle_H u-H^Uu\,d\mu=0\,.\end{aligned}$$

Hence by (4.8)

$$\mathcal{E}_1^0(u - H^U u, u - H^U u) = \mathcal{E}_1^0(u - H^U u, u - H^U u) - \int \langle \beta, \nabla(u - H^U u)\rangle_H u - H^U u \, d\mu = 0 \ ,$$

which implies that $u = H^U u$. In particular, $H^U u = 0$ $\mathcal{E}^0$–q.e. on $E \setminus U$ and thus $P_x [X_{\sigma_U} \in U] = 0$ $\mathcal{E}^0$–q.e. on $E \setminus U$.

Let $u_n \in \mathcal{F}C_b^\infty$, $n \geq 1$, be such that $U_n := \{u_n > 0\}$, $n \geq 1$, separates the points of E in the sense that if $x \neq y$ there exists $n = n(x, y)$ such that $x \in U_n$, $y \in E \setminus U_n$. Similar to the proof of the corresponding statement of 3.6 it now follows that $P_x [\Omega_d] = 0$ $\mathcal{E}^0$–q.e., where

$$\Omega_d := \{\omega | X_{t-}(\omega) \neq X_t(\omega) \text{ for some } t \in (0, \zeta(\omega))\} \ . \quad \square$$

References

[AR] Albeverio, S., Röckner, M., *Dirichlet form methods for uniqueness of martingale problems and applications*, in: Stochastic Analysis. Proceedings of Symposia in Pure Mathematics **Vol. 57** (1995), Am. Math. Soc., Rhode Island, 513-528.

[ABR] Albeverio, S., Bogachev, V.I., Röckner, M., *On uniqueness of invariant measures for finite and infinite dimensional diffusions*, Preprint 97–057, SFB 343, Bielefeld (1997).

[BKR1] Bogachev, V.I., Krylov, N., Röckner, M., *Regularity of Invariant Measures: The case of Non–constant Diffusion Part*, J. Funct. Anal. **138** (1996), 223 - 242.

[BKR2] Bogachev, V.I., Krylov, N., Röckner, M., *Elliptic Regularity and Essential Self–Adjointness of Dirichlet Operators on* $\mathbb{R}^n$, Preprint 96–038, SFB 343, Bielefeld (1996).

[BH] Bouleau, N., Hirsch, F., *Dirichlet Forms and Analysis on Wiener space*, Walter de Gruyter, Berlin, 1991.

[BR] Bogachev, V.I., Röckner, M., *Regularity of invariant measures on finite and infinite dimensional spaces and applications*, J. Funct. Anal. **133** (1995), 168 - 223.

[D] Davies, E.B., *L^1–Properties of second order elliptic operators*, Bull. London Math. Soc. **17** (1985), 417 - 436.

[FOT] Fukushima, M., Oshima, Y., Takeda, M., *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter, Berlin, 1994.

[K] Krylov, N.V., *Lectures on Elliptic and Parabolic Equations in Hölder Spaces*, Graduate Studies in Mathematics, vol. 12, American Mathematical Society, 1996.

[MR] Ma, Z.M., Röckner, M., *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*, Springer, Berlin - Heildelberg - New York, 1992.

[Na] Nagel, R. (Editor), *One–parameter Semigroups of Positive Operators*, Lecture Notes in Mathematics 1184, Springer, Berlin, 1986.

[Pa] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, Berlin, 1983.

[ReSi] Reed, M., Simon, B., *Methods of modern mathematical physics II. Fourier Analysis*, Academic Press, New York - San Francisco - London, 1975.

[RSch] Röckner, M., Schmuland, B., *Quasi–Regular Dirichlet Forms: Examples and Counterexamples*, Can. J. Math. **47 (1)** (1995), 165 - 200.

[RZ] Röckner, M., Zhang, T.S., *Uniqueness of Generalized Schrödinger Operators, Part II*, J. Funct. Anal. **119** (1994), 455 - 467.

[S] Sturm, K.-Th., *Analysis on Local Dirichlet Spaces, I. Recurrence, Conservativeness and L^p–Liouville Properties*, J. reine angew. Math. **456** (1994), 173 - 196.

[St1] Stannat, W., *The Theory of generalized Dirichlet forms and its applications in Analysis and Stochastics*, PhD-thesis, Bielefeld, publication in preparation (1996).

[St2] Stannat, W., *First Order Perturbations of Dirichlet Operators: Existence and Uniqueness*, J. Funct. Anal **141** (1996), 216 - 248.