On some Poincaré–series on hyperbolic space

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1 Introduction

Let $\mathbb{H}^n$ denote hyperbolic n-space with $d(P,Q)$ the hyperbolic distance between the points P and Q. Introduce

$$\delta(P,Q) = 2\cosh(d(P,Q))$$

then it is easily seen (see e.g. [5]) that

$$\delta(P,Q) = \frac{\sum_{i=0}^{n-2} \left(u_i(P) - u_i(Q)\right)^2 + v^2(P) + v^2(Q)}{v(P)v(Q)}$$

where $u_0, \dots, u_{n-2}, v$ are the coordinates in the upper half-space model, see section 2. From the definition follows that δ is a point pair invariant, i.e. for γ from the isometry group of $\mathbb{H}^n$ we have

$$\delta(\gamma P, Q) = \delta(P, \gamma^{-1} Q).$$

Let Γ be a discrete cofinite subgroup of $\mathrm{Iso}^+(\mathbb{H}^n)$, of orientation preserving isometries of $\mathbb{H}^n$. Then the Poincaré series built from δ with respect to Γ is defined as

$$H(P,Q,s) := \sum_{\gamma\in\Gamma} \left(\delta(P,\gamma Q)\right)^{-(n-2+s)}$$

for $\Re s$ sufficiently large. Analogous considerations as in [4], section 3.1 show that this Poincaré series converges absolutely for all P, $Q \in \mathbb{H}^n$ if $\Re s > 1$.

$H(P,Q,s)$ plays an important role for the study of the distribution of hyperbolic lattice points. Its analytic behaviour was investigated by Huber [6] for $n = 2$ and for $n = 3$ an extensive study can be found in the monograph of Elstrodt, Grunewald and Mennicke, see [4].

In the present paper we relate $H(P,Q,s)$ (for any dimension n) with certain Theta lifts of real analytic Poincaré series of Selberg type defined on $\mathbb{H}^2$. We have already considered Theta lifts of this sort in [13] and [12].

For more information about such Theta lifts generalizing Shimura's lift [16] see also the work of Niwa [14], Shintani [17], Duke [3] and Katok/Sarnak, [7].

Kudla and Millson, see [8], do similar computations in the setting of holomorphic Hilbert–Siegel modular forms but restrict to the case, that the manifold for the lifted forms is compact and is at most three–dimensional .

In the present paper we have the following situation: Let Q denote a quadratic form of signature $(1,n)$. Then for $z \in \mathbb{H}^n$, $w \in \mathbb{H}^n$ Siegel's theta function $\Theta_{Q,\mathbb{Z}^{n+1}}(z,w)$ will be defined in section 6. For ν a fundamental discriminant the real analytic Poincaré series for $\Gamma_0(N)$ of weight $(1-n)/2$ and multiplier $v_{D,n}$ is given as

$$P_\nu(z,s) = \sum_{T=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_0(N)} \overline{v_{D,n}(T) e^{\frac{i(n-1)}{2} \arg(cz+d)}} y_T^s e(\nu x_T + i|\nu| y_T), \quad \Re s > 1,$$

with $x_T + i y_T := Tz$. See sections 6,7 for the full definitions. The lifted series is defined as $P_\nu^\Theta(w,s) = \int_{\mathcal{F}_N} P_\nu(z,s) y^{\frac{n+1}{4}} \overline{\Theta(z,w)} d\mu(z)$ where $\mathcal{F}_N$ denotes a fundamental domain for $\Gamma_0(N)$.

Our main result now is the following

Theorem 1.1. *Let $\Gamma(Q)$ be the group of units for Q which is a discrete cofinite subgroup of $\mathrm{Iso}^+(\mathbb{H}^n)$, ν a fundamental discriminant integer. Let further $\mathbf{a}_i$, $i \in I$, be a full set of orbit representants for the action of $\Gamma(Q)$ on $\{\mathbf{a} \in \mathbb{Z}^{n+1} : Q[\mathbf{a}] = \nu\}$, see section 7, with isotropy group $\mathcal{P}_i$ which has finite order f_i if $\nu > 0$.*

Let the points η_i and the geodesic hyperplanes $\mathbf{g}_i$ associated to $\mathbf{a}_i$ be given as in the end of section 9, and let $d\sigma$ denote hyperbolic measure on such a hyperplane. We then have for $\Re s > (n+1)/4$

i) if $\nu > 0$

$$P_\nu^\Theta(w,s) = \frac{\Gamma(s+\frac{n-3}{4})}{f_i (4\pi\nu)^{s+\frac{n-3}{4}}} \sum_{i\in I} H(\eta_i, w, 2s - \frac{n-1}{2}), \quad w \in \mathbb{H}^n,$$

ii) if $\nu < 0$

$$\begin{aligned} P_\nu^\Theta(w,s) &= \frac{1}{2^{4s+n-4} \pi^{s+\frac{3n-7}{4}} |\nu|^{s+\frac{n-3}{4}}} \frac{\Gamma(2s+\frac{n-3}{2})}{\Gamma(s-\frac{n-1}{4})} \\ &\quad \times \sum_{i\in I} \int_{\mathcal{P}_i \backslash \mathbf{g}_i} H(\omega, w, 2s - \frac{n-1}{2}) d\sigma(\omega), \quad w \in \mathbb{H}^n. \end{aligned}$$

One should remark, that $H(P,Q,s)$ can be chosen as a kernel function in the Selberg trace formula. For $n = 3$ its spectral expansion has been computed in [4] and for Γ a discrete cofinite subgroup of $\mathrm{PSL}_2(\mathbb{C})$ it has the form

$$\begin{aligned} H(P,Q,s) &= \frac{2^s \pi}{\mathrm{vol}(\Gamma)} \frac{\Gamma(\frac{s-1}{2})\Gamma(\frac{s+1}{2})}{\Gamma(s+1)} \\ &+ 2^s \pi \sum_m \frac{\Gamma(\frac{s-\mu_m}{2})\Gamma(\frac{s+\mu_m}{2})}{\Gamma(s+1)} e_m(Q) \overline{e_m(P)} \qquad (1) \\ &+ \text{Eisenstein terms} \end{aligned}$$

where the sum is over the nonconstant L^2–eigenfunctions e_m of the Laplace–Beltrami operator with eigenvalue $\lambda_m = 1 - \mu_m^2$ which we assume to be orthonormalized.

At least if e_m is a cusp form them it can be shown that

$$e_m^{\Theta}(z) = \int_{\mathcal{F}_\Gamma} e_m(w) y \Theta_{Q,\mathbb{Z}^4}(z,w) d\mu(w)$$

is a cusp form of weight -1 on $\Gamma_0(N)\backslash\mathbb{H}^2$ where N is the level of Q, see section 6. Moreover it is again an eigenform for the Laplacian of eigenvalue $\lambda'_m = \frac{1}{4}\lambda_m$ and it possesses a Fourier expansion at infinity

$$e_m^{\Theta}(z) = \sum_{n\neq 0} \rho_m(n) W_{-\frac{1}{2}\mathrm{sgn}(n),\frac{\mu_m}{2}}(4\pi|n|y)e(nx)$$

with $W_{a,b}(z)$ the exponentially decreasing Whittaker function, and as usual $e(u) := e^{2\pi i u}$.

From this we immediately obtain from our theorem a formula for the Fourier coefficients $\rho_m(n)$ which generalizes the formula of Maass, see [10], see also [3], [15]

Corollary 1.1. *Let $\Gamma = \Gamma(Q)$ where Q is a quadratic form of signature $(1,3)$. Let e_m be a cusp form on $\Gamma\backslash\mathbb{H}^3$ as above, then*

i) if $n > 0$

$$\rho_m(n) = \frac{\sqrt{\pi}}{4f_i|n|}\sum_{i\in I} e_m(\eta_i)$$

ii) if $n < 0$

$$\rho_m(n) = \frac{1}{2\sqrt{\pi}|n|}\sum_{i\in I}\int_{\mathcal{P}_i\backslash \mathrm{g}_i} e_m(\omega) d\sigma(\omega).$$

Remark. This corollary has an obvious generalization to the case of a subgroup of finite index in $\Gamma(Q)$.

Proof. A straight forward calculation gives

$$\begin{aligned}
\int_{\mathcal{F}_\Gamma} P_n^{\Theta}(w,s)\overline{e_m(w)}d\mu(w) &= \int_{\mathcal{F}_{\Gamma_0(N)}} P_n(z,s)\overline{e_m^{\Theta}(z)}d\mu(z) \\
&= \overline{\rho_m(n)}\int_0^\infty W_{-\frac{1}{2}\mathrm{sgn}(n),\frac{\mu_m}{2}}(4\pi|n|y)e^{-2\pi i|n|y}y^{s-2}dy \\
&= \overline{\rho_m(n)}\frac{\Gamma(s-\frac{1}{2}+\frac{\mu_m}{2})\Gamma(s-\frac{1}{2}-\frac{\mu_m}{2})}{(4\pi|n|)^{s-1}\Gamma(s+\mathrm{sgn}(n)\frac{1}{2})}
\end{aligned}$$

where the last formula can be taken from integral tables, e.g. [11], p. 316.

On the other hand from our theorem and (1) we obtain for $n > 0$

$$\begin{aligned}
\int_{\mathcal{F}_\Gamma} P_n^{\Theta}(w,s)\overline{e_m(w)}d\mu(w) &= \frac{\Gamma(s)}{f_i(2\pi\nu)^s}\int_{\mathcal{F}_\Gamma}\sum_{i\in I} H(\eta_i,w,2s-1)\overline{e_m(w)}d\mu(w) \\
&= \frac{\pi}{(\pi n)^s}\frac{\Gamma(s)}{\Gamma(2s)}\Gamma(s-\frac{1}{2}-\frac{\mu_m}{2})\Gamma(s-\frac{1}{2}+\frac{\mu_m}{2})\sum_{i\in I} e_m(\eta_i)
\end{aligned}$$

and for $n<0$

$$\begin{aligned}\int_{\mathcal{F}_\Gamma} P_n^\Theta(w,s)\overline{e_m(w)}d\mu(w) &= \frac{1}{\Gamma(\frac{1}{2})(\pi|n|)^s 2^{4s-2}}\frac{\Gamma(2s)}{\Gamma(s-\frac{1}{2})} \\ &\quad\times\int_{\mathcal{F}_\Gamma}\sum_{i\in I}\int_{\mathcal{P}_i\backslash \mathrm{g}_i} H(\lambda,w,2s-1)\overline{e_m(w)}d\sigma(\lambda)d\mu(w) \\ &= \frac{\pi}{\Gamma(\frac{1}{2})(\pi|n|)^s 2^{2s-1}\Gamma(s-\frac{1}{2})}\Gamma(s-\frac{1}{2}-\frac{\mu_m}{2})\Gamma(s-\frac{1}{2}+\frac{\mu_m}{2})\end{aligned}$$

$$\times\sum_{i\in I}\int_{\mathcal{P}_i\backslash \mathrm{g}_i} e_m(\omega)d\sigma(\omega).$$

from which the statement of the corollary follows, where in the case $n>0$ we use the formula $\Gamma(2s)=\frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma(s+\frac{1}{2})$. □

2 Hyperbolic space as symmetric space

Let ${}^tQ=Q$ be the matrix belonging to a quadratic form of signature $(1,n)$ over the rational integers. This means that the enties of the matrix have to be from $\frac{1}{2}\mathbb{Z}$ and the entries on the diagonal have to be integers.

There is a real matrix C such that

$$Q[C]:={}^tCQC=\begin{pmatrix}1 & 0\\ 0 & -E_n\end{pmatrix}$$

where E_n denotes the $n-dim.$ identity matrix. $Q_C^+:=(C\,{}^tC)^{-1}$ is called a majorant for Q and the majorant space is given as

$$\mathcal{M}_C(Q):=\{Q_C^+[T^{-1}]:T\in \mathrm{SO}_0(Q)\}$$

where $\mathrm{SO}_0(Q)$ is the connected component of the special orthogonal group $\mathrm{SO}(Q)$ containing E_{n+1}.

$\mathrm{SO}_0(Q)$ acts on $\mathcal{M}_C(Q)$ in an obvious way leaving invariant the metric

$$dP^2=\frac{1}{8}\mathrm{Tr}\left(P^{-1}dP\,P^{-1}dP\right).$$

The stabilizer K_C of Q_C^+ is a maximal compact subgroup of $\mathrm{SO}_0(Q)$ and so $\mathcal{M}_C(Q)$ is the Riemannian symmetric space associated to $\mathrm{SO}_0(Q)$, K_C. Clearly $C^{-1}\mathrm{SO}_0(Q)C\cong \mathrm{SO}^+(1,n)$, the group of future preserving Lorentz transformations and $\mathcal{M}_C(Q)$ can be viewed upon as hyperbolic n-space.

3 Hyperbolic space and Clifford algebras

Let $\mathcal{C}_k$ denote the Clifford algebra associated to the quadratic space $(\mathbb{R}^k,q)$ with

$$q(x_1,\ldots,x_k)=-(x_1^2+\ldots+x_k^2).$$

This is the algebra generated over $\mathbb{R}$ by $i_1, \dots, i_k$ subject to the relations

$$i_1^2 = \dots = i_k^2 = -1, \ \ i_m i_l = -i_m i_l \text{ if } m \neq l.$$

The space $V_{k+1} = \mathbb{R} \oplus \mathbb{R} i_1 \oplus \dots \oplus \mathbb{R} i_k$ is called the space of vectors in $\mathcal{C}_k$. Of course vectors are invertible elememts of $\mathcal{C}_k$ and the set of all products of elements form V_{k+1} forms a group, the so-called Clifford group Γ_k

There is a main involution $'$ and antiinvolution $*$ in $\mathcal{C}_k$ given by

$$(i_1 \dots i_r)' = (-1)^r i_1 \dots i_r$$
$$(i_1 \dots i_r)^* = i_r \dots i_1$$

and extended linearly to all of $\mathcal{C}_k$. We further put $\bar{x} := (x')^*$. There are natural injections $\mathcal{C}_k \subset \mathcal{C}_l$ for $k \leq l$. We now choose for hyperbolic n-space the subset of vectors in $\mathcal{C}_{n-1}$

$$\mathbb{H}^n = \{u + i_{n-1} v \ : \ u = u_0 + i_1 u_1 + \dots + i_{n-2} u_{n-2} \in V_{n-1}, \, v > 0\}$$

with the metric

$$ds^2 = \frac{du_0^2 + \dots + dv^2}{v^2}.$$

In this model $\mathrm{Iso}^+(\mathbb{H}^n)$ can be realized as the so-called Vahlen group $\mathrm{SV}_{n-2}/\{I, -I\}$, see [5] for a detailed discussion.

SV_{n-2} is defined as the set of matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M(2, \mathcal{C}_{n-2})$ satisfying

i) $\alpha, \, \beta, \, \gamma, \, \delta \in \Gamma_{n-2} \cup \{0\}$,

ii) $\bar{\alpha}\beta, \, \bar{\gamma}\delta \in V_{n-1}$,

iii) $\alpha\delta^* - \beta\gamma^* = 1$.

SV_{n-2} forms a group under matrix multiplication and it acts on $\mathbb{H}^n$ via

$$gw = (\alpha w + \beta)(\gamma w + \delta)^{-1}, \ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SV}_{n-2},$$

where we understand the entries of g as elements of $\mathcal{C}_{n-1}$ because of the natural injection $\mathcal{C}_{n-2} \subset \mathcal{C}_{n-1}$.

One obtains the following explicit formulae, see [5]

$$gw = u_g + i_{n-1} v_g,$$

where

$$u_g = \frac{(\alpha u + \beta)\overline{(\gamma u + \delta)} + \bar{\gamma} v^2}{|\gamma u + \delta|^2 + |\gamma|^2 v^2} \tag{2}$$

$$v_g = \frac{v}{|\gamma u + \delta|^2 + |\gamma|^2 v^2}. \tag{3}$$

Of course there is an isomorphism

$$\psi : \mathrm{SV}_{n-2}/\{I, -I\} \mapsto \mathrm{SO}^+(1, n)$$

as orientation preserving isometry groups of two different models for hyperbolic space. ψ shall be arranged in such a way that $\psi^{-1}(\mathrm{SO}(n)) = \mathrm{Stab}(i_{n-1})$.

4 The exceptional isomorphism

We want to give a more explicit description for ψ. For this we describe a further realization of hyperbolic space via hermitean matrices, see e.g. [5]. Let for this let H_n denote the set of hermitean matrices over $\mathcal{C}_{n-2}$, i.e.

$$\mathrm{H}_n = \{\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in M(2, \mathcal{C}_{n-2}) : a, c \in \mathbb{R},\ b = b_0 + \ldots + i_{n-2}b_{n-2} \in V_{n-1}\}.$$

For $H = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$ the determinant map $\det(H) = ac - b_0^2 - \ldots - b_{n-2}^2$ defines a quadratic form on $\mathbb{R}^n$ of signature $(1, n)$.

$\mathbb{R}^+$ acts on the set

$$\mathrm{H}_n^+ = \{\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in \mathrm{H}_n : c > 0,\ \det(H) > 0\}$$

of positive definite hermitean forms by multiplication and we put

$$\tilde{\mathrm{H}}_n^+ := \mathrm{H}_n^+ / \mathbb{R}^+.$$

Analogously one defines H_n^-, $\tilde{\mathrm{H}}_n^-$.

Let further for $\sigma \in \mathrm{SV}_{n-2}$, $H \in \mathrm{H}_n$

$$[\sigma]H := \sigma H {}^t\sigma.$$

We quote the following facts from [5]:

Proposition 4.1. *i) The map $\sigma \mapsto [\sigma]H$ defines an action H_n, H_n^+ and $\tilde{\mathrm{H}}_n^+$.*

ii) Define $\Phi : \mathrm{H}_n^+ \mapsto \mathbb{H}^n$ by

$$\Phi(H) := \frac{b}{c} + \frac{\sqrt{\det(H)}}{c} i_{n-1},\ H = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}.$$

Φ induces an SV_{n-2}-equivariant bijection $\tilde{\Phi} : \tilde{\mathrm{H}}_n^+ \mapsto \mathbb{H}^n$, i.e. $\tilde{\Phi}([\sigma]H) = \sigma\tilde{\Phi}(H)$.

Further there is a natural identification of the hyperboloid

$$\mathbf{H} = \{(x_0, \ldots, x_n) \in \mathbb{R}^n :\ x_0^2 - x_2^2 - \ldots - x_n^2 = 1\}$$

with $\tilde{\mathrm{H}}_n^+$ via the map

$$\chi : \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \mapsto \frac{1}{\sqrt{ac - b\bar{b}}}(\frac{1}{2}(a + c), \frac{1}{2}(a - c), b_0, \ldots, b_{n-2})$$

which induces the isomorphism $\psi : \mathrm{SV}_{n-2}/\{\pm I\} \to \mathrm{SO}^+(1, n)$, namely

$$\psi(\sigma) = \chi[\sigma]\chi^{-1}.$$

Lemma 4.1. *ψ maps $\mathrm{Stab}(i_{n-1})$ onto $\mathrm{SO}(n)$.*

Proof. Just remark, that $\tilde{\Phi}^{-1}(i_{n-1}) = E_2$ and $\chi(E_2) = (1, 0, 0, \ldots, 0)$. $\square$

5 Iwasawa decomposition of SV_{n-2}

From the evaluation map $T \mapsto Ti_{n-1}$, $T \in \mathrm{SV}_{n-2}$ results the Iwasawa decomposition of SV_{n-2}.

Lemma 5.1. *Every element $T \in \mathrm{SV}_{n-2}$ can uniquely be written in the form*

$$T = nak$$

with $n \in N := \{\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : x \in V_{n-1}\}$, $a \in A := \{\begin{pmatrix} \sqrt{v} & 0 \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix} : v > 0\}$, $k \in \mathrm{Stab}(i_{n-1})$.

Consider for $T \in \mathrm{SV}_{n-2}$ the evaluation map

$$T \mapsto Ti_{n-1} =: u(T) + i_{n-1}v(T) =: w(T).$$

From formulae (2), (3) one finds that for the matrix

$$P_{w(T)} := \begin{pmatrix} 1 & u(T) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{v(T)} & 0 \\ 0 & \frac{1}{\sqrt{v(T)}} \end{pmatrix} \tag{4}$$

one obtains $P_{w(T)}i_{n-1} = w(T) = Ti_{n-1}$. Therefore $T^{-1}P_{w(T)} \in \mathrm{Stab}(i_{n-1})$. Since the decomposition (4) of $P_{w(T)}$ is clearly unique as element of NA the lemma is proven. $\square$

We now generally introduce for $w = u + i_{n-1}v \in \mathbb{H}^n$ as in the foregoing proof the notation

$$P_w := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{v} & 0 \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix}.$$

With this we have

Lemma 5.2. *For every $T \in \mathrm{SV}_{n-2}$ one has with some $k \in \mathrm{Stab}(i_{n-1})$*

$$TP_w = P_{Tw}k.$$

Proof. A calculation using (2), (3) shows that $TP_wi_{n-1} = P_{Tw}i_{n-1}$. This proves the lemma. $\square$

6 Siegel's Theta series

For a quadratic form Q of signature $(1, n)$ and C fixed such that $Q[C] = \begin{pmatrix} 1 & 0 \\ 0 & -E_n \end{pmatrix}$ the Siegel theta series for $z = x + iy \in \mathbb{H}^2$, $w = u + i_{n-1}v \in \mathbb{H}^n$ is defined as the series

$$\Theta_{Q,\mathbb{Z}^{n+1}}(z, w) = \sum_{\mathbf{a} \in \mathbb{Z}^{n+1}} e(xQ[\mathbf{a}])e^{-2\pi y[w,\mathbf{a}]_C}.$$

Here the pairing $[.,.]_D$ is defined for $D \in \mathrm{GL}_{n+1}(\mathbb{R})$ as follows: Let $Q_D^+ = (D\ {}^t\ D)^{-1}$, ψ the exceptional isomorphism, $\psi_D : \mathrm{SV}_{n-2}/\{\pm I\} \to D\,\mathrm{SO}^+(1,n)D^{-1}$, $A \to D\psi(A)D^{-1}$ the conjugate isomorphism and $w \to P_w$ as in section 5. Then for $\mathbf{a} \in \mathbb{R}^{n+1}$

$$\begin{aligned} [w, \mathbf{a}]_D : &= Q_D^+[\psi_D(P_w^{-1})][\mathbf{a}] \\ &= \|\psi(P_w^{-1})][D^{-1}\mathbf{a}]\|^2 \end{aligned}$$

where $\|.\|$ is the usual euclidean norm in $\mathbb{R}^{n+1}$.

We encounter some properties of this pairing:

Lemma 6.1. *Let $D \in \mathrm{GL}_{n+1}(\mathbb{R})$, λ, $\mu \in \mathbb{R}$.*

i) $[\lambda w_1 + \mu w_2, \mathbf{a}]_D = \lambda[w_1, \mathbf{a}]_D + \mu[w_2, \mathbf{a}]_D$,

ii) $[w, \lambda\mathbf{a}] = \lambda^2[w, \mathbf{a}]$.

iii) For $T \in \mathrm{SV}_{n-2}$ we have

$$[Tw, \mathbf{a}]_D = [w, T^{-1}\mathbf{a}]_D$$

where by abuse of notation we write $T^{-1}\mathbf{a}$ instead of $\psi_D(T^{-1}\mathbf{a})$.

iv) $[w, \mathbf{a}]_D = [w, ED^{-1}\mathbf{a}]_E$ *for any $E \in \mathrm{GL}_{n+1}(\mathbb{R})$.*

Proof. The first two properties are clear from the definition.

Because of lemmas 4.1 and 5.2 we find for some $k \in \mathrm{Stab}(i_{n-1})$

$$\begin{aligned} [Tw, \mathbf{a}]_D &= Q_D^+[\psi_D(P_{Tw}^{-1})][\mathbf{a}] \\ &= Q_D^+[\psi_D(kP_w^{-1}T^{-1})][\mathbf{a}] \\ &= Q_D^+[\psi_D(k)][\psi_D(P_w^{-1}][T^{-1}\mathbf{a}] \\ &= Q_D^+[\psi_D(P_w^{-1})][T^{-1}\mathbf{a}] \\ &= [w, T^{-1}\mathbf{a}]_D \end{aligned}$$

which proves (iii). Further

$$\begin{aligned} [w, \mathbf{a}]_D &= Q_D^+[\psi_D(P_w^{-1})][\mathbf{a}] \\ &= Q_E^+[ED^{-1}][\psi_D(P_w^{-1})][\mathbf{a}] \\ &= Q_E^+[\psi_E(P_w^{-1})ED^{-1}][\mathbf{a}] \\ &= Q_E^+[\psi_E(P_w^{-1})][ED^{-1}\mathbf{a}] \\ &= [w, ED^{-1}\mathbf{a}]_E \end{aligned}$$

which proves (iv). $\square$

The following lemmas describe the fundamental transformation properties of Siegel's Theta series. The statement of the next lemma is essentially taken from [3], section 4, and specialized to our situation.

Lemma 6.2. *As before Q is a quadratic form of signature $(1,n)$ Let $N = N(Q) \in \mathbb{N}$ be such that $N(2Q)^{-1}$ is an integer matrix with even diagonal entires. Let further D be defined as in [3], p.81. Namely let $\delta = |\det(2Q)|$, $\delta' = \delta$ for n odd, $\delta' = 2\delta$ for n even and write $\delta' = ef^2$ with e squarefree. Let further*

$$\omega = \begin{cases} 0, & e \equiv 1 (\mathrm{mod}\ 4) \\ 1 & \textit{otherwise} \end{cases}$$

and for n odd let $\sigma = (-1)^{\frac{n-1}{2}}$ and finally put

$$D = \begin{cases} -(-1)^{\omega(e)} 4e & n \text{ even} \\ \sigma 4^{\omega(\sigma e)} \text{odd} \end{cases}$$

Then:

$D|N$, D is a fundamental discriminant or 1 and for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ one has for $\Theta^*(z,w) := y^{\frac{n+1}{4}} \Theta_{Q,\mathbb{Z}^{n+1}}(z,w)$

$$\Theta^*(\gamma z, w) = \left(\frac{c}{d}\right)^{n-1} \left(\frac{-1}{d}\right)^{\frac{n-1}{2}} \left(\frac{|D|}{d}\right) e^{-i\frac{n-1}{2}\arg(cz+d)} \Theta^*(z,w)$$

with $\left(\frac{\cdot}{\cdot}\right)$ the Kronecker symbol. In other words: $\Theta^*(z,w)$ is for any fixed w a real analytic automorphic form for $\Gamma_0(N)$ of weight $\frac{1-n}{2}$ for the multiplier

$$v_{D,n}(\gamma) = \left(\frac{c}{d}\right)^{n-1} \left(\frac{-1}{d}\right)^{\frac{n-1}{2}} \left(\frac{|D|}{d}\right).$$

For a proof see e.g. [3], section 4. □

Fix for Q the matrix C as before. Then we have the trivial transformation property with respect to the second argument:

Lemma 6.3. *For $T \in \mathrm{SO}_0(Q,\mathbb{Z}) = \mathrm{SO}_0(Q) \cap \mathrm{SL}_{n+1}(\mathbb{Z})$ one has*

$$\Theta^*(z, Tw) = \Theta^*(z,w)$$

Proof: Clear. □

7 Theta lift of Poincaré series

Let Q, D, N be as before. For ν a fundamental discriminant we define the real analytic Poincaré series for $\Gamma_0(N)$ of weight $(1-n)/2$ and multiplier $v_{D,n}$ as

$$P_\nu(z,s) = \sum_{T=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_0(N)} \overline{v_{D,n}(T)} e^{\frac{i(n-1)}{2}\arg(cz+d)} y_T^s e(\nu x_T + i|\nu| y_T), \quad \Re s > 1$$

Here it is understood that $z = x + iy$, $Tz = x_T + iy_T$, $s = \sigma + i\tau$, Γ_∞ is the stabilizer of ∞ in $\Gamma_0(N)$, i.e. $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} : k \in \mathbf{Z}\}$.

For $\nu = 0$ this is an Eisenstein series and for $\nu \neq 0$ the above series converges for $\sigma > \frac{3}{4}$, see e.g [9]. From the Fourier expansion in the cusps it can further be seen that these series can for every $z \in \mathbb{H}^2$ be continued as a meromorphic function of s to the whole complex plane.

It is obvious that these series have the same transformation property as $y^{\frac{n+1}{2}}\Theta_{Q,\mathbb{Z}^{n+1}}(z,w)$ with respect to the first variable. For $\nu \neq 0$ they are in $L^2(\Gamma_0(N)\backslash\mathbb{H}^2, \frac{1-n}{2}, v_{D,n})$ which is the obvious Hilbert space with inner product given by

$$\langle f, g\rangle = \int_{\mathcal{F}_N} f(z)\overline{g(z)}d\mu(z)$$

where $\mathcal{F}_N$ denotes a fundamental domain for $\Gamma_0(N)$ and $d\mu(z) = \frac{dxdy}{y^2}$ is hyperbolic volume measure. We shall also use the symbol $\langle\cdot,\cdot\rangle$ in case the above integral exists though one of the functions may not be in L^2.

We then have the following

Proposition 7.1. *If* $\Re s > 3/4$ *then*

$$\begin{aligned} P_\nu^\Theta(w,s) &:= \langle P_\nu(\cdot,s), \Theta^*(\cdot,w)\rangle \\ &= \frac{\Gamma(s+\frac{n-3}{4})}{(2\pi)^{s+\frac{n-3}{4}}} \sum_{\mathbf{a}\in\mathbb{Z}^{n+1}\cap S_\nu(Q)} \frac{1}{([w,\mathbf{a}]_C + |\nu|)^{s+\frac{n-3}{4}}}, \end{aligned}$$

where we put

$$S_\nu(Q) = \{\mathbf{a} \in \mathbb{R}^{n+1} : Q[\mathbf{a}] = \nu\}.$$

Proof. Unfolding the fundamental domain we obtain

$$\begin{aligned} P_\nu^\Theta(w,s) : &= \int_{\mathcal{F}_N} P_\nu(z,s)\overline{\Theta^*(z,w)}d\mu(z) \\ &= \int_0^\infty \int_0^1 \overline{\Theta_{Q,\mathbb{Z}^{n+1}}(z,w)} e^{-2\pi|\nu|y} e(\nu x) y^{s+\frac{n+1}{4}} \frac{dxdy}{y^2} \\ &= \int_0^\infty \sum_{\mathbf{a}\in\mathbb{Z}^{n+1}\cap S_\nu(Q)} e^{-2\pi y([w,\mathbf{a}]_C+|\nu|)} y^{s+\frac{n-3}{4}} \frac{dxdy}{y} \\ &= \frac{\Gamma(s+\frac{n-3}{4})}{(2\pi)^{s+\frac{n-3}{4}}} \sum_{\mathbf{a}\in\mathbb{Z}^{n+1}\cap S_\nu(Q)} \frac{1}{([w,\mathbf{a}]_C + |\nu|)^{s+\frac{n-3}{4}}}. \end{aligned}$$

This proves the proposition. □

Clearly $P_\nu^\Theta(w,s)$ is automorphic with respect to $\Gamma(Q) := \psi_C^{-1}(SO_0(Q,\mathbb{Z}))$.

We have the following:

Lemma 7.1. *The group* $\Gamma(Q)$ *is a finitely generated discrete cofinite subgroup of* $\mathrm{SV}_{n-2}/\{-I, I\}$.

This follows by virtue of

Theorem 7.1 (Borel,Harish-Chandra). *If G is a semisimple algebraic Lie-group defined over Q, then $G_{\mathbb{Z}}$ is finitely generated and $G_{\mathbb{R}}/G_{\mathbb{Z}}$ has finite Haar measure.*

This theorem can be found in [2].

It is well known that $\Gamma(Q)$ is cocompact iff Q is anisotropic. $\psi_C(\Gamma(Q))$ acts on $S_\nu(Q) \cap \mathbb{Z}^{n+1}$. Let $\mathbf{a}_i, i \in I$ be a full set of orbit representatives. Denote by $\mathcal{P}_i$ the preimage under ψ_C of the corresponding isotropy group.

We now define for $w = u + iv \in \mathbb{H}^n$ and $\Re s$ sufficiently large

$$z_i(w, s, \nu) = \sum_{T \in \mathcal{P}_i \backslash \Gamma(Q)} \frac{1}{([T^{-1}w, \mathbf{a}_i]_C + |\nu|)^{s + \frac{n-3}{4}}}.$$

With lemma 6.1 it is clear that

$$P_\nu^\Theta(w, s) = \frac{\Gamma(s + \frac{n-3}{4})}{(2\pi)^{s+\frac{n-3}{4}}} \sum_{i \in I} z_i(w, s, \nu). \tag{5}$$

8 Normal form of hermitean matrices

The aim of this section is to prove the following

Proposition 8.1. *Let $h \in \mathcal{H}_n$ denote a hermitean matix of determinant $\nu \neq 0$. Then there is a $g_h \in SV_{n-2}$ such that*

$$[g_h]h = \begin{pmatrix} \sqrt{\nu} & 0 \\ 0 & \sqrt{\nu} \end{pmatrix}$$

for $\nu > 0$ and

$$[g_h]h = \begin{pmatrix} 0 & \sqrt{|\nu|} \\ \sqrt{|\nu|} & 0 \end{pmatrix}$$

for $\nu < 0$. g_h will explicitly be constructed during the proof.

Proof. Let as before V_{n-1} denote the space of vectors in the Clifford algebra $\mathcal{C}_{n-2}$. We write $(.,.)$ and $\|.\|$ for the usual euclidean scalar product and norm, respectively, in V_{n-1} when considered as real $n-1$-dimensional vector space.

Now let α be a nonzero element of V_{n-1} then a short computation shows that the reflection in V_{n-1}

$$s_\alpha : x \mapsto 2\frac{(x, \alpha)}{\|\alpha\|^2}\alpha \tag{6}$$

can be written as

$$s_\alpha(x) = -\frac{\alpha \bar{x} \alpha}{\alpha \bar{\alpha}} = -\frac{\alpha \bar{x} \alpha}{\|\alpha\|^2}. \tag{7}$$

Of course, for $\beta \in V_{n-1}\backslash\{0\}$ we have

$$s_{\beta-\|\beta\|}(\beta) = \|\beta\| \tag{8}$$

where we consider $\|\beta\|$ as element of V_{n-1}.

Let now

$$h = \begin{pmatrix} \alpha & \bar{b} \\ b & \gamma \end{pmatrix}$$

be a hermitean matrix over $\mathcal{C}_{n-2}$ with $b \notin \mathbb{R}$. Then

$$[\begin{pmatrix} b-\|b\| & 0 \\ 0 & \bar{b}-\|b\| \end{pmatrix}]h = \|b-\|b\|\|^2 \begin{pmatrix} \alpha & -\|b\| \\ -\|b\| & \gamma \end{pmatrix}. \tag{9}$$

Indeed, if we put $\delta := b - \|b\|$ we obtain

$$[\begin{pmatrix} \delta & 0 \\ 0 & \bar{\delta} \end{pmatrix}]h = \begin{pmatrix} \bar{\delta}\delta\alpha & \delta\bar{b}\delta \\ \bar{\delta}b\bar{\delta} & \bar{\delta}\delta\gamma \end{pmatrix}$$

This proves (9) because of (7), (8).

If T is a real symmetric 2 by 2 matrix of determinant $\nu \in \mathbb{Z}\backslash\{0\}$ with corresponding quadratic form $T(x,y) = \alpha x^2 + \beta xy + \gamma y^2 = \alpha(x+\eta_1 y)(x+\eta_2 y))$. Here η_i are real iff $\nu < 0$ and in this case we assume $\eta_1 < \eta_2$. In case $\nu > 0$ we assume $\Im\eta_1 > 0$. If we put

$$g_T = \frac{1}{\sqrt{\eta_2 - \eta_1}} \begin{pmatrix} \eta_2 & 1 \\ \eta_1 & 1 \end{pmatrix}$$

for $\nu < 0$ and

$$g_T = \frac{1}{\sqrt{\Im\eta_1}} \begin{pmatrix} 1 & 0 \\ \Re\eta_1 & \Im\eta_1 \end{pmatrix}$$

for $\nu > 0$ then it is easily calculated that

$$[g_T]T = \begin{pmatrix} \sqrt{\nu} & 0 \\ 0 & \sqrt{\nu} \end{pmatrix}$$

for $\nu > 0$ and

$$[g_T]T = \begin{pmatrix} 0 & \sqrt{|\nu|} \\ \sqrt{|\nu|} & 0 \end{pmatrix}$$

for $\nu < 0$. If we choose $T = \begin{pmatrix} \alpha & -\|b\| \\ -\|b\| & \gamma \end{pmatrix}$, then the matrix $g_h = g_T \frac{1}{\|\delta\|} \begin{pmatrix} \delta & 0 \\ 0 & \bar{\delta} \end{pmatrix}$ with $\delta = b - \|b\|$ obviously is the right one to make the statement of the proposition true. $\square$

9 Proof of Theorem 1.1

We want to prove the main result of this paper. For this we first fix the special quadratic form

$$Q_0 = \begin{pmatrix} & \frac{1}{2} & & & & \\ \frac{1}{2} & & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & & 1 \end{pmatrix}$$

of signature $(1, n)$. Obviously

$$Q_0[(a, c, b_0, \dots, b_{n-2}] = \det(\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix})$$

for $\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}) \in \mathcal{H}_n$. Further choose

$$C_0 = \begin{pmatrix} 1 & -1 & & & & \\ 1 & 1 & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & & 1 \end{pmatrix}$$

then ${}^tC_0 Q_0 C_0 = \begin{pmatrix} 1 & 0 \\ 0 & -E_n \end{pmatrix}$ and

$$Q_0^+ := (C_0 t C_0)^{-1} = \begin{pmatrix} \frac{1}{2} & & & & & \\ & \frac{1}{2} & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & & 1 \end{pmatrix}.$$

We write for short $[w, \mathbf{a}]_0 := [w, \mathbf{a}]_{C_0}$. With this we can prove

Proposition 9.1. *Assume $\nu \in \mathbb{Z}\backslash\{0\}$. Let $\mathbf{h} = (a, c, b_0, b_1, \dots, b_{n-2}) \in S_\nu(Q_0) \cap \mathbb{Z}^{n+1}$ and $h = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$ the corresponding hermitean matrix from H_n with non vanishing determinant ν. For $\nu > 0$ set $\eta = \Phi(h)$ where Φ is the map defined in section 4 while for $\nu < 0$ we define $\mathbf{g}_h$ to be the geodesic hyperplane given by*

$$\{w = u + i_{n-1} v \in \mathbb{H}^n : a + \bar{b} u + b \bar{u} + c w \bar{w} = 0\}.$$

Then we obtain for $w \in \mathbb{H}^n$ with $\delta(.,.)$ the hyperbolic distance function from the introduction *i) if* $\nu > 0$

$$[w, \mathbf{h}]_0 + \nu = 2\nu\delta(w, \eta)^2,$$

ii) if $\nu < 0$, $\Re s > 0$

$$\mathrm{vol}(S^{n-3})2^{3s-2}|\nu|^s\Gamma(\frac{n-2}{2})\frac{\Gamma(s+\frac{2-n}{2})\Gamma(s)}{\Gamma(2s)}\left([w, \mathbf{h}]_0 + |\nu|\right)^{-s} = \int_{\mathbf{g}_h} \delta(w, \omega)^{-2s} d\sigma(\omega),$$

with $d\sigma$ *denoting hyperbolic measure on the hyperplane* $\mathbf{g}_h$*.*

Proof. (i): Let g_h be as in section 8,i.e. $[g_h]h = \sqrt{\nu}E_2 =: h_\nu$. Then the corresponding element in $S_\nu(Q_0)$ is $\mathbf{h}_\nu = \sqrt{\nu}(1, 1, 0, \ldots, 0)$ and we find

$$\begin{aligned}
[w, \mathbf{h}_\nu]_0 &= \nu Q_0^+[\psi_{C_0}(P_w^{-1})][^t(1, 1, 0, \ldots, 0)] \\
&= \nu\|\psi(P_w^{-1})(C_0^{-1t}(1, 1, 0, \ldots, 0)\|^2 \\
&= \nu\|\chi[P_w^{-1}]\chi^{-1}(1, 0, 0 \ldots, 0)\|^2 \\
&= \nu\|\chi\left(P_w^{-1t}P_w^{-1}\right)\|^2 \\
&= \nu\|\chi\left(\begin{pmatrix} \frac{1+u\bar{u}}{v} & -u \\ -\bar{u} & v \end{pmatrix}\right)\|^2 \\
&= \nu\frac{(u\bar{u})^2 + v^4 + 1 + (2 + 2v^2)u\bar{u}}{2v^2} \\
&= \nu\frac{(u\bar{u} + v^2 + 1)^2}{2v^2} - \nu \\
&= 2\nu\delta(w, i_{n-1})^2 - \nu.
\end{aligned}$$

The proof of (i) follows now from lemma 6.1(iii) together with the fact that the map $\tilde{\Phi}$ is SV_{n-2}–equivariant.

For the proof of (ii) we observe that g_h maps the geodesic hyperplane $\mathbf{g}_h$ onto the hyperplane $H_0 = \{w \in \mathbb{H}^n \ : \ u_0 = 0\}$ and compute the integral

$$\begin{aligned}
&\int_{\mathbf{g}_h} \delta(w, \omega)^{-s} d\sigma(\omega) \\
&= \int_{H_0} \delta(w, g_h^{-1}\omega)^{-s} d\sigma(g_h^{-1}\omega) \\
&= \int_{H_0} \delta(g_h w, \omega)^{-s} d\sigma(\omega)
\end{aligned}$$

where we used the SV_{n-2} invariance of the hyperbolic measure $d\sigma$. Write $g_h w := w_h :=$

$u_h + i_{n-1} v_h$ then this becomes

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_0^{i\infty} \left(\frac{u_{0_h}^2 + (u_{1_h} - \mu_1)^2 + \ldots + (u_{(n-2)_h} - \mu_{n-2})^2 + v_h^2 + \lambda^2}{2 v_h \lambda} \right)^{-s} \frac{d\lambda}{\lambda^{n-1}} \prod_k d\mu_k \\
&= (2v)^s \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(u_{0_h}^2 + (u_{1_h} - \mu_1)^2 + \ldots + (u_{(n-2)_h} - \mu_{n-2})^2 + v_h^2 \right)^{-\frac{s+n-2}{2}} \prod_k d\mu_k \\
&\quad \times \int_0^{i\infty} \left(\lambda' + \frac{1}{\lambda'} \right)^{-s} \frac{d\lambda'}{\lambda'^{n-1}} \\
&= \frac{\Gamma(\frac{s-n+2}{2}) \Gamma(\frac{s+n-2}{2})}{2\Gamma(s)} \frac{(2v)^s}{(u_{0_h}^2 + v_h^2)^{\frac{s-n+2}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(1 + \frac{\mu_1^2 + \ldots + \mu_{n-2}^2}{u_{0_h}^2 + v_h^2} \right)^{-\frac{s+n-2}{2}} \prod_k d\mu_k \\
&= \frac{\Gamma(\frac{s-n+2}{2}) \Gamma(\frac{s+n-2}{2})}{2\Gamma(s)} \frac{(2v)^s}{(u_{0_h}^2 + v_h^2)^{\frac{s}{2}}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(1 + \mu_1^2 + \ldots + \mu_{n-2}^2 \right)^{-\frac{s+n-2}{2}} \prod_k d\mu_k \\
&= \frac{\Gamma(\frac{s-n+2}{2}) \Gamma(\frac{s+n-2}{2})}{2\Gamma(s)} \frac{(2v)^s}{(u_{0_h}^2 + v_h^2)^{\frac{s}{2}}} \mathrm{vol}(S^{n-3}) \int_0^{\infty} \frac{r^{n-3}}{(1+r^2)^{\frac{s+n-2}{2}}} dr \\
&= \frac{\Gamma(\frac{s-n+2}{2}) \Gamma(\frac{n-2}{2}) \Gamma(\frac{s}{2})}{4\Gamma(s)} \frac{(2v)^s}{(u_{0_h}^2 + v_h^2)^{\frac{s}{2}}} \mathrm{vol}(S^{n-3})
\end{aligned}$$

with $\mathrm{vol}(S^{n-3})$ the euclidean volume of the $n-3$–dimensional unit sphere.

Here we tacitly assumed $n > 3$. But if we use the well-known formula $\mathrm{vol}(S^{n-3}\Gamma(\frac{n-2}{2}) = 2\pi^{\frac{n}{2}-1}$ the reader may convince himself that the resulting formula stays correct for $n = 2, 3$ as well. On the other hand we have because of proposition 8.1

$$\begin{aligned}
[g_h w, \mathbf{h}]_0 + |\nu| &= [w, \psi_{C_0}(g_h^{-1}\mathbf{h})]_0 + |\nu| \\
&= [w, (0, 0, \sqrt{\nu}, 0, \ldots, 0)]_0 + |\nu|.
\end{aligned}$$

An elementary calculation shows that the latter equals $2|\nu| \|(w_h)_0\|^2 / v_h^2$ which completes the proof of the proposition. $\square$

Obviously we have $Q_0 = Q[CC_0^{-1}]$. We put $A := CC_0^{-1}$ then for $\mathbf{h} \in S_\nu(Q)$ we have $A^{-1}\mathbf{h} \in S_\nu(Q_0)$. We now denote by h_A the hermitean matrix corresponding to $A^{-1}\mathbf{h}$ as in the proof of the foregoing proposition, η_{h_A}, g_{h_A}, $\mathbf{g}_{h_A}$ respectively.

If as before $\mathcal{P}_i$ corresponds via ψ_C to the isotropy subgroup of $\mathbf{a}_i$ then it is well known that for $\nu > 0$ this is a finite group of order f_i, say, and for $\nu < 0$ $\mathcal{P}_i$ leaves invariant the geodesic hyperplane $\mathbf{g}_{a_{iA}}$.

We now put

$$\eta_i := \eta_{a_{iA}}, \quad \mathbf{g}_i := \mathbf{g}_{a_{iA}}.$$

The proof of the theorem for $\nu > 0$ is now clear.

For $\nu < 0$ some additional remarks are in order: The orbit set under the action of $\mathcal{P}_i$ on $\mathbf{g}_i$ is denoted by $\mathcal{P}_i \backslash \mathbf{g}_i$ and can be chosen to be connected and measurable.

With this we obtain

$$\begin{aligned}
&\int_{\mathcal{P}_i\backslash \mathrm{g}_i} H(\omega, w, 2s-n+2)d\sigma(\omega)\\
&= \int_{\mathcal{P}_i\backslash \mathrm{g}_i} \sum_{T\in\mathcal{P}_i\backslash\Gamma(Q)} \sum_{P\in\mathcal{P}_i} \left(\delta(PTw,\omega)\right)^{-2s} d\sigma(\omega)\\
&= \int_{\mathcal{P}_i\backslash \mathrm{g}_i} \sum_{T\in\mathcal{P}_i\backslash\Gamma(Q)} \sum_{P\in\mathcal{P}_i} \left(\delta(Tw,P^{-1}\omega)\right)^{-2s} d\sigma(\omega)\\
&= \int_{\mathrm{g}_{a_i}} \sum_{T\in\mathcal{P}_i\backslash\Gamma(Q)} \left(\delta(Tw,\omega)\right)^{-2s} d\sigma(\omega)\\
&= \mathrm{vol}(S^{n-3})\Gamma(\frac{n-2}{2})2^{3s-2}|\nu|^s\frac{\Gamma(s+\frac{2-n}{2})\Gamma(s)}{\Gamma(2s)} z_i(w, s-\frac{n-3}{4},\nu).
\end{aligned}$$

For the last equality we used the above proposition and lemma 6.1(iv). The proof of the theorem can now easily be completed. □

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