1 Introduction

This paper concerns hyperbolic lattices in dimension 3. These are, by definition, groups of orienation preserving isometries of three dimensional hyperbolic space with action properly discontinuous and volume of the orbit space finite. We use the upper half space (Poincaré) model for hyperbolic space: C

$$\mathbf{H}^3 = \{(w, r) \mid w \in \mathbf{C}, r > 0\}$$

which allows to identify the full group $Iso^+(\mathbf{H}^3)$ of orientation preserving isometries of $\mathbf{H}^3$ to $PSL(2, \mathbf{C})$. The lattices to be studied below will not contain any torsion, so we do not loose any generality when working with lattices in $SL(2, \mathbf{C})$, instead, which we shall do exclusively.

We deal with the following problem: A lattice Γ in $SL(2, \mathbf{C})$ is, after suitable conjugation, a group of matrices with entries from some algebraic number field K. We may look at K as an abstract finite degree extension of $\mathbf{Q}$ and at Γ as a subgroup of $SL(2, K)$. Each archimedian place of K, i.e. each embedding of K into $\mathbf{C}$ will induce an embedding of $SL(2, K)$ into $SL(2, \mathbf{C})$ and so create an isomorphism of Γ to some subgroup of $SL(2, \mathbf{C})$. Switching from one archimedian place of K to another one will, in general, affect traces, and if this happens at most one of them will yield, by Mostow rigidity, a lattice which then will be a reproduction of (a conjugate of) Γ or its complex conjugate $\bar{\Gamma}$. In general the field K will have many archimedian places, so the geometric requirement for Γ to be a lattice means a choice among them. For an infinite series of lattices we find the "'geometrically relevant"' archimedian place.

The source for the lattices studied is three dimensional topology: A large class of three dimensional manifolds M nowadays is known to be hyperbolic. These manifolds are all fibre bundles over the circle, the fibre being a topological surface which is obtained from a closed oriented one by (at most) finitely many punctures, see [8]. To be hyperbolic means that the universal cover $\tilde{M}$ of M is three dimensional hyperbolic space $\mathbf{H}^3$. To guarantee hyperbolicity for M so is equivalent to guaranteeing the existence of a lattice representation Γ of $\pi_1(M)$, the fundamental group of M, in $PSL(2, \mathbf{C})$ which then will reproduce M as

$$M \sim \Gamma \backslash \mathbf{H}^3.$$

For further study of a manifold M the pure existence of a lattice representation of its fundamental group does not yield sufficient information for computation of invariants like, e.g. ,volume. We in this paper explicitely exhibit, for an infinite series of (nonarithmetic) manifolds M a lattice representation of $\pi_1(M)$ (which, again by Mostow rigidity, is unique in a well defined sense). We do so by determining, from the abstract presentation of M, its so called trace field and then detecting that specific embedding of this field into $\mathbf{C}$ which leads to a faithful, discrete, and cofinite representation of $\pi_1(M)$ in $SL(2,\mathbf{C})$.

The methods used are elementary; it is expected that they carry over to larger classes of manifolds.

2 Preparatory material

We use various properties of the function

$$c_\vartheta(t) = 2\cos(t\vartheta/2)$$

for t real and ϑ a complex non real parameter. c_ϑ is even in t (and ϑ) and satisfies the Čebyšev recursion

$$c_\vartheta(t+2) + c_\vartheta(t-2) = 2\cos\vartheta \cdot c_\vartheta(t). \tag{1}$$

We have

$$c_\vartheta(0) = 2,\ c_\vartheta(2) = 2\cos\vartheta,$$
$$c_\vartheta(-1) = c_\vartheta(1) = 2\cos(\vartheta/2),$$

so, for $t = 2k$ an even integer, $c_\vartheta(2k)$ will be a polynomial as a function of $2\cos\vartheta$; similarly, for $t = 2k+1$ odd, the expression $\frac{c_\vartheta(2k+1)}{c_\vartheta(1)}$ will be a polynomial in terms of $2\cos\vartheta$. We call these polynomials $\tilde{p}_k$:

$$\begin{aligned} \tilde{p}_{2k}(2\cos\vartheta) &= c_\vartheta(2k) = 2\cos(k\vartheta), \\ \tilde{p}_{2k+1}(2\cos\vartheta) &= \tfrac{c_\vartheta(2k+1)}{c_\vartheta(1)} = \tfrac{\cos(k+\frac{1}{2})\vartheta}{\cos\frac{1}{2}\vartheta}. \end{aligned} \tag{2}$$

In the standardized notation of [6] for Čebyšev polynomials these are the polynomials

$$\tilde{p}_{2k}(z) = 2T_k(\frac{z}{2}),\ \tilde{p}_{2k+1}(z) = U_k(\frac{z}{2}) - U_{k-1}(\frac{z}{2}).$$

They have integer coefficients, with leading coefficient unity, and their degree is $|k|$ in either case, except for $\tilde{p}_0(z) = 2$, and they, by costruction, are even with respect to their index: $\tilde{p}_{-m}(z) = \tilde{p}_m(z)$.

From the various properties of $\tilde{p}_m(z)$ we mention only two, as these will be used in several places: $\tilde{p}_m(z)$, for $m \neq 0$ decomposes into linear factors over the real subfield of the field of $(2m)^{th}$ roots of unity:

$$\tilde{p}_m(z) = \prod_{1\le\nu\le|m|-1,\nu\equiv 1(2)} (z - 2\cos\frac{\nu\pi}{m}). \tag{3}$$

In addition we have a quadratic identity: Let

$$r_m(z) = \begin{cases} z-2 & \text{if m is odd} \\ z^2-4 & \text{if m is even} \end{cases}. \tag{4}$$

Then

$$\tilde{p}_m^2(z) - \tilde{p}_{m+2}(z)\tilde{p}_{m-2}(z) = -r_m(z). \tag{5}$$

Table (1) contains a short list.

Table 1:

m	$\tilde{p}_m(z)$
0	2
2	z
4	z^2-2
6	z^3-3z
8	z^4-4z^2+2
10	z^5-5z^3+5z
12	$z^6-6z^4+9z^2-2$
14	$z^7-7z^5+14z^3-7z$
16	$z^8-8z^6+20z^4-16z^2+2$
18	$z^9-9z^7+27z^5-30z^3+9z$
20	$z^{10}-10z^8+35z^6-50z^4+25z^2-2$

m	$\tilde{p}_m(z)$
1	1
3	$z-1$
5	z^2-z-1
7	z^3-z^2-2z+1
9	$z^4-z^3-3z^2+2z+1$
11	$z^5-z^4-4z^3+3z^2+3z-1$
13	$z^6-z^5-5z^4+4z^3+6z^2-3z-1$
15	$z^7-z^6-6z^5+5z^4+10z^3-6z^2-4z+1$
17	$z^8-z^7-7z^6+6z^5+15z^4-10z^3-10z^2+4z+1$
19	$z^9-z^8-8z^7+7z^6+21z^5-15z^4-20z^3+10z^2+5z-1$
21	$z^{10}-z^9-9z^8+8z^7+28z^6-21z^5-35z^4+20z^3+15z^2-5z-1$

We briefly relate these polynomials to standard Čebyšev polynomials; all relations are paraphrases of trigonometric identities: Let, for $m \geq 1$,

$$p_m(z) = U_{m-1}(\frac{z}{2}) = \prod_{1\leq\nu\leq m-1} (z - 2\cos\frac{2\nu\pi}{m}) \tag{6}$$

in the notation of [6] which is defined such that $p_m(2\cos\vartheta) = \frac{\sin m\vartheta}{\sin\vartheta}$ identically in ϑ. We put $p_{-m}(z) = -p_m(z)$ and $p_0(z) = 0$. Then

$$p_{m+1}(z) + p_{m-1}(z) = zp_m(z), \tag{7}$$

$$p_m^2(z) - p_{m+1}(z)p_{m-1}(z) = 1. \tag{8}$$

For A any unimodular 2x2 matrix these polynomials allow to write

$$A^m = p_m(trA)A - p_{m-1}(trA)E \tag{9}$$

for arbitrary integral m; here E is the 2x2 unit matrix. Let

$$d_m(z) = gcd(p_m(z), p_{m+1}(z) - 1) = gcd(p_m(z), p_{m-1}(z) + 1),\ m \neq 0 \tag{10}$$

where the greatest common divisor is taken with leading coefficient equal to that of $p_m(z)$. $d_m(z)$ is an integer coefficient polynomial which, for positive m factors in the following way:

$$d_m(z) = \prod_{1 \leq \nu < \frac{m}{2}} (z - 2\cos\frac{2\nu\pi}{m}). \tag{11}$$

We recover

$$\tilde{p}_m(z) = \frac{p_m(z)}{d_m(z)},\ m \neq 0. \tag{12}$$

Furthermore

$$\tilde{p}_{m-2}(z) = \frac{p_{m-1}(z) + 1}{d_m(z)},\ \tilde{p}_{m+2}(z) = \frac{p_{m+1}(z) - 1}{d_m(z)} \tag{13}$$

where again $m \neq 0$. Finally

$$\begin{array}{rcl} \tilde{p}_{2k+1}(z) & = & p_{k+1}(z) - p_k(z), \\ \tilde{p}_{2k}(z) & = & p_{k+1}(z) - p_{k-1}(z), \\ \tilde{p}_{2k+5}(z) - \tilde{p}_{2k+1}(z) & = & (-1)^{k+1}(z-2)\tilde{p}_{2k+3}(-z), \\ \tilde{p}_{2k+4}(z) - \tilde{p}_{2k}(z) & = & (z^2-4)p_{k+1}(z), \\ \tilde{p}_{k+2}^2(z) - \tilde{p}_k^2(z) & = & r_k(z)p_{k+1}(z). \end{array} \tag{14}$$

Our main result is achieved by a careful but elementary study of the polynomial

$$f_m(z) = \tilde{p}_{m+2}^2(z) - r_m(z) \tag{15}$$

for $m \geq 1$. The main theorem below will contain the fact that the algebraic number field generated by a suitable root of $f_m(z)$ will contain the trace field of a faithful lattice representation Γ_m of $\pi_1(M_m)$, M_m being the manifold defined in the next section, see (20). At the time being we can guarantee irreducibility of $f_m(z)$ over the rationals for only finitely many values of m; nevertheless the lattices will all be nonarithmetic with the exception of two of them, compare [2].

We define

$$K_m = \begin{cases} \{z \in \mathbf{C} \mid |r_m(z)| = (2\sin\frac{\pi}{2m})^2\} & \text{if m is odd} \\ \{z \in \mathbf{C} \mid |r_m(z)| = (2\sin\frac{\pi}{m})^2, Re(z) \geq 0\} & \text{if m is even} \end{cases}. \tag{16}$$

K_m is a simple closed curve. It is symmetric with respect to the real axis and, if m is even intersects the real axis at

$$a_m = 2\cos\frac{\pi}{m} \text{ and } b_m = 2\sqrt{1+\sin^2(\frac{\pi}{m})},$$

bounding a convex domain. In case m is odd K_m is simply the circle with center 2, intersecting the real axis at $a_m = 2\cos\frac{\pi}{m}$ and $b_m = 2(2-\cos\frac{\pi}{m})$, see figure (8) for the case $m = 13$.

Lemma 1 *The polynomial $f_m(z)$ defined in (15) has, for $m \geq 1$, no real zeroes.*

Proof. If there were any real zeroes of $f_m(z)$ then there would be some on the real half line $z \geq 2$: For m odd both summands in (15) composing $f_m(z)$ are nonnegative if $z \leq 2$, and for m even $f_m(z)$ is an even function. We disprove their existence. ¿From the factoring of $\tilde{p}_{m+2}(z)$ in (3) one sees that $\tilde{p}^2_{m+2}(z)$ increases strictly monotonically on $z \geq 2$, starting out from

$$\tilde{p}^2_{m+2}(2) = \begin{cases} 1 & \text{for m odd} \\ 4 & \text{for m even} \end{cases}.$$

The same statement is true for $r_m(z)$, with starting value $r_m(2) = 0$, see (4). Consider first the case of m odd. Then on the intervall $2 < z \leq 3$ we have

$$\tilde{p}^2_{m+2}(z) > 1 = r_m(3) \geq r_m(z),$$

so $f_m(z)$ has no zeroes there. On the half line $z \geq 3$ each factor $z - 2\cos\frac{\nu\pi}{m+2}$ obviously is at the same time greater than one and greater than $z-2 = r_m(z)$. So $\tilde{p}^2_{m+2}(z) > r_m(z)$ there.

In case m is even from monotonicity we have

$$\tilde{p}^2_{m+2}(z) > 4 = r_m(\sqrt{8}) \geq r_m(z)$$

on the intervall $2 < z \leq \sqrt{8}$. Now the linear factors which compose $\tilde{p}_{m+2}(z)$ come up in pairs $\{z \pm 2\cos\frac{\nu\pi}{m+2}\}, 1 \leq \nu \leq m/2, \nu$ odd, with one additional factor z in case $m \equiv 0 \bmod 4$. For this factor we shall have to make use only of the trivial statement that its value is ≥ 1 on the half line $z \geq 2$. For the remaining pairs of factors we again prove that their contribution to the product representing $\tilde{p}^2_{m+2}(z)$ is greater than 1 and greater than $r_m(z) = z^2 - 4$: For each relevant value of ν we have on $z \geq \sqrt{8}$:

$$(z - 2\cos\frac{\nu\pi}{m+2})(z + \cos\frac{\nu\pi}{m+2}) = z^2 - (2\cos\frac{\nu\pi}{m+2})^2 \geq 8 - 4 \geq 1,$$

and

$$(z - 2\cos\frac{\nu\pi}{m+2})^2(z + 2\cos\frac{\nu\pi}{m+2})^2 = (z^2 - (2\cos\frac{\nu\pi}{m+2})^2)^2$$

$$= (z^2 - (2\cos\frac{\nu\pi}{m+2})^2 - \frac{1}{2})^2 + z^2 - (2\cos\frac{\nu\pi}{m+2})^2 - \frac{1}{4}$$

$$\geq (8 - 4 - \frac{1}{2})^2 - \frac{1}{4} + z^2 - 4 = 12 + z^2 - 4 > z^2 - 4.$$

So there is no real root of f_m on the right of $\sqrt{8}$ if m is even. This proves the lemma.

Remark: The polynomial $\tilde{p}_{m+2}(z)$ has, for $m \geq 3$, exactly one root in the interiour of K_m, namely the number $2\cos\frac{\pi}{m+2}$, (and this root, of course, is simple) and it has no root on K_m itself: Take any root $2\cos\frac{\nu\pi}{m+2}$ with ν odd, $1 \leq \nu \leq \mathrm{m}+1$. To lie inside or on K_m means (see (16))

$$\begin{array}{ll} 2 - 2\cos\frac{\nu\pi}{m+2} \leq (2\sin\frac{\pi}{2m})^2 & \text{for m odd,} \\ 4 - (2\cos\frac{\nu\pi}{m+2})^2 \leq (2\sin\frac{\pi}{m})^2 & \text{for m even.} \end{array}$$

This means, for m odd that $\sin^2\frac{\nu\pi}{2(m+2)} \leq \sin^2\frac{\pi}{2m}$, equivalently, as $0 < \frac{\nu\pi}{2(m+2)} < \frac{\pi}{2}$: $\frac{\nu}{m+2} \leq \frac{1}{m}$ whence $(\nu-1) \leq 2$. This, for odd $\nu, 1 \leq \nu \leq m+1$, is satisfied only by $\nu = 1$, and in this case this inequality and also the preceding ones are strict. Similarly, for m even, the requirement on ν means $\sin^2\frac{\nu\pi}{m+2} \leq \sin^2\frac{\pi}{m}$. Remember that K_m entirely lies on the right of the imaginary axis, so we again have $\frac{\nu}{m+2} \leq \frac{1}{m}$, or, equivalently, $(\nu - 1)m \leq 2$, so $\nu = 1$ as in the previous case, and all inequalities are strict.

3 A series of manifolds

Let M be a three dimensional (oriented) manifold which fibers over the circle with fibre an oriented surface F whose fundamental group $\pi_1(F)$ is finitely generated. Then the fundamental group of M has the following structure:

$$\pi_1(M) \cong \pi_1(F) \rtimes \langle\mu\rangle. \tag{17}$$

Here, for $\omega \in \pi_1(F)$, $\mu\omega\mu^{-1} = f(\omega)$ where f is an automorphism of $\pi_1(F)$ induced by an orientation preserving autohomeomorphism of F. μ will, in what follows, have infinite order. We specialize to F to be the once punctured torus, so

$$\pi_1(F) = \langle \xi, \eta \rangle \tag{18}$$

is free of rank two. We choose the generators ξ and η such that the homotopy class of a loop around the puncture is the commutator $\lambda = \eta^{-1}\xi^{-1}\eta\xi$. Those automorphisms f of $\langle \xi, \eta \rangle$ which are induced by an orientation preserving autohomeomorphism of F will conjugate λ. We shall only deal with automorphisms which leave λ invariant; they form a subgroup $Aut_\lambda(\langle \xi, \eta \rangle)$ of the full automorphism group which is very well understood, see [4], [7]. In addition it is easy to decide whether or not the assignment

$$f : \begin{array}{c} \xi \\ \eta \end{array} \longrightarrow \begin{array}{c} \alpha \\ \beta \end{array}$$

with some $\alpha, \beta \in \langle \xi, \eta \rangle$ given extends to such an automorphism, see [4]. The two automorphisms

$$f_1 : \begin{array}{c} \xi \\ \eta \end{array} \longrightarrow \begin{array}{c} \xi \\ \xi\eta \end{array}, \quad f_2 : \begin{array}{c} \xi \\ \eta \end{array} \longrightarrow \begin{array}{c} \eta\xi \\ \eta \end{array} \tag{19}$$

generate $Aut_\lambda(\langle \xi, \eta \rangle)$, and this group surjects onto $SL(2, \mathbf{Z})$ if f is assigned its monodromy

$$Aut_\lambda(\langle \xi, \eta \rangle) \ni f \longrightarrow \bar{f} \in \mathrm{Aut}\ \mathbf{Z}^2,$$

i.e. the action induced by it on $\langle \xi, \eta \rangle^{ab}$. Denote, for $f \in Aut_\lambda(\langle \xi, \eta \rangle)$, the three manifold constructed by using f by $M(f)$. It is well known that $M(f)$ is hyperbolic if and only if its monodromy has trace by absolute value greater than 2. The subject of this paper will be the series of manifolds

$$M_m = M(f_2 \circ f_1^m) \tag{20}$$

where $m \geq 1$ is any integer. The automorphism $f_2 \circ f_1^m$ is described by

$$f_2 \circ f_1^m : \begin{array}{c} \xi \\ \eta \end{array} \longrightarrow \begin{array}{c} \eta\xi \\ (\eta\xi)^m\eta \end{array}.$$

Its monodromy is

$$\overline{f_2 \circ f_1^m} = \begin{pmatrix} 1 & m \\ 1 & m+1 \end{pmatrix},$$

so M_m is hyperbolic with fundamental group (17)

$$\pi_1(M_m) \cong \langle \xi, \eta, \mu \mid \mu\xi\mu^{-1} = \eta\xi, \mu\eta\mu^{-1} = (\eta\xi)^m\eta \rangle. \tag{21}$$

Note that we have

$$\mu\lambda = \mu\eta^{-1}\xi^{-1}\eta\xi = \lambda\mu. \tag{22}$$

4 Character varieties

A (two dimensional unimodular complex) character of an abstract group G is defined to be a function

$$s : G \longrightarrow \mathbf{C}$$

satisfying

$$\begin{gathered} s(e) = 2, \;\; s(g^2) = s^2(g) - 2, \\ s(g_1g_2g_3) + s(g_3g_2g_1) + s(g_1)s(g_2)s(g_3) \\ -s(g_1)s(g_2g_3) - s(g_2)s(g_3g_1) - s(g_3)s(g_1g_2) = 0 \end{gathered} \tag{23}$$

for all $g_1, g_2, g_3, g \in G$, e being the neutral element of G. There are many straightforward consequences of (23) such as

$$s(g_1g_2) = s(g_2g_1), \;\; s(g_1^{-1}g_2) + s(g_1g_2) = s(g_1)s(g_2)$$

for arbitrary $g_1, g_2 \in G$. They will be used frequently later in this paragraph; they, e.g., allow to express the value of a character polynomially in terms of its values on short generator products (with no more than three factors). The most important consequence which is not quite straightforward is

$$\begin{gathered} 4(s(g_1g_2g_3) - s(g_3g_2g_1))(s(g_4g_5g_6) - s(g_6g_5g_4)) \\ +det \begin{pmatrix} 2s(g_1g_4) - s(g_1)s(g_4) & 2s(g_1g_5) - s(g_1)s(g_5) & 2s(g_1g_6) - s(g_1)s(g_6) \\ 2s(g_2g_4) - s(g_2)s(g_4) & 2s(g_2g_5) - s(g_2)s(g_5) & 2s(g_2g_6) - s(g_2)s(g_6) \\ 2s(g_3g_4) - s(g_3)s(g_4) & 2s(g_3g_5) - s(g_3)s(g_5) & 2s(g_3g_6) - s(g_3)s(g_6) \end{pmatrix} \\ = 0 \end{gathered} \tag{24}$$

for $g_1, g_2, g_3, g_4, g_5, g_6 \in G$, see [3].

The characters of G are in one-one correspondence to the conjugacy classes of semisimple representations φ of G in $SL(2, \mathbf{C})$ via

$$s(g) = tr\varphi(g) \text{ for all } g \in G.$$

The character variety of the rank two free group $G = \langle \xi, \eta \rangle$ is well known to be $\mathbf{C}^3$; this identification is established by

$$(\xi, \eta) \longrightarrow (s(\xi), s(\eta), s(\eta\xi)) \in \mathbf{C}^3. \tag{25}$$

We now turn to the character variety of $\pi_1(M_m)$. For the present purpose it is natural to restrict attention to those characters on $\pi_1(M_m)$ which are irreducible in the strong sense that they remain irreducible when restricted to the free subgroup $\langle \xi, \eta \rangle$ of $\pi_1(M_m)$. We for short call these characters strongly irreducible.

Proposition 1 *The strongly irreducible characters of $\pi_1(M_m)$ form a set which is Zariski dense on a projective algebraic curve X_m defined over the integers. X_m, as a compact Riemann surface, may be realized as the twosheeted cover of some complex projective line, with $m+3$ ramification points if m is odd, and $m+2$ such points if m is even. So it has genus $\frac{m+1}{2}$ in the odd, and $\frac{m}{2}$ in the even case.*

The proof will require some trace calculus; it will yield complete information about the location of the ramification points and the location of the points of X_m which do not correspond to any strongly irreducible character of $\pi(M_m)$. Let $(x,y,z) = (s(\xi), s(\eta), s(\eta\xi)) \in \mathbf{C}^3$ as in (25) be the coordinates of a character s on $\langle \xi, \eta \rangle$ as a subgroup of $\pi_1(M_m)$. We assume

$$k(x,y,z) = xyz - x^2 - y^2 - z^2 + 4 \neq 0 \tag{26}$$

which is eqivalent to s being irreducible. We extract from the abstract group presentation (21) of $\pi_1(M_m)$ the conditions on (x,y,z) which allow an extension to $\pi_1(M_m)$, again called s.

$$\mu\xi\mu^{-1} = \eta\xi \text{ implies } x = z, \tag{27}$$

$$\mu\eta\mu^{-1} = (\eta\xi)^m\eta \text{ implies } y(p_{m+1}(z) - 1) = zp_m(z), \tag{28}$$

$$\mu\eta\xi\mu^{-1} = (\eta\xi)^m\eta^2\xi \text{ implies } yp_{m+2}(z) = z(p_{m+1}(z) + 1). \tag{29}$$

Extracting common divisors in lines (28), (29) leads to

$$d_m(z)(y\tilde{p}_{m+2}(z) - z\tilde{p}_m(z)) = 0,$$

$$d_{m+2}(z)(y\tilde{p}_{m+2}(z) - z\tilde{p}_m(z)) = 0,$$

see (13). The factorization of $d_m(z)$ given in (11) shows that $d_m(z)$ and $d_{m+2}(z)$ are coprime, so we infer

$$y = z\frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)} \tag{30}$$

with z arbitrary complex, not a zero of $\tilde{p}_{m+2}(z)$.

Any character s irreducible on $\langle \xi, \eta \rangle$ with the conditions (27) and (30) now certainly may be extended to a character s on all of $\pi_1(M_m)$. We determine its extension which is of course unique up to a sign at $s(\mu)$. (This fact reflects the hyperellipticity of X_m.) We set

$$(t,u,v,w) = (s(\mu), s(\mu^{-1}\xi), s(\mu^{-1}\eta), s(\mu^{-1}\eta\xi)) \tag{31}$$

and draw, from the presentation (21) of $\pi_1(M_m)$, the algebraic relations to z: We rewrite the first relator $\mu\xi\mu^{-1} = \eta\xi$ in two different ways:

$$\xi\mu^{-1}\xi^{-1} = \mu^{-1}\eta, \text{ so } v = t, \tag{32}$$

$$\eta^{-1}\mu\xi = \xi\mu \text{ so, after some trace calculation, } u = w. \tag{33}$$

We similarly rewrite the second relation $\mu\eta\mu^{-1} = (\eta\xi)^{-1}\eta$:

$$\eta\mu^{-1} = \mu^{-1}(\eta\xi)^m\eta, \text{ so, after some trace calculation, } p_m(z)u = (p_{m-1}(z)+1)v.$$

In this last expression we cancel $d_m(z)$, getting, for z not a root of $\tilde{p}_m(z)$:

$$u = w = \frac{\tilde{p}_{m-2}(z)}{\tilde{p}_m(z)}t. \tag{34}$$

So u, v,and w are linearly relatet to t with rational coefficients in z involved. The final algebraic relation between t and z comes from (24): We use this for $g_1 = g_4 = \mu^{-1}, g_2 = g_5 = \eta, g_3 = g_6 = \xi$. It leads, again after some trace calculus, to

$$\begin{array}{rcl} \tilde{p}_{m+2}(z)t^2 = (-1)^{\frac{m-1}{2}}\tilde{p}_m^2(z)\tilde{p}_{m+4}(-z) & & \text{for odd m,} \\ \tilde{p}_{m+2}(z)t^2 = \tilde{p}_m^2(z)p_{\frac{m+4}{2}}(z) & & \text{for even m.} \end{array} \tag{35}$$

It should be mentioned that actually a factor $y-2$ has been cancelled in between. But this is legal, as $y = 2$ would have lead to non strongly irreducible characters on $\pi_1(M_m)$:

$$k(z, 2, z) = 2z^2 - z^2 - 4 - z^2 + 4 = 0,$$

see (26). The curve (35) is the curve X_m in (z,t)-space of the theorem. It is easy to check that the polynomials $\tilde{p}_{m+2}(z)$ and $\tilde{p}_{m+4}(-z)$, for odd m, do have no common zero, similarly the polynomials $\tilde{p}_{m+2}(z)$ and $p_{\frac{m+4}{2}}(z)$ in the even case, see (3) and (6). So the curve X_m has $\frac{m+1}{2} + \frac{m+3}{2} + 1$ points of ramification when being projected onto the z-axis, including one at infinity, in the odd situation, and $\frac{m+2}{2} + \frac{m+2}{2}$ such points in the even situation. Again by (3) the polynomials $\tilde{p}_m(z)$ and $\tilde{p}_{m+2}(z)$ are coprime. So the equation (35) defining X_m ensures that the singularities of the rational function $\frac{t}{\tilde{p}_m(z)}$ at the zeroes of its denominator are removable. So (34) makes sense also there. This proves the proposition.

We may make the statement of the proposition more precise:

Proposition 2 *The variety of strongly irreducible characters on $\pi_1(M_m)$ is the affine algebraic curve X_m of (35) in (z,t)-space from which the points*

(z,t) with $\tilde{p}_{m+2}(z)=0$ have been removed. Collecting (27), (30), (32), (33), (34), the relation between characters and points on X_m is established by

$$\begin{aligned} & X_m \backslash \{(z,t) \mid \tilde{p}_{m+2}(z)=0\} \ni (z,t) \\ \longrightarrow\ & (s(\xi), s(\eta), s(\eta\xi), s(\mu), s(\mu^{-1}\xi), s(\mu^{-1}\eta), s(\mu^{-1}\eta\xi)) \\ = & (z, z\tfrac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)}, z, t, \tilde{p}_{m-2}(z)\tfrac{t}{\tilde{p}_m(z)}, t, \tilde{p}_{m-2}(z)\tfrac{t}{\tilde{p}_m(z)}) \in \mathbf{C}^7. \end{aligned} \tag{36}$$

We finish this section by defining a zero dimensional algebraic subset of X_m whose points are candidates for representing (characters of) lattice representations of $\pi_1(M_m)$: This group contains $\langle\lambda,\mu\rangle$ which, by (22), is a rank two free abelian subgroup. This in any lattice representation is forced to be a group of parabolic matrices forming the stabilizer of some cusp. Since λ is a commutator and the trace of a commutator in $SL(2,\mathbf{C})$ to be two can be achieved only trivially we have to impose, by geometric reasons, the condition $s(\lambda)=-2$.

Proposition 3 *If $(z,t)\in X_m$ defines a faithful lattice character on $\pi_1(M_m)$ then $f_m(z)=0$. (Remember the definition of $f_m(z)$ in (15).)*

Proof. We compute, for $\lambda=\eta^{-1}\xi^{-1}\eta\xi$,:

$$tr\ s(\lambda) = 2-k(x,y,z) = 2-(xyz-x^2-y^2-z^2+4) = -2$$

which is the "'Markoff equation"'

$$xyz = x^2+y^2+z^2,$$

so, by (27) and (30),

$$z^3\frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)} = 2z^2 + z^2\frac{\tilde{p}_m^2(z)}{\tilde{p}_{m+2}^2(z)}.$$

Now the factor z^2 may be cancelled as we are looking for faithful representations, only.

The remaining terms constitute, after multiplying by $\tilde{p}^2_{m+2}(z)$, the condition $f_m(z)=0$.

5 Two special cases

We interrupt the general discussion in order to settle the cases $m=1$ and $m=2$. These are the only cases in which the lattice representation of $\pi_1(M_m)$ is arithmetic, see [2]. Their combinatorial and geometric shape smoothly fits into the general frame of section 7; they are excluded from the following only in order to simplify discussion and notation.

First $m = 1$. The equation for z is quadratic:

$$f_1(z) = \tilde{p}_3^2(z) - (z-2) = z^2 - 3z + 3$$

has the roots $z = \frac{3 \pm i\sqrt{3}}{2}$; they both lead to a lattice representation of $\pi_1(M_1)$. We choose $z_1 = \frac{3-i\sqrt{3}}{2}$. The quotient of hyperbolic space with respect to this lattice is well known to be topologically the complement of the figure eight knot in S^3, the three sphere. In figure (1) we have reproduced a (standard) fundamental domain, it consists of two ideal tetrahedra with vertices $\{\infty, 0, \frac{1-i\sqrt{3}}{2}, 1\}$ and $\{\infty, 0, 1, \frac{1+i\sqrt{3}}{2}\}$, respectively. The faces are paired in the following manner:

$$\begin{matrix} \infty \\ \frac{1+i\sqrt{3}}{2} \\ 0 \end{matrix} \longrightarrow \begin{matrix} \frac{1-i\sqrt{3}}{2} \\ 1 \\ 0 \end{matrix}, \quad \begin{matrix} \infty \\ \frac{1-i\sqrt{3}}{2} \\ 1 \end{matrix} \longrightarrow \begin{matrix} 0 \\ 1 \\ \frac{1+i\sqrt{3}}{2} \end{matrix}, \quad \begin{matrix} \infty \\ 0 \\ \frac{1-i\sqrt{3}}{2} \end{matrix} \longrightarrow \begin{matrix} \infty \\ \frac{1+\sqrt{3}}{2} \\ 1 \end{matrix}.$$

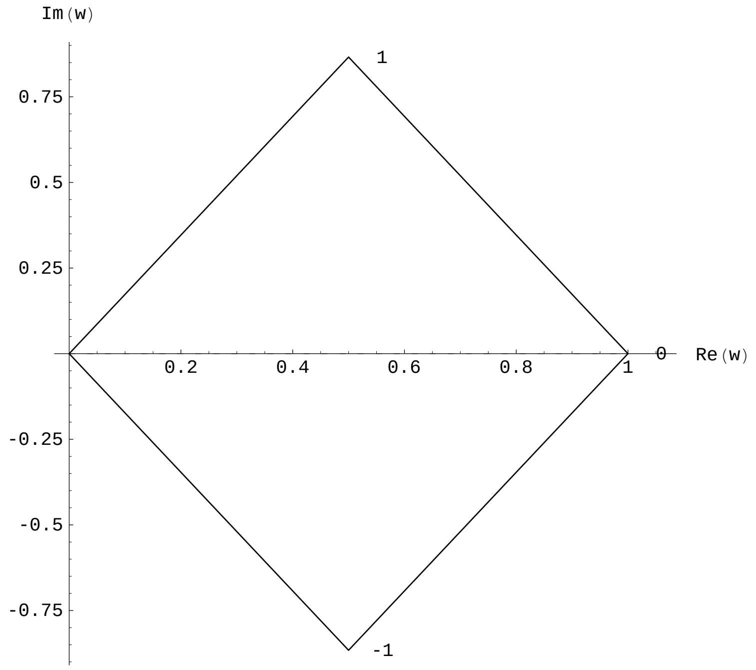

Figure 1

The case $m = 2$ leads to a biquadratic equation:

$$f_2(z) = \tilde{p}_4^2(z) - (z^2 - 4) = z^4 - 5z^2 + 8$$

has roots $z^2 = \frac{5 \pm i\sqrt{7}}{2}$. Each such root defines a lattice representation of $\pi_1(M_2)$; its quotient is, topologically, the complement of a knot in the lense space with fundamental group having order 2. The construction of a fundamental domain is based on the selection of the root $z_2 = \sqrt{\frac{5+4\sqrt{2}}{2}} - \sqrt{\frac{-5+4\sqrt{2}}{2}} i$ of $f_2(z)$. It is composed from three ideal tetrahedra with vertices $\{\infty, 0, w_{2,-1}, w_{2,0}\}$, $\{\infty, 0, w_{2,0}, w_{2,1}\}$, and $\{\infty, 0, w_{2,1}, w_{2,2}\}$, respectively, see figure (2); the vertex $w_{2,j}$ is denoted by label j, $-1 \leq j \leq 2$. Here the complex numbers $w_{2,-1}, w_{2,0}, w_{2,1}, w_{2,2}$ are the following:

$$\begin{array}{rcl} w_{2,-1} = \frac{z_2^2-2}{z_2} & = & .47807 - .691777i \\ w_{2,0} = \frac{z_2}{2} & = & .81612 - .202617i \\ w_{2,1} = \frac{2}{z_2} & = & 1.15417 + .286544i \\ w_{2,2} = \frac{z_2}{z_2^2-2} & = & 676094 + .978322i \end{array}.$$

The identification among the faces is the following:

$$\begin{array}{c} \infty \\ w_{2,1} \\ 0 \end{array} \longrightarrow \begin{array}{c} w_{2,-1} \\ w_{2,0} \\ 0 \end{array}, \quad \begin{array}{c} \infty \\ w_{2,-1} \\ w_{2,0} \end{array} \longrightarrow \begin{array}{c} 0 \\ w_{2,0} \\ w_{2,1} \end{array}, \quad \begin{array}{c} \infty \\ w_{2,0} \\ w_{2,1} \end{array} \longrightarrow \begin{array}{c} 0 \\ w_{2,1} \\ w_{2,2} \end{array}, \quad \begin{array}{c} \infty \\ 0 \\ w_{2,-1} \end{array} \longrightarrow \begin{array}{c} \infty \\ w_{2,2} \\ w_{2,1} \end{array}.$$

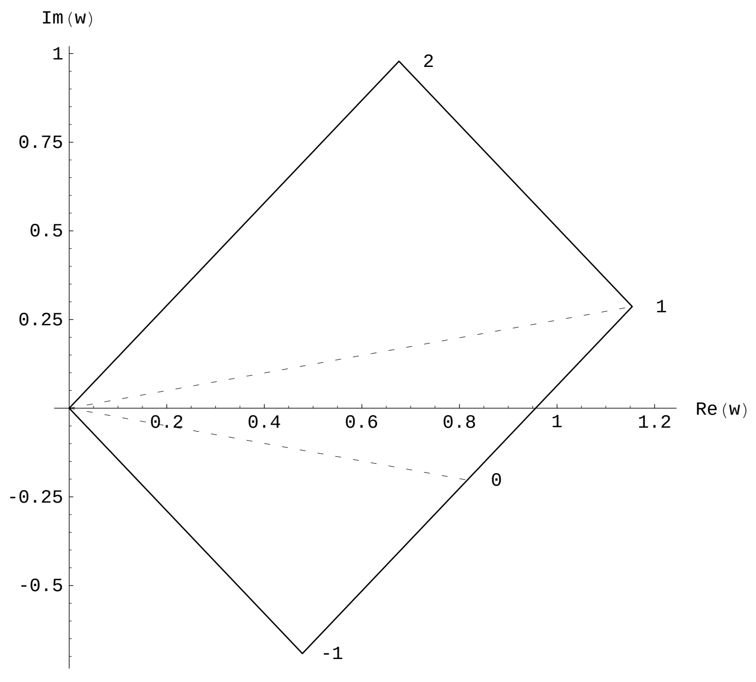

Figure 2

We, from now on assume m to be at least 3.

6 Isolation of a root of $f_m(z)$

We compare, for z on K_m, $|\tilde{p}^2_{m+2}(z)|$ to

$$|r_m(z)| = \begin{cases} (2\sin\frac{\pi}{2m})^2 & \text{if m is odd,} \\ (2\sin\frac{\pi}{m})^2 & \text{if m is even} \end{cases},$$

assuming $m \geq 3$. From (5) we have

$$\tilde{p}^2_{m+2}(a_m) = \tilde{p}_m(a_m)\tilde{p}_{m+4}(a_m) - r_m(a_m) = -r_m(a_m)$$

where $a_m = 2\cos\frac{\pi}{m}$ is, as in the text following (16), the smaller of the two real numbers where K_m intersects the real axis. So we have $|\tilde{p}^2_{m+2}(a_m)| = |r_m(a_m)|$, but, as $0 < a_m < 2$,

$$f_m(a_m) = \tilde{p}^2_{m+2}(a_m) - r_m(a_m) = -2r_m(a_m) \neq 0. \tag{37}$$

We observe that $|\tilde{p}^2_{m+2}(z)| > |r_m(a_m)|$ for all z on K_m except a_m. This is evident for odd m: Each factor $|z - 2\cos\frac{\nu\pi}{m+2}|$ entering into $|\tilde{p}_{m+2}(z)|$ grows strictly monotonically when moving z along K_m from a_m to b_m. For m even the same observation can be made with the exception of the factor $|z - 2\cos\frac{\pi}{m+2}|$ whose root is the only root of $\tilde{p}_{m+2}(z)$ inside K_m. But combining this factor with $|z - 2\cos\frac{(m+1)\pi}{m+2}| = |z + 2\cos\frac{\pi}{m+2}|$ allows to estimate

$$|z^2 - (2\cos\frac{\pi}{m+2})^2| \geq |z^2 - 4| - |4 - (2\cos\frac{\pi}{m+2})^2|$$

$$= (2\sin\frac{\pi}{m})^2 - (2\sin\frac{\pi}{m+2})^2 = (2\cos\frac{\pi}{m+2})^2 - (2\cos\frac{\pi}{m})^2$$

$$= |a_m^2 - (2\cos\frac{\pi}{m+2})^2|$$

which is the value taken at the initial point a_m. As now $m \geq 4$ there will be at least $\frac{m+2}{2} \geq 3$ linear factors composing $|\tilde{p}_{m+2}(z)|$, so there is at least one more factor which guarantees

$$|\tilde{p}^2_{m+2}(z)| > |r_m(z)| \text{ for } z \text{ on } K_m, z \neq a_m.$$

This proves, together with (37) that $f_m(z) \neq 0$ on K_m.

We now can state, as an easy consequence from complex function theory:

Proposition 4 *For $m \geq 3$, the polynomial $f_m(z) = \tilde{p}^2_{m+2}(z) - r_m(z)$ has exactly two roots $z_m, \bar{z}_m$ in the interior of K_m. They are nonreal and simple.*

We select z_m to have negative imaginary part.

We denote by s_m the character on $\pi_1(M_m)$ which is, up to a sign at the t-coordinate, determined by z_m, see (35), (36). The major part of the rest of this paper is devoted to showing that s_m (and its complex conjugate as well) is a character of a lattice representation of $\pi_1(M_m)$.

Proof of the proposition. The constituent $\tilde{p}^2_{m+2}(z)$ of $f_m(z)$ has exactly one zero inside K_m, namely $2\cos\frac{\pi}{m+2}$, and its order is 2. The polynomial $f_m(z)$ may be viewed as a small perturbation of $\tilde{p}^2_{m+2}(z)$ in the sense that

$$\tilde{p}^2_{m+2}(z) - f_m(z) = r_m(z) \neq \tilde{p}^2_{m+2}(z)$$

and

$$|\tilde{p}^2_{m+2}(z) - f_m(z)| = |r_m(z)| \leq |\tilde{p}^2_{m+2}(z)|$$

on the simple closed curve K_m. So $f_m(z)$ has as many roots inside K_m as $\tilde{p}^2_{m+2}(z)$, counted with multiplicity. By lemma (1) they are not real. This proves the proposition.

We have to localize z_m still a bit more precisely. The function $\vartheta \longrightarrow z(\vartheta) = 2\cos\vartheta$ maps the vertical strip $0 < Re\ \vartheta < \pi$ in the complex ϑ-plane schlicht to the complex z-plane slit along $|Re\ z| \geq 2$. We denote the inverse of this map by $z \longrightarrow \vartheta(z) = \arccos(\frac{z}{2})$ with $\alpha(z) = Re\ \vartheta(z), \beta(z) = Im\ \vartheta(z)$. Figures (3) and (4) show the position of z_m and ϑ_m for $3 \leq m \leq 19$ with values taken from table (2).

Table 2:

m	z_m	ϑ_m	$\frac{\pi}{m+2}$
3	1.727155-.228442i	.565023+.211738i	.628319
5	1.837447-.094171i	.420988+.114965i	.44878
7	1.893939-.047383i	.334679+.072066i	.349066
9	1.925942-.02699i	.277283+.049279i	.2856
11	1.945594-.016757i	.236452+.03576i	.241661
13	1.958444-.011087i	.205973+.027102i	.20944
15	1.967272-.007702i	.182382+.021232i	.1848
17	1.973583-.005562i	.163597+.017074i	.165347
19	1.978244-.004144i	.148293+.014023i	.1496
4	1.792106-.14197i	.48268+.152344i	.523599
6	1.869977-.065551i	.373023+.089815i	.392699
8	1.912015-.035312i	.303347+.059072i	.314159
10	1.936873-.021074i	.255274+.041717i	.261799
12	1.952654-.013537i	.220178+.030986i	.224399
14	1.963247-.009191i	.19347+.023901i	.19635
16	1.970678-.006517i	.172485+.018984i	.174533
18	1.976081-.004784i	.155572+.015437i	.15708

We abbreviate $\vartheta(z_m) = \vartheta_m$, $\alpha(z_m) = \alpha_m$, $\beta(z_m) = \beta_m$. We have $0 < \alpha_m < \pi$, $\beta_m > 0$, the latter from the normalization $Im\ z_m < 0$ by the addition theorems of the trigonometric functions.

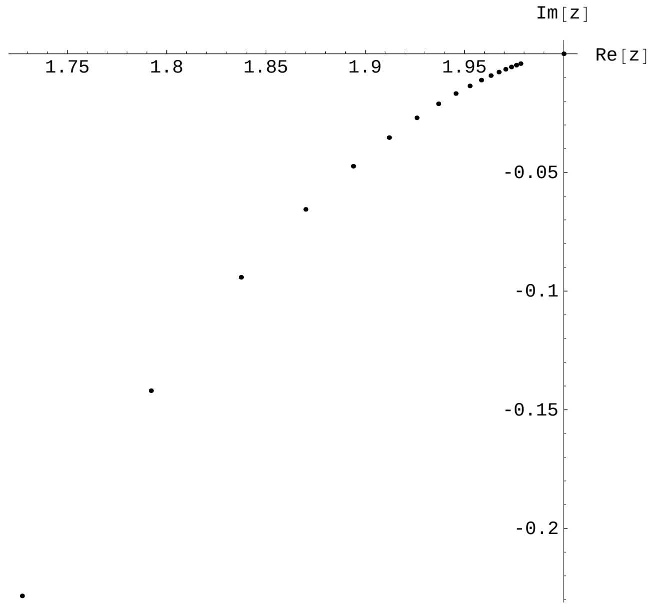

Figure 3

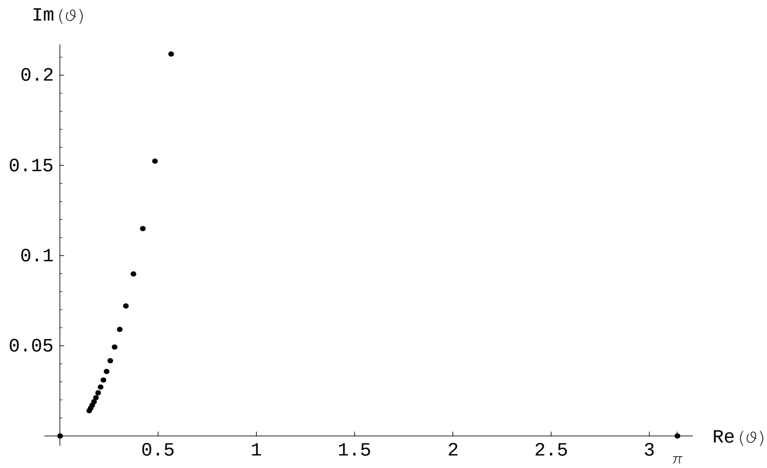

Figure 4

Lemma 2 *For $m \geq 3$ we have* $0 < \alpha_m = Re\, \arccos \frac{z_m}{2} < \frac{\pi}{m+2}$.

Proof. We have, for z complex, neither ≥ 2 nor ≤ -2:

$$r_m(z) = \begin{cases} z - 2 = 2(\cos\vartheta(z) - 1) = -4\sin^2\frac{\vartheta(z)}{2} & \text{for m odd,} \\ z^2 - 4 = 4(\cos^2\vartheta(z) - 1) = -4\sin^2\vartheta(z) & \text{for m even.} \end{cases}$$

So z to lie in the interiour of K_m, not being real ≥ 2, means

$$\begin{array}{ll} |\sin \frac{\vartheta(z)}{2}| < \sin \frac{\pi}{2m}, & \text{if m is odd,} \\ |\sin \vartheta(z)| < \sin \frac{\pi}{m}, & \text{if m is even.} \end{array} \tag{38}$$

But

$$|\sin \vartheta| = |\sin \alpha \cosh \beta + i \cos \alpha \sinh \beta| \geq |\sin \alpha|$$

for ϑ arbitrary complex (with real and imaginary parts α and β, respectively) which by (38) implies

$$\begin{array}{ll} \sin \frac{\alpha(z)}{2} < \sin \frac{\pi}{2m} & \text{for odd m,} \\ \sin \alpha(z) < \sin \frac{\pi}{m} & \text{for even m.} \end{array}$$

So $\alpha(z) < \frac{\pi}{m}$ for all relevant values of z, especially

$$\alpha_m = \alpha(z_m) = \alpha(\bar{z}_m) < \frac{\pi}{m}. \tag{39}$$

We sharpen this to $\alpha_m < \frac{\pi}{m+2}$ for all $m \geq 3$:

The complex number ϑ_m satifies the equation $f_m(2\cos\vartheta_m) = 0$ which, explicitely written, is

$$\tilde{p}^2_{m+2}(2\cos\vartheta_m) = r_m(2\cos\vartheta_m) = \left\{ \begin{array}{ll} -4\sin^2 \frac{\vartheta_m}{2} & \text{for m odd, and} \\ -4\sin^2 \vartheta_m & \text{for m even.} \end{array} \right. \tag{40}$$

But, by definition (2)

$$\tilde{p}_{m+2}(2\cos\vartheta_m) = \left\{ \begin{array}{ll} \frac{\cos\frac{m+2}{2}\vartheta_m}{\cos\frac{1}{2}\vartheta_m}, & \text{if m is odd,} \\ 2\cos\frac{m+2}{2}\vartheta, & \text{if m is even.} \end{array} \right.$$

So, from (40) we have in both cases

$$\cos\frac{m+2}{2}\vartheta_m = \varepsilon i \sin\vartheta_m$$

with $\varepsilon = \pm 1$. We split this into real and imaginary parts and get

$$\cos\alpha_m \sinh\beta_m = -\varepsilon\cos\frac{m+2}{2}\alpha_m \cosh\frac{m+2}{2}\beta_m, \tag{41}$$

$$\sin\alpha_m \cosh\beta_m = -\varepsilon\sin\frac{m+2}{2}\alpha_m \sinh\frac{m+2}{2}\beta_m. \tag{42}$$

¿From $\alpha_m > 0, \beta_m > 0$ we deduce

$$\varepsilon\cos\frac{m+2}{2}\alpha_m < 0,$$

$$\varepsilon \sin \frac{m+2}{2}\alpha_m < 0.$$

But the estimate (39) implies, for $m \geq 3$:

$$0 < \frac{m+2}{2}\alpha_m < \frac{m+2}{2m}\pi < \pi,$$

so $\varepsilon = -1$ from (42) and $\cos \frac{m+2}{2}\alpha_m > 0$ from (41). So $\frac{m+2}{2}\alpha_m < \frac{\pi}{2}$ which is the statement of the lemma.

Lemma 3 *Let $m \geq 3$, and let the real number t range over the closed intervall $-\frac{m}{2} \leq t \leq \frac{m}{2}$. Then the expression $v_m(t) = \frac{\cos t\vartheta_m}{\sin \vartheta_m}$ will assume only values with real part positive and imaginary part negative.*

Proof. We split numerator and denominator into real and imaginary parts:

$$\cos t\vartheta_m = \cos t\alpha_m \cosh t\beta_m - i \sin t\alpha_m \sinh t\beta_m,$$

$$\sin \vartheta_m = \sin \alpha_m \cosh \beta_m + i \cos \alpha_m \sinh \beta_m.$$

Remember from lemma (2) that $\sin t\alpha_m > 0$, $\cos t\alpha_m > 0$ throughout the intervall $|t| \leq \frac{m}{2}$, and that we have chosen β_m to be positive. The statement of the lemma is obviously true for $v_m(0) = \frac{1}{\sin \vartheta_m}$. We show that $v_m(t)$ does not leave the open lower right hand quarter of the complex v-plane: Reaching the real axis at $t = t_0$ would, after an elementary computation, lead to

$$\tan t_0\alpha_m \tanh t_0\beta_m = -\cot \alpha_m \tanh \beta_m,$$

and reaching the imaginary axis to

$$\tan t_0\alpha_m \tanh t_0\beta_m = -\tan \alpha_m \coth \beta_m.$$

No such equality is possible which becomes evident just from checking signs on either side. So the lemma is proven.

7 An ideal polyhedron

The curve S_m defined by

$$w_m(t) = \frac{c_{\vartheta_m}(2t-1)}{c_{\vartheta_m}(2t+1)} = \frac{\cos(t-\frac{1}{2})\vartheta_m}{\cos(t+\frac{1}{2})\vartheta_m},\ t \in \mathbf{R}$$

is, as ϑ_m is complex non real, a spiral in the complex w-plane. The Möbius transformation defined by the matrix

$$A_m = \begin{pmatrix} 0 & 1 \\ -1 & 2\cos\vartheta_m \end{pmatrix}$$

creates a shift on it:

$$A_m w_m(t) = \frac{c_{\vartheta_m}(2t+1)}{-c_{\vartheta_m}(2t-1) + 2\cos\vartheta_m \cdot c_{\vartheta_m}(2t+1)} = w_m(t+1). \tag{43}$$

The spiral winds about the two fixed points of A_m which are $e^{\pm i\vartheta_m}$; they are, at the same time, the eigenvalues of A_m.

Lemma 4 *That part of the spiral S_m which is parametrized by $-\frac{m+1}{2} \le t \le \frac{m+1}{2}$ for $m \ge 3$ arbitrary integral entirely lies in the half plane $Re\ w > 0$.*

Proof. We have

$$Re\ w_m(t) = \frac{1}{2} \frac{\cos(t-\frac{1}{2})\vartheta_m \cos(t+\frac{1}{2})\bar{\vartheta}_m + \cos(t+\frac{1}{2})\vartheta_m \cos(t-\frac{1}{2})\bar{\vartheta}_m}{|\cos(t+\frac{1}{2})\vartheta_m|^2}$$

which leads to

$$2|\cos(t+\frac{1}{2})\vartheta_m|^2 Re\ w_m(t) = \cos 2t\alpha_m \cosh\beta_m + \cos\alpha_m \cosh 2t\beta_m. \tag{44}$$

So only $|t|$ matters. In the part $0 \le t < \frac{\pi}{4\alpha_m}$ of the interval $0 \le t \le \frac{m+1}{2}$ both summands are positive. For $\frac{\pi}{4\alpha_m} \le t \le \frac{m+1}{2}$ we have first

$$\frac{\pi}{2} \le 2t\alpha_m \le (m+1)\alpha_m < \frac{m+1}{m+2}\pi = \pi - \frac{\pi}{m+2,}$$

so

$$|\cos 2t\alpha_m| < \cos\frac{\pi}{m+2} < \cos\alpha_m, \tag{45}$$

and second

$$2t \ge \frac{\pi}{2\alpha_m} > 1,$$

so

$$\cosh 2t\beta_m > \cosh\beta_m \tag{46}$$

there. (45) and (46) together imply that the possibly negative summand on the right side of (44) cannot destroy positivity of the sum.

We now construct an ideal polyhedron P_m in hyperbolic space which will be shown to be a fundamental domain of a lattice Γ_m which, as an abstract group, is isomorphic to $\pi_1(M_m)$. P_m will be composed from $m+1$ ideal tetrahedra. For this purpose we position $m+2$ points on the spiral S_m: For m odd we set

$$w_{m,j} = w_m(j) = \frac{\tilde{p}_{2j-1}(z_m)}{\tilde{p}_{2j+1}(z_m)}, \quad -\frac{m+1}{2} \le j \le \frac{m+1}{2}, \tag{47}$$

and for even m

$$w_{m,j} = w_m(j - \frac{1}{2}) = \frac{\tilde{p}_{2j-2}(z_m)}{\tilde{p}_{2j}(z_m)}, \quad -\frac{m}{2} \le j \le \frac{m+2}{2}. \qquad (48)$$

Notice that, in both cases, these points have positive real parts which is evident for m odd and easy to check for m even, remembering the discussion of the spiral S_m.

The following lemma is the main ingredient of the subject. It shows that the root z_m of the polynomial f_m is the right choice among all roots in order to create a discrete subgroup of $SL(2, \mathbf{C})$.

Lemma 5 *Let $m \ge 3$ be an arbitrary integer. Then*

$$arg\ w_{m,j} < arg\ w_{m,j+1}$$

for all relevant values of j, i.e.

$$\begin{array}{ll} -\frac{m+1}{2} \le j \le \frac{m-1}{2} & \textit{for } m \textit{ odd,} \\ -\frac{m}{2} \le j \le \frac{m}{2} & \textit{for } m \textit{ even.} \end{array}$$

Here the argument is, for complex w with positive real part, understood to lie in $-\frac{\pi}{2} < argw < \frac{\pi}{2}$.

Proof. We show

$$Im \frac{w_{m,j+1}}{w_{m,j}} > 0$$

which is equivalent to the assertion of the lemma. We have, for odd m,

$$\frac{w_{m,j+1}}{w_{m,j}} = \frac{\tilde{p}^2_{2j+1}(z_m)}{\tilde{p}_{2j+3}(z_m)\tilde{p}_{2j-1}(z_m)}$$

$$= \frac{\tilde{p}^2_{2j+1}(z_m)}{\tilde{p}^2_{2j+1}(z_m) + r_m(z_m)} = \frac{1}{1 + \frac{r_m(z_m)}{\tilde{p}^2_{2j+1}(z_m)}} = \frac{1}{1 + \frac{r_m(2\cos\vartheta_m)}{\tilde{p}^2_{2j+1}(2\cos\vartheta_m)}}.$$

But

$$\frac{\tilde{p}^2_{2j+1}(2\cos\vartheta_m)}{r_m(2\cos\vartheta_m)} = -(\frac{\cos(j+\frac{1}{2})\vartheta_m}{\sin\vartheta_m})^2,$$

see (2), (4). Here $j + \frac{1}{2}$ remains in the intervall $-\frac{m}{2} \le j + \frac{1}{2} \le \frac{m}{2}$, so by lemma (4) this quotient, for all relevant values of j, has positive imaginary part. So its inverse has negative imaginary part which remains true after the number 1 has been added. But this is the inverse of the quotient studied. So the lemma is proven for m odd. The reasoning when dealing with even m is exactly the same.

Let $T_{m,j}$, for fixed $m \geq 3$, be the ideal tetrahedron with vertices $\{0, w_{m,j}, w_{m,j+1}, \infty\}$; here j runs, as in the previous lemma over the integers $-\frac{m+1}{2} \leq j \leq \frac{m-1}{2}$ or $-\frac{m}{2} \leq j \leq \frac{m}{2}$, depending on the parity of m. They all meet in the hyperbolic line $w = 0$, and, by the above discussion, any two of them have no more points in common except that two consecutive ones, $T_{m,j}$ and $T_{m,j+1}$, intersect in a common face which is the ideal triangle with vertices $\{0, w_{m,j+1}, \infty\}$. So they compose an ideal hyperbolic polyhedron P_m:

$$P_m = \begin{cases} \bigcup_{-\frac{m+1}{2} \leq j \leq \frac{m-1}{2}} T_{m,j} & \text{if m is odd,} \\ \bigcup_{-\frac{m}{2} \leq j \leq \frac{m}{2}} T_{m,j} & \text{if m is even.} \end{cases}$$

See figures (1), (2), (5), and (9); they show, for $m = 1, 2, 5, 13$, the projection of P_m from the point at infinity into the complex w-plane. The one dimensional boundary pieces are all hyperbolic lines: vertical ones which meet the w-plane in 0 or one of the $w_{m,j}$'s and others, one type with feet 0 and some $w_{m,j}$ and one more type with feet two consecutive $w_{m,j}$'s. We adopt the notation "'$\overline{p,q}$"' for the hyperbolic line with feet $p, q \in \mathbf{C} \cup \{\infty\}$ and "' $\sphericalangle \overline{p,q}$"' for the spacial angle in the interiour of P_m along the line $\overline{p,q}$. We observe that the tetrahedra $T_{m,j}$ and $T_{m,-(j+1)}$ are (hyperbolically) congruent to each other ($0 \leq j \leq \frac{m-1}{2}$ resp. $\frac{m}{2}$); the congruence is simply achieved via multiplication by the complex number $w_{m,j} w_{m,j+1}$ or its inverse, respectively. There are more symmetries:

$$w_{m,j+1} - w_{m,j} = w_{m,-(j+1)} - w_{m,-(j+2)}, \quad \begin{cases} 0 \leq j \leq \frac{m-3}{2} & \text{for m odd,} \\ 0 \leq j \leq \frac{m-2}{2} & \text{for m even.} \end{cases} \tag{49}$$

Namely, for m odd,

$$\begin{aligned} w_{m,-(j+1)} - w_{m,-(j+2)} &= \frac{\tilde{p}_{-2j-3}(z_m)}{\tilde{p}_{-2j-1}(z_m)} - \frac{\tilde{p}_{-2j-5}(z_m)}{\tilde{p}_{-2j-3}(z_m)} \\ &= \frac{-r_m(z_m)}{\tilde{p}_{2j+1}(z_m)\tilde{p}_{2j+3}(z_m)} = w_{m,j+1} - w_{m,j}. \end{aligned}$$

For m even the computation is the same. We also observe

$$\text{m odd} : A_m w_{m,\frac{m+1}{2}} = \frac{1}{2} w_{m,\frac{m+1}{2}}, \; A_m^{-1} w_{m,-\frac{m+1}{2}} = 2 w_{m,-\frac{m+1}{2}}, \tag{50}$$

$$\text{m even} : A_m w_{m,\frac{m+2}{2}} = \frac{1}{2} w_{m,\frac{m+2}{2}}, \; A_m^{-1} w_{m,-\frac{m}{2}} = 2 w_{m,-\frac{m}{2}}. \tag{51}$$

We prove only the second statement for m odd, as this will be used below:

$$A_m^{-1} w_{m,-\frac{m+1}{2}} = \frac{z_m \frac{\tilde{p}_{m+2}(z_m)}{\tilde{p}_m(z_m)} - 1}{\frac{\tilde{p}_{m+2}(z_m)}{\tilde{p}_m(z_m)}} = \frac{\tilde{p}_{m+4}(z_m)}{\tilde{p}_{m+2}(z_m)} = \frac{\tilde{p}_{m+4}(z_m)\tilde{p}_m(z_m)}{\tilde{p}_{m+2}(z_m)\tilde{p}_m(z_m)}$$

$$= \frac{\tilde{p}^2_{m+2}(z_m) + r_m(z_m)}{\tilde{p}_{m+2}(z_m)\tilde{p}(z_m)} = \frac{2\tilde{p}^2_{m+2}(z_m)}{\tilde{p}_{m+2}(z_m)\tilde{p}_m(z_m)} = 2w_{m,-\frac{m+1}{2}}.$$

In addition we note the following angular relations; they become transparent when looking at pictures (1), (2), (5), (9).

Table 3 :

	m odd	m even
1	$\sphericalangle\overline{0, w_{m,-\frac{m-1}{2}}} + \sphericalangle\overline{\infty, w_{m,-\frac{m+1}{2}}} = \pi$	$\sphericalangle\overline{0, w_{m,-\frac{m-2}{2}}} + \sphericalangle\overline{\infty, w_{m,-\frac{m}{2}}} = \pi$
2	$\sphericalangle\overline{\infty, w_{m,-\frac{m+1}{2}}} + \sphericalangle\overline{\infty, w_{m,\frac{m-1}{2}}} = \pi$	$\sphericalangle\overline{\infty, w_{m,-\frac{m}{2}}} + \sphericalangle\overline{\infty, w_{m,\frac{m}{2}}} = \pi$
3	$\sphericalangle\overline{\infty, 0} + \sphericalangle\overline{\infty, w_{m,\frac{m+1}{2}}} = \pi$	$\sphericalangle\overline{\infty, 0} + \sphericalangle\overline{\infty, w_{m,\frac{m+2}{2}}} = \pi$
4	$\sphericalangle\overline{0, w_{m,\frac{m+1}{2}}} = \sphericalangle\overline{\infty, w_{m,-\frac{m+1}{2}}}$	$\sphericalangle\overline{0, w_{m,\frac{m+2}{2}}} = \sphericalangle\overline{\infty, w_{m,-\frac{m}{2}}}$
5	$\sphericalangle\overline{0, w_{m,-\frac{m+1}{2}}} = \sphericalangle\overline{\infty, w_{m,\frac{m+1}{2}}}$	$\sphericalangle\overline{0, w_{m,-\frac{m}{2}}} = \sphericalangle\overline{\infty, w_{m,\frac{m+2}{2}}}$
6	$\sum_{j=-\frac{m+1}{2}}^{\frac{m-1}{2}} \sphericalangle\overline{w_{m,j}, w_{m,j+1}} = \sphericalangle\overline{\infty, 0}$	$\sum_{j=-\frac{m}{2}}^{\frac{m}{2}} \sphericalangle\overline{w_{m,j}, w_{m,j+1}} = \sphericalangle\overline{\infty, 0}$

We restrict to comments for m odd: A_m^{-1} maps the four points

$$\begin{matrix} w_{m,-\frac{m-3}{2}} \\ w_{m,-\frac{m-1}{2}} \\ w_{m,-\frac{m+1}{2}} \\ 0 \end{matrix} \quad \text{to} \quad \begin{matrix} w_{m,-\frac{m-1}{2}} \\ w_{m,-\frac{m+1}{2}} \\ 2w_{m,-\frac{m+1}{2}} \\ \infty \end{matrix} ,$$

see (43), (5). So $\sphericalangle\overline{0, w_{m,-\frac{m-1}{2}}}$ equals the angle at the point $w_{m,-\frac{m+1}{2}}$ in the Euclidian triangle with vertices $\{w_{m,-\frac{m-1}{2}}, w_{m,-\frac{m+1}{2}}, 2w_{m,-\frac{m+1}{2}}\}$. But this angle obviously with $\sphericalangle\overline{\infty, w_{m,-\frac{m+1}{2}}}$ adds up to π. This is point 1 in the above list. The second and third points are justified by (49), special case $j = \frac{m-3}{2}$, and the fact that $\overline{w_{m,\frac{m-1}{2}}, w_{m,\frac{m+1}{2}}}$ is a Euclidian translate of $\overline{w_{m,-\frac{m+1}{2}}, 0}$ (by the complex number $w_{m,\frac{m+1}{2}}$). The points 4 and 5 come from the observation that $T_{m,-\frac{m+1}{2}}$ and $T_{m,\frac{m-1}{2}}$ are congruent: After having mapped the two tetrahedra isometrically to each other the angles which are asserted to be equal play the rôle of opposite angles. Point 6 finally is trivial: Each $\sphericalangle\overline{w_{m,j}, w_{m,j+1}}$ becomes, when replaced by its opposite in the tetrahedron $T_{m,j}$, a constituent of $\sphericalangle\overline{\infty, 0}$.

8 The main result

We still need one preparatory proposition.

Let $m \geq 3$ integral and z arbitrary complex, but not a root of $\tilde{p}_{m+2}(z)$, and define three Mőbius transformations

$$A = A_m(z) = \begin{pmatrix} 0 & 1 \\ -1 & z \end{pmatrix},$$

$$B = B_m(z) = \begin{pmatrix} 1 & 0 \\ \frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)} & 1 \end{pmatrix}, \; C = C_m(z) = \begin{pmatrix} 1 & \frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)} \\ 0 & 1 \end{pmatrix}.$$

Proposition 5 *If z is a root of $f_m(z)$ then*

$$ACA^{-1} = B^{-1}, \; CBC^{-1}B^{-1} = A^m.$$

Proof. The first relation is easily verified, it is actually independent of the specific choice of z. Establishing the second relation requires some explicit matrix calculation. We compute

$$CBC^{-1}B^{-1} = \begin{pmatrix} 1 + (\frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)})^2 + (\frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}})^4 & -(\frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)})^3 \\ (\frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)})^3 & 1 - (\frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)})^2 \end{pmatrix} \tag{52}$$

and

$$A^m = \begin{pmatrix} -p_{m-1}(z) & p_m(z) \\ -p_m(z) & p_{m+1}(z) \end{pmatrix}. \tag{53}$$

We compare (52) and (53) elementwise using (14). First

$$r_m(z)\left(p_{m+1}(z) - 1 + (\frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)})^2\right) = r_m(z)p_{m+1}(z) - \tilde{p}_{m+2}^2(z) + \tilde{p}_m^2(z) = 0.$$

This is the identity claimed in the lower right hand position. From this follows

$$\left(\frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)}\right)^3 = -(p_{m+1}(z) - 1)\frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)}$$

$$= -(p_{m+1}(z) - 1)\frac{p_m(z)}{p_{m+1}(z) - 1} = -p_m(z),$$

which settles the identities in the off diagonal positions. Finally

$$p_{m-1}(z) + 1 + \left(\frac{\tilde{p}(z)}{\tilde{p}_{m+2}(z)}\right)^2 + \left(\frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)}\right)^4$$

$$= p_{m-1}(z) + 1 - (p_{m+1}(z) - 1)\left(1 + (\frac{\tilde{p}_m(z)}{\tilde{p}_{m+2}(z)})^2\right)$$

$$= d_m(z)\left(\tilde{p}_{m-2}(z) - \tilde{p}_{m+2}(z)\frac{r_m(z) + \tilde{p}_{m+2}(z)\tilde{p}_{m-2}(z) - r_m(z)}{r_m(z)}\right) = 0.$$

(Here we have used (13)). So the proposition is proven.

We can now state and prove the main result of this paper: Let, for $m \geq 3$ integral, z_m the complex number defined in proposition (4), $A_m = A_m(z_m), B_m = B_m(z_m), C_m = C_m(z_m)$, M_m the manifold defined in (20) with fundamental group as presented in (21), and P_m the polyhedron constructed in section 7.

Theorem 1 *The group Γ_m generated by $\{A_m, B_m, C_m\}$ is a lattice in $SL(2, \mathbf{C})$ with fundamental domain P_m. It is, as an abstract group, isomorphic to $\pi_1(M_m)$. Its character s_m is determined by the coordinates*

$$s_m(\xi) = z_m,\ \ s_m(\eta) = z_m\frac{\tilde{p}_m(z_m)}{\tilde{p}_{m+2}(z_m)},\ \ s_m(\eta\xi) = z_m,$$

$$s_m(\mu) = 2,\ s_m(\mu^{-1}\xi) = 2\frac{\tilde{p}_{m-2}(z_m)}{\tilde{p}_m(z_m)},\ s_m(\mu^{-1}\eta) = 2,\ s_m(\mu^{-1}\eta\xi) = 2\frac{\tilde{p}_{m-2}(z_m)}{\tilde{p}_m(z_m)}.$$

The basis of the proof is, of course, Poincaré 's criterion. We note where the Möbius transformations A_m, B_m, and C_m move the vertices of P_m:

$$A_m : \begin{array}{c} \infty \\ w_{m,j} \end{array} \longrightarrow \begin{array}{c} 0 \\ w_{m,j+1} \end{array}$$

where $\frac{m+1}{2} \leq j \leq \frac{m-1}{2}$ for odd m, and $-\frac{m}{2} \leq j \leq \frac{m}{2}$ for m even.

$$B_m : \begin{array}{c} 0 \\ \infty \\ w_{m,\frac{m+1}{2}} \end{array} \longrightarrow \begin{array}{c} 0 \\ w_{m,-\frac{m+1}{2}} \\ w_{m,\frac{m-1}{2}} \end{array} \text{ resp. } \begin{array}{c} 0 \\ \infty \\ w_{m,\frac{m+2}{2}} \end{array} \longrightarrow \begin{array}{c} 0 \\ w_{m,-\frac{m}{2}} \\ w_{m,-\frac{m-2}{2}} \end{array}$$

$$C_m : \begin{array}{c} \infty \\ 0 \\ w_{m,-\frac{m+1}{2}} \end{array} \longrightarrow \begin{array}{c} \infty \\ w_{m,\frac{m+1}{2}} \\ w_{m,\frac{m-1}{2}} \end{array} \text{ resp. } \begin{array}{c} \infty \\ 0 \\ w_{m,-\frac{m}{2}} \end{array} \longrightarrow \begin{array}{c} \infty \\ w_{m,\frac{m+2}{2}} \\ w_{m,\frac{m}{2}} \end{array}$$

where the first alternative for B_m and C_m is for odd, the second for even m.

The proof of the theorem now is easy. We have, among the edges of P_m to be identified, $m-1$ pairs:

$$\overline{\infty, w_{m,j}} \xrightarrow{A_m} \overline{0, w_{m,j+1}} \xrightarrow{A_m^{-1}} \overline{\infty, w_{m,j}}$$

for $-\frac{m-1}{2} \leq j \leq \frac{m-3}{2}$ (m odd) or $-\frac{m-2}{2} \leq j \leq \frac{m-2}{2}$ (m even). Here $\sphericalangle\overline{\infty, w_{m,j}} + \sphericalangle\overline{0, w_{m,j+1}} = 2\pi$ by construction. There are two more equivalence classes of edges: First

$$\overline{\infty, w_{m,-\frac{m-1}{2}}} \xrightarrow{C_m} \overline{\infty, w_{m,\frac{m-1}{2}}} \xrightarrow{A_m} \overline{0, w_{m,\frac{m+1}{2}}}$$

$$\xrightarrow{B_m} \overline{0, w_{m,-\frac{m-1}{2}}} \xrightarrow{A_m^{-1}} \overline{\infty, w_{m,-\frac{m+1}{2}}}$$

for odd m and

$$\overline{\infty, w_{m,-\frac{m}{2}}} \xrightarrow{C_m} \overline{\infty, w_{m,\frac{m}{2}}} \xrightarrow{A_m} \overline{0, w_{m,\frac{m+2}{2}}}$$

$$\xrightarrow{B_m} \overline{0, w_{m,-\frac{m-2}{2}}} \xrightarrow{A_m^{-1}} \overline{\infty, w_{m,-\frac{m}{2}}}$$

for even m. The second class is, again noted separately for the two pairities of m:

$$\overline{\infty, 0} \xrightarrow{C_m} \overline{\infty, w_{m,\frac{m+1}{2}}} \xrightarrow{B_m} \overline{w_{m,-\frac{m+1}{2}}, w_{m,-\frac{m-1}{2}}} \xrightarrow{A_m} \overline{w_{m,-\frac{m+1}{2}+1}, w_{m,-\frac{m+1}{2}+2}} \xrightarrow{A_m}$$

$$\ldots \xrightarrow{A_m} \overline{w_{m,-\frac{m+1}{2}+j-1}, w_{m,m-\frac{m+1}{2}+j}} \xrightarrow{A_m} \overline{w_{m,-\frac{m+1}{2}+j}, w_{-\frac{m+1}{2}+j+1}} \xrightarrow{A_m}$$

$$\ldots \xrightarrow{A_m} \overline{w_{m,-\frac{m+1}{2}+m}, w_{m,-\frac{m+1}{2}+m+1}} \xrightarrow{C_m^{-1}} \overline{w_{m,\frac{m+1}{2}}, 0} \xrightarrow{B_m} \overline{\infty, 0}$$

(m odd), and

$$\overline{\infty, 0} \xrightarrow{C_m} \overline{\infty, w_{m,\frac{m+2}{2}}} \xrightarrow{B_m} \overline{w_{m,-\frac{m}{2}}, w_{m,-\frac{m-2}{2}}} \xrightarrow{A_m} \overline{w_{m,-\frac{m}{2}+1}, w_{m,-\frac{m}{2}+2}} \xrightarrow{A_m}$$

$$\ldots \xrightarrow{A_m} \overline{w_{m,-\frac{m}{2}+j-1}, w_{m,-\frac{m}{2}+j}} \xrightarrow{A_m} \overline{w_{m,-\frac{m}{2}+j}, w_{m,-\frac{m}{2}+j+1}} \xrightarrow{A_m}$$

$$\ldots \xrightarrow{A_m} \overline{w_{m,-\frac{m}{2}+m}, w_{m,-\frac{m}{2}+m+1}} \xrightarrow{C_m^{-1}} \overline{w_{m,-\frac{m}{2}}, 0} \xrightarrow{B_m^{-1}} \overline{\infty, 0}$$

(m even). In all situations the angles, by table (3), add up to 2π. Checking the gluing conditions on the edges now requires to remember proposition (5), only. So $\Gamma_m \backslash \mathbf{H}^3$ is a complete hyperbolic three dimensional manifold with fundamental group Γ_m which, by Poincaré's criterion has the following presentation:

$$\Gamma_m \cong \langle A_m, B_m, C_m \mid A_m^{-1} B_m A_m C_m,\ B_m^{-1} C_m^{-1} A_m^m B_m C_m \rangle.$$

But this group is isomorphic to $\pi_1(M_m)$:

$$\varphi_m : \begin{array}{c} \xi \\ \eta \\ \mu \end{array} \longrightarrow \begin{array}{c} A_m \\ C_m B_m \\ C_m \end{array}$$

is easily seen to extend to an isomorphism. Its inverse is determined by

$$\varphi_m^{-1}: \begin{array}{c} A_m \\ B_m \\ C_m \end{array} \longrightarrow \begin{array}{c} \xi \\ \mu^{-1}\eta \\ \mu \end{array} .$$

So $\Gamma_m \backslash \mathbf{H}^3$ and M_m are homeomorphic.

9 Illustration, examples, remarks

We treat two examples: the cases $m = 5$ and $m = 13$. Let first $m = 5$. We have

$$f_5(z) = \tilde{p}_7^2(z) - r_5(z) = z^6 - 2z^5 - 3z^4 + 6z^3 + 2z^2 - 5z + 3.$$

It is irreducible. Table (4) contains some numerical material relevant in this case, especially the numerical values of the roots z_5, z_5', z_5'' of $f_5(z)$, (imaginary parts taken negative). Figure (5) shows the shape of the polyhedron P_5 viewed from the point at infinity. It is composed from 6 ideal tetrahedra with vertices located at $\{\infty, 0, w_{5,-3}, w_{5,-2}, w_{5,-1}, w_{5,0}, w_{5,1}, w_{5,2}, w_{5,3}\}$. We have taken as a label for $w_{m,j}$ the index j, $-3 \leq j \leq 3$. Figures (6) and (7) show the "'polyhedra"' P_5', P_5'' constructed using, in total analogy, formula (47), but replacing z_5 by z_5' or z_5'', respectively.

Table 4:

z_5	=	1.83745-.094171 i	$\vartheta_5 = \arccos \frac{z_5}{2}$	=	.420985+.114926 i
z_5'	=	.483571-.496547 i	$\vartheta_5' = \arccos \frac{z_5'}{2}$	=	1.33433+.252683 i
z_5''	=	-1.32102-.315775 i	$\vartheta_5'' = \arccos \frac{z_5''}{2}$	=	2.27417+.20557 i

$e^{i\vartheta_5}$	=	.813566+.364278 i	$e^{-i\vartheta_5}$	=	1.02388-.458449 i
$e^{i\vartheta_5'}$	=	.18196+.7551 i	$e^{-i\vartheta_5'}$	=	.301615-1.25165 i
$e^{i\vartheta_5''}$	=	-.526609+.620949 i	$e^{-i\vartheta_5''}$	=	-.794409-.936724 i

j	$w_{5,j}$	$w_{5,j}'$	$w_{5,j}''$
3	.878575 + 1.02982 i	.199392 1 +.1677 i	.577953 +1.10523 i
2	1.35799 + .467828 i	.341481 + .335576 i	-.949481 +.394726 i
1	1.17919 +.1326 i	-1.00618 + .967442 i	-.423015 +.057551 i
0	1	1	1
-1	.83745 -.094171 i	-.516429 -.496547 i	-2.32102 -.315775 i
-2	.65826 -.226771 i	1.48975 -1.46399 i	-.898005 -.373326 i
-3	.479459 -.561999 i	.14209 -.832133 i	-.371539 -.710501 i

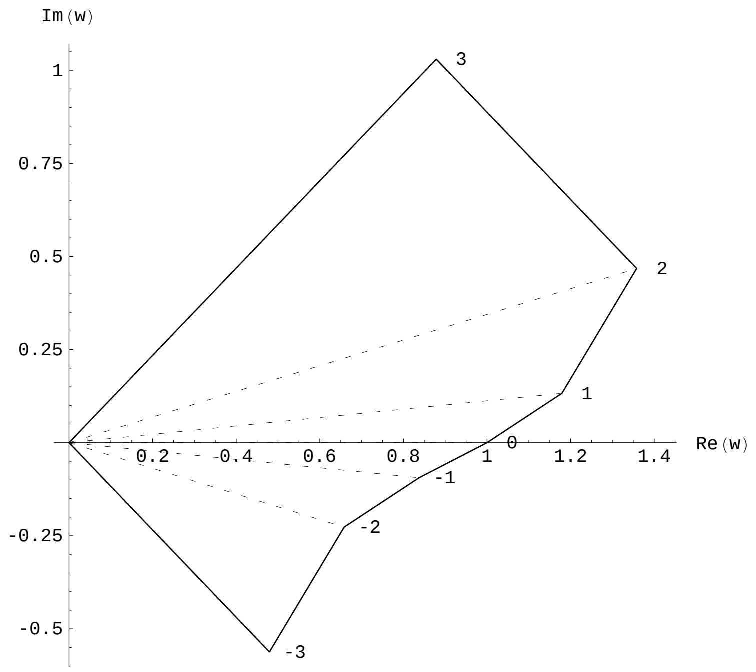

Figure 5

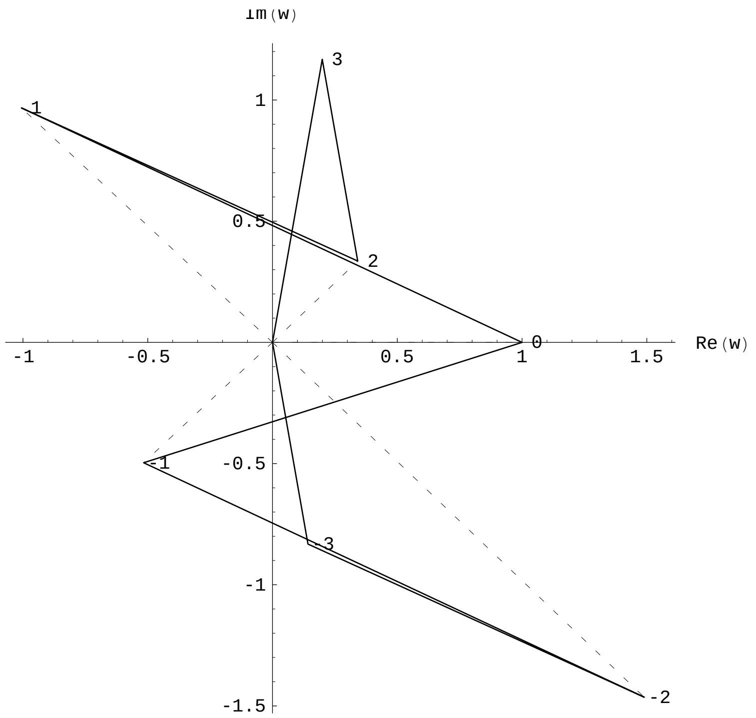

Figure 6

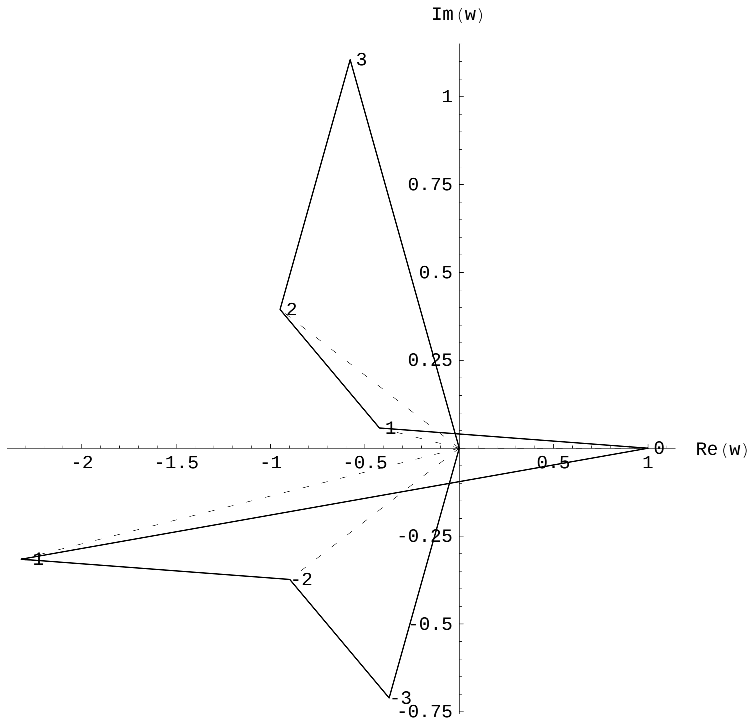

Figure 7

$m = 13$ leads to the polynomial

$$f_{13}(z) = \tilde{p}_{15}^2 - r_{13}(z) = z^{14} - 2z^{13} - 11z^{12} + 22z^{11} + 46z^{10} - 92z^9$$

$$-91z^8 + 182z^7 + 86z^6 - 172z^5 - 34z^4 + 68z^3 + 4z^2 - 9z + 3.$$

Maple asserts that it is also irreducible. Its roots are (imaginary parts taken negative)

1.95844-.011087i	1.63062-.08632i	1.01476-.181169i
-.212857-.215849i	-1.35374-.13558i	-1.83439-.041984i

Figure (8) shows how K_{13} isolates $z_{13} = 1.95844 - .011087i$ from the other roots. The location of $w_{13,j}, -7 \le j \le 7$, is contained in table (5), and figure (9) shows the polyhedron P_{13}.

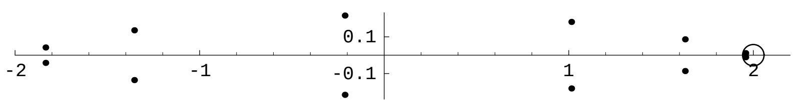

Figure 8

Table 5:

j	$w_{13,j}$
7	.973902+1.01484i
6	1.46617+.501877i
5	1.34793+.197894i
4	1.23222+.095532i
3	1.15175+.051455i
2	1.09193+.027625i
1	1.04322+.011068i
0	1

j	$w_{13,j}$
-1	.958444-.011087i
-2	.915225-.023155i
-3	.866516-.038712i
-4	.806695-.062542i
-5	.726225-.106619i
-6	.610513-.208981i
-7	.492271-.512964i

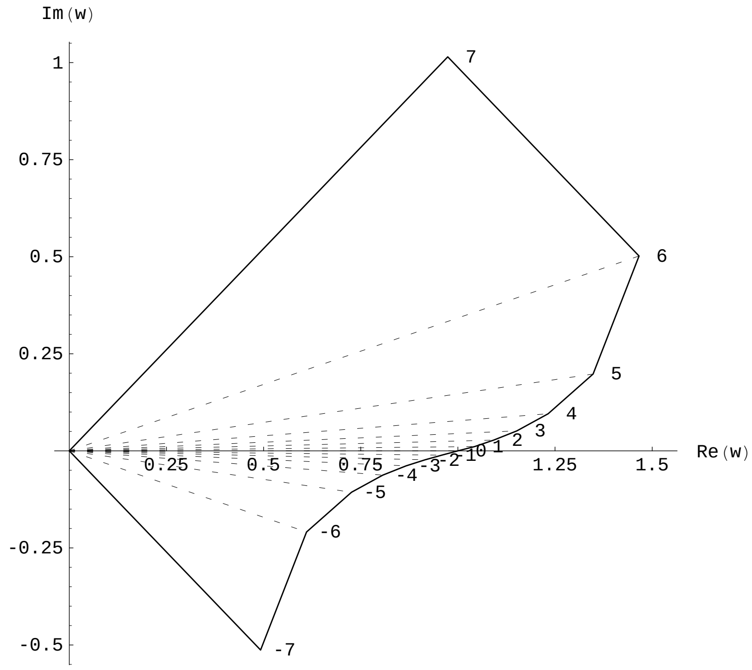

Figure 9

It is worthwhile noting that the character s_m on $\pi_1(M_m)$ has a limit as $m \longrightarrow \infty$, compare [5]:

$$z_\infty = \lim_{m\to\infty} z_m = 2$$

obviously. We set $A_\infty = \lim_{m\to\infty} A_m = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$. The trace $y_m = z_m \frac{\tilde{p}_m(z_m)}{\tilde{p}_{m+2}(z_m)}$ satisfies the Markoff equation of section 4:

$$z_m^2 y_m = 2z_m^2 + y_m^2,$$

so $y_\infty = \lim_{m\to\infty}$ solves the equation

$$y_\infty^2 - 4y_\infty + 8 = 0.$$

So $y_\infty = 2(1 \pm i)$. By a computation similar to the one in the proof of lemma (4) we can show

$$Im\ y_m > 0 \text{ for all } m \geq 1,$$

so $y_\infty = 2(1+i)$, namely:

$$Im\ y_m = Im \frac{\tilde{p}_{m-2}(z_m)}{\tilde{p}_{m+2}(z_m)} = Im \frac{\cos \frac{m-2}{2}\vartheta_m}{\cos \frac{m+2}{2}\vartheta_m}$$

which leads to

$$2|\cos \frac{m+2}{2}\vartheta_m|^2 Im\ y_m = \sin m\alpha_m \sinh 2\beta_m + \sin \alpha_m \sinh 2m\beta_m$$

whose summands, by lemma (2), both are obviously positive.

The polyhedron P_m behaves controllably when sending m to infinity; it will become an ideal polyhedron P_∞ composed from infinitely many ideal tetrahedra: Let

$$w_k = \begin{cases} \frac{k^2+k+1+i}{k^2+1}, & k \geq 0, \\ \frac{k^2+k+1-i}{k^2+1}, & k < 0. \end{cases}$$

These points lie on two circles in the upper respectively lower half of the complex w-plane, being tangent to the real axis at $w = 1$:

$$w_k \in \begin{cases} \{w \in \mathbf{C} \mid |w - \frac{2+i}{2}| = \frac{1}{2}, Re\ w > 1\}, & k \geq 0, \\ \{w \in \mathbf{C} \mid |w - \frac{2-i}{2}| = \frac{1}{2}, Re\ w < 1\}, & k < 0. \end{cases}$$

Then, by an easy computation,

$$A_\infty^{-1} w_k = w_{k+1} \text{ for } k \geq 0,\ A_\infty w_k = w_{k-1} \text{ for } k < 0.$$

Furthermore, $B_\infty = \lim_{m\to\infty} B_m = \begin{pmatrix} 1 & 0 \\ 1+i & 1 \end{pmatrix}$, $C_\infty = \lim_{m\to\infty} C_m = \begin{pmatrix} 1 & 1+i \\ 0 & 1 \end{pmatrix}$ act as follows:

$$B_\infty : \begin{matrix} 0 \\ \infty \\ 1+i = w_0 \end{matrix} \longrightarrow \begin{matrix} 0 \\ \frac{1-i}{2} = w_{-1} \\ \frac{3-i}{5} = w_{-2} \end{matrix},\quad C_\infty : \begin{matrix} \infty \\ 0 \\ \frac{1-i}{2} = w_{-1} \end{matrix} \longrightarrow \begin{matrix} \infty \\ 1+i = w_0 \\ \frac{3+i}{2} = w_1 \end{matrix}.$$

Call Γ_∞ the group generated by $\{A_\infty, B_\infty, C_\infty\}$; it is an arithmetic lattice with fundamental polyhedron P_∞.

In figure (10) we have drawn a more standard fundamental domain for this situation; it shows actually a fundamental domain for a conjugate of Γ_∞ which makes it compatible with the notation and pictures in [1]: For $T = \begin{pmatrix} 1 & -2 \\ 0 & -2 \end{pmatrix}$ put

$$X = TA_\infty T^{-1} = \begin{pmatrix} 2 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix},$$

$$Y = TC_\infty B_\infty T^{-1} = \begin{pmatrix} -1 & \frac{3+i}{2} \\ -2+2i & 3-2i \end{pmatrix},$$

$$M = TC_\infty T^{-1} = \begin{pmatrix} 1 & \frac{-1+i}{2} \\ 0 & 1 \end{pmatrix}.$$

Then

$$L = Y^{-1}X^{-1}YX = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, \; MXM^{-1} = \begin{pmatrix} 1+i & \frac{1}{2} \\ 2 & 1-i \end{pmatrix}.$$

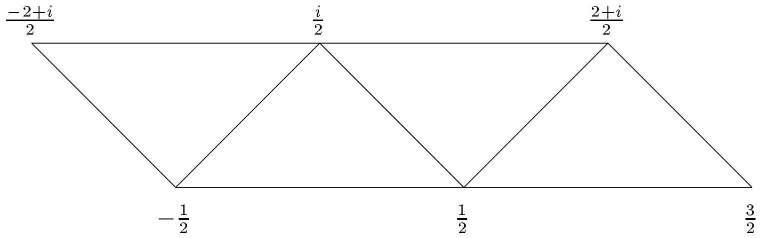

Figure 10

The fundamental domain consists of four ideal tetrahedra; we note the action on the vertices of the four isometries which pair the faces:

$$X: \begin{array}{c} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{i}{2} \end{array} \longrightarrow \begin{array}{c} \frac{3}{2} \\ \frac{1}{2} \\ \frac{2+i}{2} \end{array}, \; MXM^{-1}: \begin{array}{c} -\frac{1}{2} \\ \frac{i}{2} \\ \frac{-2+i}{2} \end{array} \longrightarrow \begin{array}{c} \frac{1}{2} \\ \frac{i}{2} \\ \frac{2+i}{2} \end{array},$$

$$M: \begin{array}{c} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{array} \longrightarrow \begin{array}{c} \frac{-2+i}{2} \\ \frac{i}{2} \\ \frac{2+i}{2} \end{array}, \; L: \begin{array}{c} -\frac{1}{2} \\ \frac{-2+i}{2} \end{array} \longrightarrow \begin{array}{c} \frac{3}{2} \\ \frac{2+i}{2} \end{array}.$$

We verify Poincaré's criterion and derive a presentation of Γ_∞: The edges of the fundamental domain in figure (10) again are all hyperbolic lines. The ones with both feet finite fall into two classes along which the angle and gluing conditions are guaranteed by the choice of the matrices X and M; the group relator is read off from the angular condition along the class of vertical edges with feet in the four vertices of the parallelogram which group theoretically is nothing but $LM = ML$:

$$\overline{-\frac{1}{2}, \frac{1}{2}} \xrightarrow{X} \overline{\frac{3}{2}, \frac{1}{2}} \xrightarrow{M} \overline{\frac{2+i}{2}, \frac{i}{2}} \xrightarrow{(MXM^{-1})^{-1}} \overline{-\frac{2+i}{2}, \frac{i}{2}} \xrightarrow{M^{-1}} \overline{-\frac{1}{2}, \frac{1}{2}},$$

$$\overline{-\frac{1}{2}, \frac{-2+i}{2}} \xrightarrow{MXM^{-1}} \overline{\frac{1}{2}, \frac{2+i}{2}} \xrightarrow{X^{-1}} \overline{\frac{1}{2}, \frac{i}{2}}$$

$$\xrightarrow{(MXM^{-1})^{-1}} \overline{-\frac{1}{2}, \frac{i}{2}} \xrightarrow{X} \overline{\frac{3}{2}, \frac{2+i}{2}} \xrightarrow{L^{-1}} \overline{-\frac{1}{2}, \frac{-2+i}{2}},$$

$$\overline{\infty, -\frac{1}{2}} \xrightarrow{L} \overline{\infty, \frac{3}{2}} \xrightarrow{M} \overline{\infty, \frac{2+i}{2}} \xrightarrow{L^{-1}} \overline{\infty, \frac{-2+i}{2}} \xrightarrow{M^{-1}} \overline{\infty, -\frac{1}{2}}.$$

This yields the presentation

$$\Gamma_\infty \cong \langle M, X \mid MXMX^{-1}M^{-1}X^{-1}MXM^{-1}X^{-1}M^{-1}XMXM^{-1}X^{-1} \rangle.$$

$\Gamma_\infty \backslash \mathbf{H}^3$ is well known to be the Whitehead link complement, see [5]. The fact that the fundamental domain P_∞ comes up in a natural manner is the source of interesting analytic phenomena; these will be presented elsewhere.

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