Generalized Neuwirth Groups and Seifert Fibered Manifolds *

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Abstract

The topological properties of the generalized Neuwirth groups

$$\Gamma_n^k = \langle x_1, \ldots, x_n \mid x_i x_{i+1} \ldots x_{i+n-2} = x_{i+n-1}^k \quad i = 1, \ldots n \rangle$$

are discussed. For example, we shall demonstrate that the group Γ_n^k is the fundamental group of the Seifert fibered space

$$\Sigma_n^k \;=\; (0 \text{ o } 0 \mid -1;\ \underbrace{(k+1,1),\ (k+1,1),\ \ldots,(k+1,1)}_{n \text{ times}}).$$

Moreover we discuss some other invariants and algebraic properties of the above groups.
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1 Introduction

Let $\mathbb{F}_n$ be the free group on free generators $x_1, x_2, \ldots, x_n$. Let $\theta : \mathbb{F}_n \to \mathbb{F}_n$ be the authomorphism for which $\theta(x_i) = x_{i+1},\ i = 1, 2, \ldots, n-1$; and $\theta(x_n) = x_1$. For any word $w \in \mathbb{F}_n$, we define (cf. [9]) a group $G_n(w) = \mathbb{F}_n/R$ where R is the normal closure in $\mathbb{F}_n$ of the set $\{w, \theta(w), \ldots, \theta^{n-1}(w)\}$. A group G is said to have a *cyclic presentation* if $G = G_n(w)$ for some n and

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$w \in \mathbb{F}_n$. Among the well-studied examples of cyclically presented groups are the *Fibonacci groups* $F(r,n) = G_n(x_1x_2\cdots x_rx_{r+1}^{-1}), (n \geq 3, r \geq 2)$ (cf. [9]).

Our aim is to study a class of cyclically presented groups which are the fundamental groups of 3-manifolds. We mention (cf.[18]), that it is impossible to find an algorithm for determinig whether arbitrary finite presentations present a fundamental group of a 3-manifold.

In [13] Neuwrith considers the groups $\Gamma_n = F(n-1,n)$, $n \geq 3$ as fundamental groups $\pi_1(\overline{M}_n)$ of closed orientable 3-manifolds. It was proven in [4] that the *Neuwirth manifold* $\overline{M}_n$ is PL–homeomorphic to the Seifert fibered space

$$\Sigma_n \;=\; (0\ \mathrm{o}\ 0 \mid -1;\ \underbrace{(2,1),\ (2,1),\ \ldots,(2,1)}_{n\ \text{times}}).$$

In the present paper we introduce a family of cyclically presented groups $\Gamma_n^k = G_n(x_1x_2\cdots x_{n-1}x_n^{-k})$ with $n \geq 3$ and $k \geq 1$. We shall call these groups *generalized Neuwirth groups.* We show (Theorem 3.1) that Γ_n^k are the fundamental groups of closed orientable 3-manifolds M_n^k. Moreover we shall prove (Theorem 3.2) that M_n^k is homeomorphic to the Seifert fibred space Σ_n^k, where

$$\Sigma_n^k \;=\; (0\ \mathrm{o}\ 0 \mid -1;\ \underbrace{(k+1,1),\ (k+1,1),\ \ldots,(k+1,1)}_{n\ \text{times}})$$

in respect to notations from [11]. As a corollary (Corollary 3.1) we show that the groups Γ_n^k are automatic for $(n,k) \neq (4,1)$ and we give a formula (Corollary 3.2) for the Casson–Walker–Lescop invariant of the manifolds Σ_n^k.

2 An example

Let Γ_3^2 be the generalized Neuwirth group. From a definition:

$$\Gamma_3^2 \;=\; \langle x_1, x_2, x_3 \mid x_1x_2 = x_3^2,\ x_2x_3 = x_1^2,\ x_3x_1 = x_2^2\rangle.$$

We start from the following simple observation.

Proposition 2.1. *The group Γ_3^2 is the fundamental group of a compact orientable flat 3-manifold.*

Proof: Let P_3^2 be the unit cube in the Euclidean 3-space $\mathbb{E}^3$ with notations of vertices according to Fig. 1.

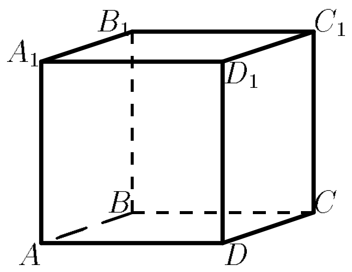

Fig. 1. The cube P_3^2.

Consider the following orientation-preserving Euclidean isometries which identify pairs of faces of the polyhedron P_3^2:

$$\begin{array}{llll} a: & A_1ABB_1 & \to & B_1C_1CB, \\ b: & A_1B_1C_1D_1 & \to & D_1DCC_1, \\ c: & A_1D_1DA & \to & ABCD. \end{array}$$

Edges of P_3^2 are divided in three classes of equivalent:

$$\begin{array}{llllllllll} x_1: & A_1A & \xrightarrow{a} & B_1C_1 & \xrightarrow{b} & DC & \xrightarrow{c^{-1}} & AD & \xrightarrow{c^{-1}} & A_1A, \\ x_2: & A_1B_1 & \xrightarrow{b} & D_1D & \xrightarrow{c} & BC & \xrightarrow{a^{-1}} & B_1B & \xrightarrow{a^{-1}} & A_1B_1, \\ x_3: & A_1D_1 & \xrightarrow{c} & AB & \xrightarrow{a} & C_1C & \xrightarrow{b^{-1}} & D_1C_1 & \xrightarrow{b^{-1}} & A_1D_1. \end{array}$$

Therefore, by Poincare theorem [19] we get that the group generated by a, b and c has the polyhedron P_3^2 as its fundamental domain, and has the following presentation:

$$\langle a, b, c \mid ab = c^2,\ bc = a^2,\ ca = b^2 \rangle,$$

that is isomorphic to Γ_3^2. Since all dihedral angles of P_3^2 are equal to $\pi/2$, and all cycles of edges are of length four, we conclude, that the quotient-space $\mathbb{E}^3/\Gamma_3^2$ is a flat 3-manifold. $\square$

Let us denote $M_3^2 = \mathbb{E}^3/\Gamma_3^2$. We recall [21], that there are only six compact orientable flat 3-manifolds. Computing the first homology group $H_1(M_3^2) = \mathbb{Z}_3 \times \mathbb{Z}$, we see that M_3^2 is the flat manifolds $\mathcal{G}_3$ in notations from [21]. According to [14, p. 138], $M_3^2 = (0 \text{ o } 0 \mid -1;\ (3,1),(3,1),(3,1))$ and can be obtained as the T^2 bundle over S^1 with matrix of the characteristic map $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ of order 3.

Proposition 2.2. *Let $Out(\Gamma_3^2)$ be the outer automorphism group of Γ_3^2, then $Out(\Gamma_3^2)$ is a dihedral group of order twelve.*

Proof: Since Γ_3^2 is the fundamental group of the flat 3-manifold $\mathcal{G}_3$ we have the short exact sequence

$$0 \longrightarrow \mathbb{Z}^3 \longrightarrow \Gamma_3^2 \longrightarrow \mathbb{Z}_3 \longrightarrow 0.$$

From the definition (cf. [21]) the action of $\mathbb{Z}_3$ on $\mathbb{Z}^3$ is given by a matrix

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let N denotes the normalizer in $GL(3, \mathbb{Z})$ of the above matrix. Then N acts in a natural way on $H^2(\mathbb{Z}_3, \mathbb{Z}^3)$. Let $\alpha \in H^2(\mathbb{Z}_3, \mathbb{Z}^3)$ denotes the cohomology class giving rise to the above extension, and let N_α denotes its stabilizer in N. Then $\mathbb{Z}_3$ is a normal subgroup of N_α, and in virtue of Theorem V.1.1 from [5, p. 172] we have the short exact sequence

$$0 \longrightarrow H^1(\mathbb{Z}_3, \mathbb{Z}^3) \longrightarrow Out(\Gamma_3^2) \longrightarrow N_\alpha/\mathbb{Z}_3 \longrightarrow 0.$$

It is not difficult to see that $H^1(\mathbb{Z}_3, \mathbb{Z}^3) \simeq \mathbb{Z}_3$ and $N_\alpha/\mathbb{Z}_3 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (cf. [3]). In order to get our result, it is enough to remark that the group $Out(\Gamma_3^2)$ is not abelian.

Indeed, let us consider two isometries of the cube P_3^2: the rotation ρ of order 3 about the line passing through vertices A_1 and C, and the rotation σ of order 2 about the line passing through midpoints of edges AB and C_1D_1. These symmetries induce automorphisms of the group Γ_3^2 such that $\rho : a \to b, b \to c, c \to a$ and $\sigma : a \to c^{-1}, b \to b^{-1}, c \to a^{-1}$. It is easy to check that automorphisms ρ, σ don't commute and are not inner. $\square$

Remark, that under the action of the group Γ_3^2 we get the tesselation of $\mathbb{E}^3$ by the right-angled unit cube P_3^2. Applying results of Grayson on the growth function of the tesselation of the euclidean space by a right-angled cube [7], we get

Proposition 2.3. *The growth function for the group Γ_3^2 in respect to the set of generators $S = \{x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}\}$ is*

$$f_S(t) = \left(\frac{1+t}{1-t}\right)^3 = 1 + 6t + 18t^2 + 38t^3 + 66t^4 + 102t^5 + \dots.$$

3 The properties of the groups Γ_n^k

It was shown in [13] that groups $\Gamma_n^1 = \Gamma_n$ are fundamental groups of 3-manifolds. In this section, we will demonstrate that this same is true for groups Γ_n^k, $k \geq 2$.

Let us recall (cf. [17]) that a *squashable complex* C on the 2-sphere $S^3 = \partial B^3$ is a combinatorial complex together with a grouping of its 2-cells in oppositely oriented identifiable pairs. The identification squashes the ball-sphere pair (B^3, S^2) into a combinatorial complex pair $(M, K) = (B^3/\sim, S^2/\sim)$. The resulting quotient map $q = q(C) : (B^3, S^3) \to (M, K)$ is called a *squashing map* associated with C. The quotient complex M is in general an orientable closed connected pseudo-manifold in the sense of [16]. It was shown in [16] that M is a closed orientable 3-manifold if and only if the Euler characteristic of M vanishes. Furthermore, the embedded 2-complex is a *spine* of M, i.e. M minus the open 3-cell $q(B^3)$ collapses onto K. If K has a single 0-cell, then K is the canonical 2-complex associated with a finite presentation

$$\langle x_1, x_2, \ \ldots \ , x_n \mid r_1, r_2, \ \ldots \ r_n \rangle$$

of the fundamental group $\pi_1(K) \cong \pi_1(M)$. The 1–skeleton K^1 of K is a bouquet of n oriented circles, also denotes by $x_1, x_1, \ldots, x_n$. The 2-cells c_j of K corresponds bijectively to the relators r_j which determine loops in K as the corresponding attaching maps $h_j : \partial B^2 \to K^1$.

Consider the complex P_n^k with $n(n+k-1)$ edges and $2n$ faces each of which is a $(n+k-1)$–gon. The combinatorial structure of P_n^k is clear from Fig. 2, where the complex P_5^3 for the group Γ_5^3 is pictured. Let A_i and A_i', $i = 1, \ldots, n$, be faces with edges numerated by

$$\{i, i+1, \ldots, i+n-2, \underbrace{i+n-1, \ldots, i+n-1}_{k}\},$$

with subscripts reduced modulo n, such that the corresponding relation r_i is $x_i x_{i+1} \cdots x_{i+n-2} = x_{i+n-1}^k$. As we see, the enumerations of edges of faces of A_i and A_i' induce opposite orientations of faces.

Denote by M_n^k the complex obtained under the squashing map associated with P_n^k. As we see, the pseudo-manifold M_n^k has one 0-cell, n 1-cells, n 2-cells and one 3-cell. Hence its Euler characteristic vanishes and M_n^k is a manifold.

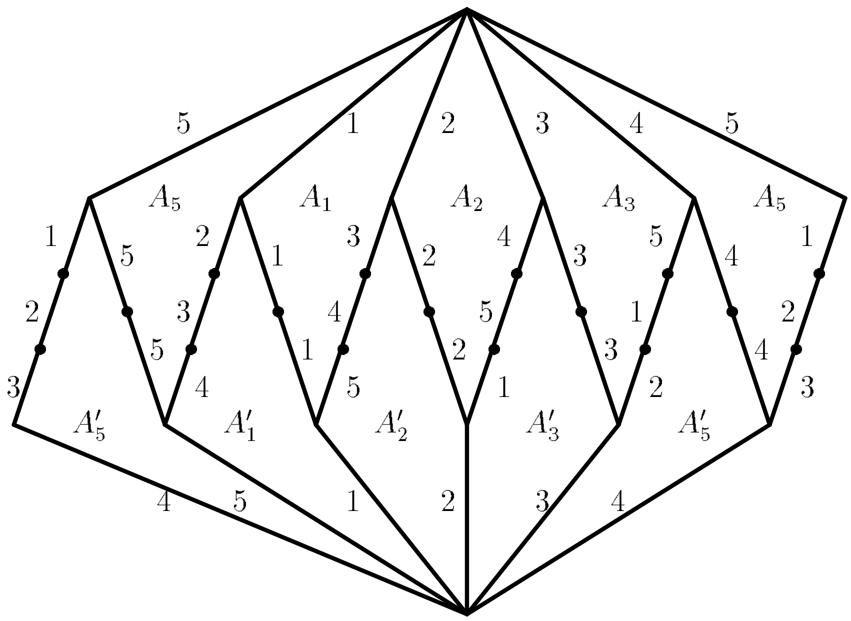

Fig. 2. The polyhedron P_5^3.

Thus, we get the following statement.

Theorem 3.1. *For $n \geq 3$ and $k \geq 1$ the group Γ_n^k is the fundamental group of the orientable closed 3-manifold M_n^k.* $\square$

We recall that a Seifert fibered 3-manifold is completely determined by a system of invariants (cf. [4, p. 180] and [11, ch. 4])

$$(\epsilon\ g\ \epsilon\prime \mid b;\ (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_r, \beta_r))$$

where

(1) g is the genus of the closed connected surface (called the orbit-manifold) obtained from the Seifert manifold by identifying each fiber to a point,

(2) $\epsilon = \epsilon\prime = 0$ if the Seifert manifold and its orbit-manifold are orientable,

(3) $b = -(e_0 + \Sigma_{i=1}^r \frac{\beta_i}{\alpha_i}) \in \mathbb{Z}$, where e_0 is the rational Euler number of the Seifert fibration,

(4) (α_j, β_j) are pairs of coprime integers ($0 < \beta_j < \alpha_j$) such that $\beta_j/\alpha_j \in (\mathbb{Q}/\mathbb{Z})^*$ (here $\mathbb{Q}$ is the field of rational numbers and $(\mathbb{Q}/\mathbb{Z})^*$ is the group $\mathbb{Q}/\mathbb{Z}$ minus zero) characterize the holonomy of the exceptional fibers.

The fundamental group of a Seifert fiber space with the above invariants is well known (cf. [11], [14, p. 91]). A standard presentation for its in oriented case is following,

$$\langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_r, h \mid [a_i, h] = [b_i, h] = 1 \quad (1 \le i \le g),$$
$$[q_i, h] = 1 \quad (1 \le i \le r), \quad q_i^{\alpha_i} h^{\beta_i} = 1 \quad (1 \le i \le r),$$
$$q_1 q_2 \cdots q_r [a_1, b_1] \cdots [a_g, b_g] = h^b \rangle.$$

Let us denote by Σ_n^k the Seifert 3-manifold with invariants

$$(0 \text{ o } 0 \mid -1; \underbrace{(k+1, 1),\ (k+1, 1),\ \ldots, (k+1, 1)}_{n \text{ times}}).$$

From above we have

$$\pi_1(\Sigma_n^k) \ = \ \langle q_1, q_2, \ \ldots \ , q_n, h \mid q_1 q_2 \cdots q_n = h^{-1}$$
$$[q_i, h] = 1, \quad q_i^{k+1} h = 1, \quad i = 1, \ldots, n \rangle. \quad (\star)$$

Since $\Gamma_n^k = G_n(x_1 x_2 \ldots x_{n-1} x_n^{-k})$, we see

$$\Gamma_n^k \ = \ \langle x_1, x_2, \ldots, x_n \mid$$
$$x_1 x_2 \cdots x_{n-1} = x_n^k,\ x_2 x_3 \cdots x_n = x_1^k,\ \ldots\ , x_n x_1 \ldots x_{n-2} = x_{n-1}^k \rangle. \quad (\star\star)$$

We have.

Lemma 3.1. *The group Γ_n^k is isomorphic to the group $\pi(\Sigma_n^k)$.*

Proof: To obtain the presentation $(\star\star)$ from $(\star)$ we identify $x_i = q_i$, $i = 1, \ldots, n$. From $q_i^{k+1} = h^{-1}$ we have $h = x_1^{-(k+1)} = \ldots = x_n^{-(k+1)}$. From the relation $q_1 \cdots q_n = h^{-1}$ we get $x_1 \cdots x_n = x_n^{k+1}$, so $x_1 \cdots x_{n-1} = x_n^k$. Thus, $x_2 \cdots x_n = x_1^{-1} x_n^{k+1}$, and using $x_n^{k+1} = x_1^{k+1}$ we get $x_2 \cdots x_n = x_1^k$. By the same way we will get all relations from $(\star\star)$.

Conversely, to obtain the presentation $(\star)$ from $(\star\star)$ we identify $q_i = x_i$, $i = 1, \ldots, n$, and $h = x_n^{-(k+1)}$. Therefore, $q_1 \cdots q_n = h^{-1}$. From the relations $x_2, \cdots x_{n-1} = x_1^{-1} x_n^k$ and $x_2, \cdots x_{n-1} = x_1^k x_n^{-1}$ we have $x_1^{k+1} = x_n^{k+1} = h^{-1}$. Analogously, we will get $h = x_1^{-(k+1)} = \ldots = x_n^{-(k+1)}$. Therefore, $[q_i, h] = 1$ and $q_i^{k+1} h = 1$. $\square$

From the standard calculations one can find the first integral homology group of M_n^k. Here $\mathbb{Z}_0 = \mathbb{Z}$.

Lemma 3.2. *The group $H_1(M_n^k, \mathbb{Z})$ is generated by $(n-1)$ elements and is isomorphic to*

$$\underbrace{\mathbb{Z}_{k+1} \oplus \mathbb{Z}_{k+1} \oplus \ldots \oplus \mathbb{Z}_{k+1}}_{n-2} \oplus \mathbb{Z}_{(k+1)\ |n-k-1|}.$$

The next theorem generalizes the Theorem 4 of [4].

Theorem 3.2. *The manifold M_n^k, $n \geq 3$, $k \geq 1$, is homeomorphic to the Seifert fibered 3-manifold Σ_n^k.*

Proof: We recall that if fundamental groups of two large Seifert manifolds are isomorphic, then the manifolds are homeomorphic [14, p.97]. From the definition of the Seifert manifold Σ_n^k we get that it is *large* (see [14, p.91]) for $n \geq 3$ and $k \geq 2$ with one exceptional case $(n,k) = (3,1)$ for which the proof is given in [4]. By Lemma 3.1, groups $\pi_1(\Sigma_n^k)$ and $\pi(M_n^k) \cong \Gamma_n^k$ are isomorphic. To prove the theorem we will demonstrate that the manifold M_n^k is a large Seifert manifold. By Lemma 3.2 for its Heegaard genus we have $h(M_n^k) \geq n-1$. Furthermore, (cf. [1]) the Heegaard genus Σ_n^k is exactly $n-1$. According to [14, p.92] the group Γ_n^k is a non-trivial free product with amalgamation and the subgroup generated by the element $h = x_1^{-(k+1)} = \ldots = x_n^{-(k+1)}$ is the unique maximal cyclic normal subgroup of Γ_n^k. Further $\langle h \rangle$ is infinite and is contained into the center of Γ_n^k, whence Γ_n^k has a non-trivial center.

Since M_n^k is prime and different from $S^1 \times S^2$ (use $h(M_n^k) > 3$), the manifold M_n^k is irreducible (i.e., any embedded 2-sphere in M_n^k bounds a 3-cell). By Theorem 6.2 of [2, p.77], it follows that M_n^k is homeomorphic to a Seifert fibered manifold with orientable orbit-manifold. Since M_n^k is irreducible and Γ_n^k is infinite, the manifold M_n^k is aspherical, i.e., $\pi_j(M_n^k) = 0$ for $j \geq 2$ (see Lemma 1.1.5 of [20]). Thus M_n^k is a $K(\Gamma_n^k, 1)$–space with infinite torsion-free fundamental group, whence M_n^k is large [14, p. 91]. Therefore large Seifert manifolds Σ_n^k and M_n^k are homeomorphic. $\square$

Corollary 3.1. *The groups Γ_n^k are automatic for $(n,k) \neq (4,1)$.*

Proof: Recall [15], that the geometry on a Seifert fibred manifold is completely determined by the Euler characteristic of the base orbifold and the

Euler number of the fibration. Calculating these values for manifolds Σ_n^k, one can see that Σ_3^1 is a spherical manifold, Σ_3^2 is an euclidean manifold, Σ_4^1 is a nilpotent manifold, for $k > 2$ manifolds Σ_{k+1}^k are $\mathbb{H}^2 \times \mathbb{R}$-manifolds, and all others are $\widetilde{SL}_2$-manifolds.

Hence it is enough to apply Theorem 12.2.5 from [6, p. 296]. $\square$

At the end of this section we shall calculate, due to results of Lescop [10], the Casson – Walker – Lescop λ invariant of above considered manifolds.

Corollary 3.2. *Denote by λ_n^k the Casson – Walker – Lescop invariant of the manifold M_n^k. Then the following formula hold:*

$$\lambda_n^{p-1} = \begin{cases} \frac{p^{n-2}}{24}\left[n^2p^2 - n(3p^3 + p^2 + 2p - 1) + 2p^2(p^2+2)\right] & \text{if } p < n, \\ \frac{n^{n-1}}{24}(n^2 - 2n - 1) & \text{if } p = n, \\ -\frac{p^{n-2}}{24}\left[n^2p^2 - n(3p^3 - 5p^2 + 2p - 1) + 2p^2(p^2 - 3p + 2)\right] & \text{if } p > n. \end{cases}$$

Proof: By [10, p. 97] the invariant λ of a Seifert fibered space

$$(0 \text{ o } 0 \mid b;\ (\alpha_i, \beta_i), i = 1, \ldots, n)$$

is given by the following formula:

$$\lambda \ = \ \left(\frac{s(e)}{24}\left(2 - n + \sum_{i=1}^{n}\frac{1}{\alpha_i^2}\right) \ + \ \frac{|e|\, e}{24} \ - \ \frac{e}{8} \ - \ \frac{|e|}{2}\sum_{i=1}^{n} s(\beta_i, \alpha_i)\right)\ \prod_{i=1}^{n} \alpha_i,$$

where we use the Euler number of the fibration $e = b + \sum_{i=1}^{n}\frac{\beta_i}{\alpha_i}$, and the Dedekind sum

$$s(q,p) = \sum_{i=1}^{|p|}\left(\left(\frac{i}{p}\right)\right)\left(\left(\frac{qi}{p}\right)\right), \qquad \text{with} \quad ((x)) = \begin{cases} 0 & \text{if } x \in \mathbb{Z}, \\ x - [x] - \frac{1}{2} & \text{otherwise,} \end{cases}$$

where $[x]$ denotes the integer part of x, and

$$s(e) = \begin{cases} \text{sign } (e) & \text{if } e \neq 0, \\ -1 & \text{if } e = 0. \end{cases}$$

For our manifolds $M_n^k = \Sigma_n^k$ we have: $b = -1$, $\alpha_i = k+1$, $\beta_i = 1$, $e = -1 + \frac{n}{k+1}$, and

$$s(1, k+1) \ = \ \sum_{i=1}^{k}\left(\frac{i}{k+1} - \frac{1}{2}\right)^2 = \frac{k(k-1)}{12(k+1)}.$$

Let us introduce a notation $p = k+1$. If $p = n$ then $e = 0$ and

$$\lambda_n^{n-1} \;=\; -\frac{1}{24}\,\left(2 - n + \frac{1}{n}\right)\;\; n^n.$$

If $e \neq 0$ (i.e. $p \neq n$) then

$$\lambda_n^{p-1} = \left(\frac{\epsilon}{24}\left(2 - n + \frac{n}{p^2}\right) + \frac{\epsilon(n-p)^2}{24} - \frac{n-p}{8} - \frac{\epsilon(n-p)(p-1)(p-2)}{24p}\right) p^n,$$

where $\epsilon = \mathrm{sign}(e)$. Simplifying, we will get the statement of the proposition. $\square$

Corollary 3.3. *If $\overline{M}_n$, $n \geq 3$, is a Neuwirth manifold, then*

$$\lambda(\overline{M}_n) = \frac{n^{n-3}}{12}\,\left[4n^2 - 31n + 48\right].$$

As it was remarked above, the manifold M_4^1 is the only one Nil-manifold in the family introduced by Neuwirth [13], and we get $\lambda_4^1 = -2$. For spherical manifold M_3^1 we have $\lambda_3^1 = -3/4$. The manifold M_3^2 is flat and $\lambda_3^2 = 3/4$. Since manifolds M_n^k are two-fold branched coverings of the three-domensional sphere branched over Montesinos links, invariant λ_n^k for rational homology spheres M_n^k also can be calculated due to [12] from Jones polynomials and signatures of Montesinos links.

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