Dirichlet Operators with Variable Coefficients in L^p-spaces of Functions of Infinitely Many Variables

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0. Introduction

In this paper we study the (minus-)generator $\mathcal{A}_p$ of the C_0-semigroup $e^{\mathcal{A}_p t}$ on $L^p(X,\mu)$ associated with a classical symmetric diffusion form $\mathcal{E}$ in $L^2(X,\mu)$, on a (infinite dimensional) locally convex state space X endowed with a finite differentiable Borel measure μ. It is a standard fact that $\mathcal{E}$ associates a C_0 sub-Markovian semigroup of contractions on $L^2(X,\mu)$ and the semigroup interpolates to all the L^p-spaces, $1 \le p < \infty$. Our main concern is the strong uniqueness for

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$(\mathcal{A}_p, \mathcal{D})$, where $\mathcal{D}$ is a lineal of smooth cylindric functions in $Dom\mathcal{A}_p$. One says that $(\mathcal{A}_p, \mathcal{D})$ is strongly unique if $\mathcal{A}_p$ is the only extension of $\mathcal{A}_p \restriction \mathcal{D}$ generating a C_0-semigroup on $L^p(X, \mu)$. In the case $p = 2$ it is equivalent to the essential self-adjointness of $\mathcal{A}_2 \restriction \mathcal{D}$.

The problem of strong uniqueness was addressed in many papers such as e.g. [AKR92], [AKR93], [AKR95], [BKR97], [DaPTu], [KT91], [L94], [L99], [LSe92], [LR99], [LT93], [S95], [W85] (see also references therein), for a survey we refer to [R98]. However, in all papers except for [KT91] and [LT93] the authors consider the case when X is a separable Hilbert space and the bilinear form of the coefficients is constant over X and associates a bounded operator with discrete spectrum on the intermediate Hilbert space. In [LT93] the authors consider finite dimensional X while the results of [KT91] do not include the case when $\mathcal{A}_p$ is the generalized Schrödinger operator.

The aim of this paper is to investigate the uniqueness problem for $\mathcal{A}_p$ in the case when X is a separable locally convex Hausdorff space rather than a separable Hilbert space and the bilinear form of the coefficients varies over X and need not be bounded on the intermediate Hilbert space. The method we use is inherited from [LSe92]. We show that $Range(\lambda + \mathcal{A}_p) \restriction \mathcal{D}$ is dense in $L^p(X, \mu)$ for $\lambda > 0$. Since $\mathcal{A}_p \restriction \mathcal{D}$ is accretive, strong uniqueness follows. To prove density of the range, we derive (dimension-independent) a-priori estimates of the gradient and Hessian of a cylinder solution of the correspondent elliptic equation. The idea of these estimates goes back to [KoSe90, Appendix]. For a development of the method to weighted spaces we refer to [LSe92], [LT93] and [L94].

We need rather complex framework to formulate our general uniqueness result, so we do it in the next section. Here we present the result for the case $dim\ X < \infty$. Then our framework is as follows. Let $\mu(dx) = \rho(x)dx$ with a weakly differentiable $\rho > 0$, $\beta := \nabla\rho/\rho$ and $\mathcal{E}$ be the closure of the Markovian form

$$\mathcal{E}(u, v) = \int_X \nabla u \cdot A(x) \cdot \nabla v \rho(x) dx, \quad u, v \in C_b^2(X),$$

where $A(x)$ is a bounded measurable weakly differentiable matrix-valued function, positively defined uniformly in $x \in X$. If $\beta, \nabla A \in L^{2\vee p}(X, \mu)$ then

$$\mathcal{A}_p \supset -(\nabla + \beta) \cdot A \cdot \nabla \restriction C_b^2(X).$$

Our result on strong uniqueness reads as follows.

Theorem 0.1. *Under the above assumptions, $(\mathcal{A}_p, C_b^\infty(X))$ is strongly unique in $L^p(X, \rho dx)$ provided*

(1) $A \in C_b^{0,\delta}(X),\ 0 < \delta \le 1;\ \nabla A, \beta \in L^{\frac{2q}{2-q}}(X, \rho dx),$ $\quad for\ 1 \le p \le q \le 6/5;$

(2) $A \in C_b^{0,\delta}(X),\ 0 < \delta \le 1;\ \nabla A, \beta \in L^{2q}(X, \rho dx),$ $\quad for\ 2 < p \le q < \infty;$

(3) $\nabla A, \beta \in L^{2p}(X, \rho dx),$ $\quad for\ 1 \le p \le q,\ 3/2 < q \le 2.$

For $p = 2$ Theorem 0.1 gives the result of [LT93].

The paper is organized as follows. In Section 1 we present the framework and formulate the main result. In Section 2 we prove the main theorem based upon a-priori smoothness estimates which are derived in Section 3. Some tedious technicalities are collected in Appendix.

1. Framework and the Main Results

Let E be a lineal of a countable dimension, $(\cdot,\cdot)_0$ be an inner product on E. Let the Hilbert space $\big(H_0, |\cdot|_0\big)$ be the completion of E in the pre-Hilbert topology, the Polish space $\hat{E}$ be the completion of E in the $\sigma(E,E)$-topology and X be a locally convex Hausdorff vector space such that E is dense in$X$ and the linear functional $(\cdot,h)_0$ is continuous in X for all $h \in E$. Then $\hat{E}^* = E$, $X \subset \hat{E}$ densely and continuously and $E \subset X^*$ densely in the w^*-topology. The $(E,\hat{E})$, (X^*,X) duality forms coincide on $E\times X$ and both coincide with the inner product in H_0 on $E \times H_0$ and $(X^*\cap H_0)\times(X\cap H_0)$, respectively. Therefore we will not distinguish between them when it does not lead to confusions. A similar background space is considered in [E97, Ch. 5]. By P (with a subindex) we further on denote a finite rank projector $\hat{E} \to E$ orthogonal in H_0.

Recall that, given $d \in \mathbb{N}$, $C_b^{m,\alpha}(\mathbb{R}^d)$, $m \in \mathbb{N}$, $0 < \alpha \le 1$ denotes class of m-times differentiable functions on $\mathbb{R}^d$ whose derivatives up to the order m are bounded continuous functions and the derivatives of order m are uniformly Hölder-continuous with the exponent α. Now, given a sub-lineal $F \subset E$, let

$$\mathcal{F}C_b^{m,\alpha}(F) = \Big\{x \mapsto f\big((l_1,x)_0,\dots,(l_d,x)_0\big);\ d\in\mathbb{N},\ f\in C_b^{m,\alpha}(\mathbb{R}^d), l_1,\dots,l_d \in F\Big\}.$$

For $u \in \mathcal{F}C_b^1(E)$, $u(x) = f\big((l_1,x)_0,\dots,(l_d,x)_0\big)$, Du denotes the Fréchet derivative

$$Du(x) = \sum_{k=1}^{n} (\partial_k f)\big((l_1,x)_0,\dots,(l_d,x)_0\big) l_k \in E \subset H_0,\ x \in X.$$

If, moreover, $u \in \mathcal{F}C_b^2(E)$, D^2u denotes the Hessian, the Fréchet derivative on Du:

$$D^2u(x) = \sum_{km=1}^{n} (\partial^2_{km} f)\big((l_1,x)_0,\dots,(l_d,x)_0\big) l_k l_m \in E\otimes E \subset B(H_0),\ x \in X.$$

Let μ be a finite measure on the weak Borel σ-algebra (the latter coincides with the strong one when X is a Souslin space, see [Ba70, Exp. 8,7]), $\operatorname{supp}\mu = X$. For $1 \le p \le \infty$, let $L^p = L^p(X,\mu)$. Denote $\langle f\rangle = \int f\,d\mu$ and $\langle f_1, f_2, \dots, f_n\rangle = \langle f_1 f_2 \dots f_n\rangle$. Our main assumption on μ is that it is differentiable along E in the sense that there exists a measurable function $\beta : X \to \hat{E}$, $(h,\beta)_0 \in L^2$ for every $h \in E$, such that

$$\int (h, \nabla\phi(x))_0 \mu(dx) = -\int \phi(x)\big(h, \beta(x)\big)_0 \mu(dx) \quad \forall \phi \in \mathcal{F}C_b^1(E), h \in E. \tag{1.1}$$

Following to [B95 Sect.7.1.8], we say that $u \in L^1$ is (weakly) differentiable along E if there exists a measurable function $\nabla u : X \to \hat{E}$, $(h, \nabla u)_0 \in L^2$ for every $h \in E$ such that, for all $h \in E$,

$$\int (h, \nabla\phi)_0 u d\mu = -\int \phi(h, \nabla u + u\beta)_0 d\mu \quad \forall \phi \in \mathcal{F}C_b^1(E). \tag{1.2}$$

Obviously, $\nabla u = Du$ if $u \in \mathcal{F}C_b^1(E)$. Higher derivatives along E are defined inductively.

Let $\boldsymbol{\nabla}$ be the operator mapping a weakly differentiable $\hat{E}$-valued function ξ to $\boldsymbol{\nabla}\xi(x) : E \to \hat{E}$ by

$$(g, \boldsymbol{\nabla}\xi(x)h)_0 = \Big(g, \nabla\big(h, \xi\big)_0\Big)_0(x), \quad \forall g, h \in E.$$

Observe that $\boldsymbol{\nabla}\xi(x) : \hat{E} \to \hat{E}$ if ξ is E-valued and $\boldsymbol{\nabla}\xi(x) : \hat{E} \to E$ if ξ is E-valued cylindric. Obviously, $\boldsymbol{\nabla}\nabla u = D^2 u$ if $u \in \mathcal{F}C_b^2(E)$. For a cylindric E-valued differentiable ξ we set

$$(\nabla, \xi)_0 = \sum_k (e_k, \boldsymbol{\nabla}\xi(x)e_k)_0$$

where (e_k) is a basis in E orthonormal in H_0. It is easy to see that $(\nabla, \xi)_0$ does not depend on choice of (e_k).

Let $a(x)[\cdot,\cdot]$ be a measurable field over X of positively defined symmetric bilinear forms on E. We assume that, for every finite rank projection $P : \hat{E} \to E$, $a(x)[Ph] \geq c_P |Ph|_0$ $\forall h \in E$, $x \in X$ with some $c_P > 0$ independent of x, and that, for all $h \in E$, $a[h]$ is in L^∞ and weakly differentiable along E.

Define

$$\begin{aligned} A(x) :& E \to \hat{E}, \quad \big(g, A(x)h\big)_0 := a(x)\big[g, h\big]; \\ A^{\boldsymbol{\nabla}}(x) :& E \to L(E \to \hat{E}) \quad \Big(f, A^{\boldsymbol{\nabla}}(x)\big[g\big]h\Big)_0 := \Big(f, \nabla a\big[g, h\big]\Big)_0(x). \end{aligned}$$

Observe that $\boldsymbol{\nabla}(A\xi) = (\boldsymbol{\nabla}\xi)A + A^{\boldsymbol{\nabla}}[\xi]$ for an E-valued weakly differentiable ξ.

Let $\big(H_a(x), |\cdot|_a(x), \big(\cdot,\cdot\big)_a(x)\big)$ denote the completion of the pre-Hilbert spaces $\big(E, a(x)[\cdot,\cdot]\big)$, $x \in X$. We denote $\mathrm{tr}_0\{\cdot\} := \mathrm{tr}_{H_0}\{\cdot\}$ and $\mathrm{tr}_a\{\cdot\} := \mathrm{tr}_{H_a}\{\cdot\}$. Observe that $\mathrm{tr}_0\{G\} = \mathrm{tr}_a\{G\}$ for every finite rank operator $G : E \to E$. By σ_0^2, σ_a^2 we denote the Hilbert-Schmidt space over H_0, H_a, respectively.

We assume that, for a.a. $x \in X$, $\beta(x) \in H_a(x)$ and $PA^{\nabla}(x)P$ belongs to the space of bounded operators $H_a(x) \to \sigma^2\big(H_a(x)\big)$ for every P. We denote by $[PA^{\nabla}P]_a(x)$ the norm of $PA^{\nabla}(x)P$ in this space. In the diagonal case

$$a(x)[h] = \sum_m \delta^{(m)}(x)|(e_m, h)_0|^2$$

where $\delta^{(m)} > 0, m \in \mathbb{N}$ and $\{e_m\} \subset E$ is a Schauder basis in H_0 ,

$$[A^{\nabla}]_a = \sup_P [PA^{\nabla}P]_a = \sup_m |\nabla \ln \delta^{(m)}|_a = \sup_m \frac{|\nabla \delta^{(m)}|_a}{\delta^{(m)}}.$$

Moreover, we assume that, for every $h \in E$, a.a. $x \in X$, $\mathrm{tr}_a\{A^{\nabla}(x)[h]\}$ is finite, and define $\partial a(x) \in \hat{E}$ by

$$\big(h, \partial a(x)\big)_0 := \mathrm{tr}_0\{A^{\nabla}(x)[h]\}\big(= \mathrm{tr}_a\{A^{\nabla}(x)[h]\}\big), \ \forall h \in E.$$

Further on we assume that there is a sequence (P_n) of finite rank projectors $\hat{E} \to E$, $P_n \nearrow Id$ on H_0 as $n \to \infty$, such that, for every $P \in (P_n)$,

$$a[Ph] \equiv \mathbb{E}_\mu\big(a[Ph]\big|P\big), \ PA^{\nabla}[h - Ph]P \equiv 0 \quad \forall h \in E. \tag{1.3}$$

For such P, let $a_P[\cdot] = a[P\cdot]$. Define A_P, A_P^{∇} and ∂a_P in the same way as A, A^{∇} and ∂a, respectively. Then $A_P = PAP$, $A_P^{\nabla}[\cdot] = A^{\nabla}[P\cdot]P = PA^{\nabla}[P\cdot]P = PA^{\nabla}[\cdot]P$ and $\partial a_P = P\partial a$. Let $\partial a^P \in PE$ be such that $(Ph, \partial a^P)_a = (h, \partial a_P)_0$. We assume that $|\partial a^P|_a$, $|\beta|_a \in L^2$ for all $P \in (P_n)$.

Consider the operator

$$\mathcal{L}v = -\mathrm{tr}_a\{D^2vA\} - (\nabla v, \beta)_a - (\nabla v, \partial a)_0 \qquad v \in \mathcal{F}C_b^2(E).$$

Given $v \in \mathcal{F}C_b^\infty(E)$, let $P \in (P_n)$ be such that $v \equiv v \circ P$. Then $P\nabla v = \nabla v$. Hence $\mathcal{L}v = -\mathrm{tr}_a\{D^2vA\} - (\nabla v, \beta + \partial a^P)_a \in L^2$ since $|\partial a^P|_a$, $|\beta|_a \in L^2$. So $\mathcal{L}$ is a densely defined operator in L^2. Integration by parts gives that

$$\langle u, \mathcal{L}v\rangle = \int a[\nabla u, \nabla v]d\mu, \ u, v \in \mathcal{F}C_b^\infty(E).$$

Hence the form

$$\mathcal{E}[u, v] = \int a[\nabla u, \nabla v]d\mu, \ u, v \in \mathcal{F}C_b^\infty(E).$$

is a closable Markovian form in L^2, in the sense of [FOT94, 1.1, $\mathcal{E}4$]. We will not distinguish between $\mathcal{E}$ and its closure unless it leads to confusions.

It follows from the Beurling-Deny criteria (see e.g. [FOT94, Th. 1.4.1]) that $\mathcal{E}$ associates a family of consistent sub-Markovian C_0-semigroups of contractions $e^{-\mathcal{A}_p t}$ on L^p, $1 \le p < \infty$. By the construction, $\mathcal{A}_2 \supset \mathcal{L}$ (in fact, $\mathcal{A}_2$ is the Friedrich extension of $\mathcal{L}$). Moreover, $\mathcal{A}_p \supset \mathcal{L}$ provided $|\beta|_a, |\partial a^P|_a \in L^{2\vee p}$ for all $P \in (P_n)$. Indeed, under this assumption $\mathcal{L}v \in L^p \cap L^2$ for every $v \in \mathcal{F}C_b^\infty(E)$. Therefore

$$\mathcal{L}v = L^p\text{-}\lim_{t\to 0} \frac{1}{t}\int_0^t e^{-\mathcal{A}_p s}\mathcal{L}v ds.$$

On the other hand,

$$\int_0^t e^{-\mathcal{A}_p s}\mathcal{L}v ds = \int_0^t e^{-\mathcal{A}_2 s}\mathcal{L}v ds = \int_0^t e^{-\mathcal{A}_2 s}\mathcal{A}_2 v ds = v - e^{-\mathcal{A}_2 t}v = v - e^{-\mathcal{A}_p t}v.$$

So $\mathcal{L}v = L^p\text{-}\lim\limits_{t\to 0} \frac{1}{t}(v - e^{-\mathcal{A}_p t}v)$ and $\mathcal{A}_p \supset \mathcal{L}$.

Under these assumptions our result on strong uniqueness reads as follows.

Theorem 1.1. $(\mathcal{A}_p, \mathcal{F}C_b^\infty(X))$ *is strongly unique in* $L^p(X, \rho dx)$ *provided*

$$(1) a \in \mathcal{F}C^{0,\delta}(E);\ |\partial a^P|_a \forall P, |\beta|_a \in L^{\frac{2q}{2-q}}, \qquad for\ 1 \le p \le q \le 2;$$
$$(2) a \in \mathcal{F}C^{0,\delta}(E); \sup_n \left\| [P_n A^{\nabla} P_n]_a + |\beta|_a \right\|_{2q} < \infty, \qquad for\ 2 < p \le q < \infty;$$
$$(3) \sup_n \left\| [P_n A^{\nabla} P_n]_a + |\beta|_a \right\|_{2q} < \infty, \qquad for\ 1 \le p \le q,\ 3/2 < q \le 2.$$

2. Proof of the uniqueness.

Since $\mathcal{L}$ is accretive, it suffices to show that the range of operator $(\lambda + \mathcal{L})$ is dense in L^p. The core of the proof is a-priori estimates of ∇v and $D^2 v$ for the solution $v \in \mathcal{F}C_b^2(E)$ of the second order elliptic differential equation

$$\lambda v - \mathrm{tr}_a\{D^2 v A\} - (\nabla v, b)_a = f. \tag{2.1}$$

Here $\lambda > 0$ and b is an E-valued $\mathcal{F}C_b(E)$-smooth function, $0 \ne f \in \mathcal{F}C_b(E)$. Actually (2.1) is a finite dimensional equation. Indeed, let P be such that (1.3) holds, $v = v \circ P$, $f = f \circ P$ and $b = Pb \circ P$. Then

$$\lambda v - \mathrm{tr}_0\{D^2 v A_P\} - (\nabla v, A_P b)_0 = f. \tag{2.1$'$}$$

Therefore $\|v\|_\infty \le \frac{\|f\|_\infty}{\lambda}$, due to the maximum principle. Observe that if $a, b, f \in \mathcal{F}C_b^{0,\delta}(E)$ then for every P be such that (1.3) holds, $f = f \circ P$ and $b = Pb \circ P$ there exists unique solution $v \in \mathcal{F}C_b^2(E)$ such that $v \equiv v \circ P$, see, e.g., [K96, Th. 4.3.2]).

The a-priori estimates are given in the following two theorems.

Theorem 2.1. *Let P be such that (1.3) holds, $v = v \circ P$, $f = f \circ P$ and $b = Pb \circ P$. Assume that $(b - \beta - \partial a^P)(x) \in H_a(x)$ μ-a.e. and $|b - \beta - \partial a^P|_a \in L^2(X, \mu)$. Then, for every $\lambda > 0$, there exists $C_\lambda > 0$, which depends only on λ , such that*

$$\| \, |Dv|_a \|_2 \le C_\lambda \left[\|f\|_2 + \|f\|_\infty \| \, |b - \beta - \partial a^P|_a \|_2 \right]. \tag{2.2}$$

Theorem 2.2. *Let P be such that (1.3) holds, $v = v \circ P$, $f = f \circ P$ and $b = Pb \circ P$. Assume that $\beta(x), (b - \partial a^P)(x) \in H_a(x)$ and $[PA^\nabla P]_a(x)$ is finite μ-a.e.*

For $p > \frac{3}{2}$, let $|b - \partial a^P|_a$, $|\beta|_a$ and $[PA^\nabla P]_a$ belong to $L^{2p}(X, \mu)$, so that

$$M_p := \| \, |b - \partial a^P|_a + |\beta|_a + [PA^\nabla P]_a \, \|_{2p} < \infty.$$

Then, for every $\lambda > 0$, there exists $C_{p,\lambda} > 0$, which depends only on p and λ, such that

$$\| \, |Dv|_a \|_{2p} \le C_{p,\lambda} \left[\sqrt{\|f\|_\infty \|f\|_p} + \|f\|_\infty M_p \right]. \tag{2.3}$$

Moreover, for $p = 2$,

$$\| \, |D^2 v A|_{\sigma_a^2} \|_2 \le C_\lambda \left[\|f\|_2 + \|f\|_\infty M_2^2 \right], \tag{2.4}$$

and, for $\frac{3}{2} < p < 2$,

$$\| \, |D^2 v A|_{\sigma_a^2} \|_p \le C_{p,\lambda} \left[\|f\|_p + \|f\|_\infty M_p^2 + \|f\|_\infty (\mu(X))^{\frac{1}{p}} \right]. \tag{2.5}$$

The proof is given in the next section.

Proof of Theorem 1.1. We present the proof of part (3) only. It is the most complex part; (1) and (2) can be proved by a simple modification of the argument.

We show that, for every $\lambda > 0$ and $f \in \mathcal{F}C_b^\infty(E)$ there is a net $(v_{nm}) \subset \mathcal{F}C_b^\infty(E)$ such that

$$\lim_{n\to\infty} \limsup_{m\to\infty} \langle (\lambda + \mathcal{L}) v_{nm}, g \rangle = 0 \qquad \forall g \in L^{q'}.$$

From this it will follow that $\mathrm{Range}(\lambda + \mathcal{L}) \restriction \mathcal{F}C_b^\infty(E)$ is dense in L^q and hence in every L^p, $1 \le p \le q$. Since $\mathcal{L} \restriction \mathcal{F}C_b^\infty(E)$ is accretive, it will prove the claim.

For $n \in \mathbb{N}$ let $\beta_n \in \mathcal{F}C_b^\infty(E)$ be an E-valued function such that $\| \, |\beta - \beta_n|_a \|_{2q} \le 1/n$. Then $\| \, |\beta_n|_a \|_{2q} \le \| \, |\beta|_a \|_{2q} + 1/n$.

Let P_n be such that (1.3) holds and $f = f \circ P_n$, $\beta_n = P_n \beta_n \circ P_n$. Let ν_n denote the Lebesgue measure on $P_n E$ and J_{nm} denote the standard mollifier on $L^1_{loc}(P_n E, \nu_n)$. Since $\beta \in L^{2q}$, $q \ge 1$, for every differentiable $g \in L^\infty$ such that $g = \mathbb{E}_\mu(g|P_n)$ and $\nabla g \in L^{2q}$, $J_{nm} g \to g$ in measure (both μ and ν_n) and $\nabla J_{nm} g \to \nabla g$ in L^{2q} as

$m \to \infty$. We refer to [CF96, Th. 2.7] for the proof in the case $q = 1$. An obvious modification based on [ARZ93, Prop. 1.5] extends this proof to the general case.

Let $a_n[h] := a[P_n h]$ and $a_{nm}[h] := a[h] + (J_{nm} - 1)a_n[h]$, $h \in E$. (Observe that $a_n[h] = \mathbb{E}_\mu\big(a_n[h]\big|P_n\big) \in L^\infty$, due to (1.3), so a_{nm} is well defined.) Without loss of generality we assume further on that $a_{nm}[h] \to a[h]$ and $\nabla a_{nm}[h] \to \nabla a[h]$ a.e. as $m \to \infty$ for every $h \in E$. Let $A_n(A_{nm})$, $A_n^{\boldsymbol{\nabla}}(A_{nm}^{\boldsymbol{\nabla}})$ and $\partial a_n(\partial a_{nm})$ be defined in the same way as A, $A^{\boldsymbol{\nabla}}$ and ∂a, respectively. Then $P_n A_{nm} P_n = J_{nm} A_n$, $P_n \partial a_{nm} = J_{nm} \partial a_n$.

By the assumption, $c_n^{-1} P_n \geq A_n(x) \geq c_n P_n$ a.e., $c_n > 0$, in the sense of self-adjoint operators on H_0. Since J_{nm} is Markovian, $c_n^{-1} P_n \geq P_n A_{nm}(x) P_n \geq c_n P_n$, with the same c_n. Let $v_{nm} \in \mathcal{F}C_b^\infty(P_n E)$ be the solution of the uniformly elliptic equation (over $P_n E$)

$$\lambda v_{nm} - (\beta_n + \nabla, P_n A_{nm} P_n \nabla v_{nm})_0 = f.$$

(We refer to [K96, Ch IV] for the theory of such equations.) Then

$$\begin{aligned}(\lambda + \mathcal{L})v_{nm} - f =&\operatorname{tr}_0\{D^2 v_{nm}(A_{nm} - A)\} + (\nabla v_{nm}, \partial a_{nm} - \partial a)_0 \\ &+ (\nabla v_{nm}, (A_{nm} - A)\beta_n)_0 + (\nabla v_{nm}, \beta_n - \beta)_a \\ =&\operatorname{tr}_0\{D^2 v_{nm}(J_{nm} - 1)A_n\} + (\nabla v_{nm}, \partial a_{nm} - \partial a_n)_0 \\ &+ (\nabla v_{nm}, ((J_{nm} - 1)A_n)\beta_n)_0 + (\nabla v_{nm}, \beta_n - \beta)_a.\end{aligned} \tag{2.6}$$

Consider the first term in the RHS of (2.6). By the Swartz inequality,

$$\big|\operatorname{tr}_0\{D^2 v_{nm}(J_{nm} - 1)A_n\}\big| \leq \big|D^2 v_{nm}\big|_{\sigma_0^2}\big|(J_{nm} - 1)A_n\big|_{\sigma_0^2}.$$

Since $c_n^{-1} P_n \geq P_n A_{nm}(x) P_n \geq c_n P_n$, we have

$$\big|D^2 v_{nm} A_{nm}\big|_{\sigma^2_{a_{nm}}} \equiv \big|(P_n A_{nm} P_n)^{1/2} D^2 v_{nm} |(P_n A_{nm} P_n)^{1/2}\big|_{\sigma_0^2} \geq c_n \big|D^2 v_{nm}\big|_{\sigma_0^2}.$$

Hence, by the Hölder inequality, for every $g \in L^{q'}$, we obtain

$$\big|\langle \operatorname{tr}_0\{D^2 v_{nm}(J_{nm} - 1)A_n\}, g\rangle\big| \leq c_n^{-1} \|\, |D^2 v_{nm} A_{nm}|_{\sigma_a^2}\|_q \big\|g\big|(J_{nm} - 1)A_n\big|_{\sigma_0^2}\big\|_{q'}.$$

Due to (2.4-5),

$$\|\, |D^2 v_{nm} A_{nm}|_{\sigma^2_{a_{nm}}}\|_q \leq C_{q,\lambda,f} + C_{q,\lambda,f} \big\|\, |\beta_n|_{a_{nm}} + |\beta|_{a_{nm}} + [P_n A_{nm}^{\boldsymbol{\nabla}} P_n]_{a_{nm}}\big\|_{2q}^2 \tag{2.7}$$

We pass to the limit in m in the RHS of (2.7). It is easy to check that, as $m \to \infty$,

$$|\beta_n|_{a_{nm}} \to |\beta_n|_a,\ |\beta|_{a_{nm}} \to |\beta|_a,\ [P_n A_{nm}^{\boldsymbol{\nabla}} P_n]_{a_{nm}} \to [P_n A^{\boldsymbol{\nabla}} P_n]_a \ a.e.$$

Furthermore, $|\beta_n|_{a_{nm}} \le c_n^{-1/2}|\beta_n|_0 \le c_n^{-1}|\beta_n|_a$ and

$$|\beta|_{a_{nm}} \le |P_n\beta|_{a_{nm}} + |\beta - P_n\beta|_{a_{nm}} \le c_n^{-1}|P_n\beta|_a + |\beta - P_n\beta|_a.$$

Hence, by the Lebesgue theorem, $\big\| \, |\beta_n|_{a_{nm}} + |\beta|_{a_{nm}} \big\|_{2q} \to \big\| \, |\beta_n|_a + |\beta|_a \big\|_{2q}$.

Consider $[P_n A^{\nabla}_{nm} P_n]_{a_{nm}}$. We have $[P_n A^{\nabla}_{nm} P_n]_{a_{nm}} \le c_n^{-1}[P_n A^{\nabla}_{nm} P_n]_0$. Since $[P_n A^{\nabla}_{nm} P_n]_0 \to [P_n A^{\nabla} P_n]_0$ in L^{2q} as $m \to \infty$, there is a $\Phi \in L^{2q}$ such that $[P_n A^{\nabla}_{nm} P_n]_0 \le \Phi$. Hence, by the Lebesgue theorem

$$\limsup_m \| \, |D^2 v_{nm} A_{nm}|_{\sigma^2_{a_{nm}}} \|_q \le C_{q,\lambda,f} + C_{q,\lambda,f} \big\| \, |\beta_n|_a + |\beta|_a + [P_n A^{\nabla} P_n]_a \big\|^2_{2q}$$

Also, $\big|(J_{nm} - 1)A_n\big|_{\sigma_0^2} \to 0$ a.e. as $m \to \infty$ and bounded uniformly. Hence, by the Lebesgue theorem, $g\big|(J_{nm} - 1)A_n\big|_{\sigma_0^2} \to 0$ in $L^{q'}$ as $m \to \infty$ for all $g \in L^{q'}$. Thus

$$\lim_m \langle \mathrm{tr}_0\{D^2 v_{nm}(J_{nm} - 1)A_n\}, g\rangle = 0 \qquad \forall g \in L^{q'}.$$

By a similar argument we prove that

$$\big\|(\nabla v_{nm}, (\partial a_{nm} - \partial a_n)_0 + (\nabla v_{nm}, ((J_{nm} - 1)A_n)\beta_n)_0\big\|_q \to 0$$

as $m \to \infty$ and

$$\limsup_m \big\|(\nabla v_{nm}, \beta_n - \beta)_a\big\|_q \le C_{q,\lambda,f} \| \, |\beta_n - \beta|_a \|_{2q}(1 + \| \, |\beta_n|_a + |\beta|_a + [P_n A^{\nabla} P_n]_a \big\|_{2q}).$$

Thus, for every $g \in L^{q'}$,

$$\lim_n \limsup_m |\langle(\lambda + \mathcal{L})v_{nm}, g\rangle| \le \lim_n C_{q,\lambda,f} \|g\|_{q'} \| \, |\beta_n - \beta|_a \|_{2q}(1 + \| \, |\beta|_a + [P_n A^{\nabla} P_n]_a \big\|_{2q}) = 0,$$

since $\sup_n \|[P_n A^{\nabla} P_n]_a\big\|_{2q} < \infty$. □

3. Proof of A-Priori Estimates

Throughout this section b is an E-valued $\mathcal{F}C_b(E)$-smooth function, $0 \ne f \in \mathcal{F}C_b(E)$, $\lambda > 0$, $v \in \mathcal{F}C_b^2(E)$ is a solution of the equation

$$\lambda v - \mathrm{tr}_a\{D^2 v A\} - (\nabla v, b)_a = f. \tag{3.1}$$

Then by the maximum principle $\|v\|_\infty \le (1/\lambda)\|f\|_\infty$.

Let P is such that (1.3) holds, $v \equiv v \circ P$, $f \equiv f \circ P$ and $b \equiv Pb \circ P$. For $a_P := a[P\cdot]$, define A_P and A_P^{∇} in the same way as A and A^{∇}, respectively. Let $\Delta_{a_P} v := (\nabla, A_P \nabla v)_0$ and $\partial a^P(x) \in H_a(x)$ be such that

$$(\partial a^P, h)_a(x) = \mathrm{tr}_a\{PA^{\nabla}[h]P\}(x) \equiv \mathrm{tr}_0\{A_P^{\nabla}(x)[h]\}.$$

Let $b_a = b - \partial a^P$. Then one can rewrite (3.1) as follows

$$\lambda v = \Delta_{a_P} v + (\nabla v, b_a)_a + f \tag{3.1$'$}$$

For $p \ge 1$, let $|\nabla v|^2_{a,\varepsilon} := |\nabla v|^2_a + \varepsilon^2$ with $\varepsilon = 0$ for $p \ge 2$ and $\varepsilon > 0$ otherwise. Set $\chi = |\nabla v|^{p-2}_{a,\varepsilon}$.

Lemma 3.1. *There exists C_λ which depends only on λ, such that*

$$\| \, |\nabla v|_a \|_2 \le C_\lambda \|f\|_2 + C_\lambda \|f\|_\infty \| \, |b_a - \beta|_a \|_2. \tag{3.2}$$

Moreover, for every $p > 1$, if $|b_a|, |\beta|_a \in L^{2p}$, then there exists $C_{p,\lambda}$ which depends only on p and λ, such that

$$\|v\|_p \le C_{p,\lambda} \|f\|_p + C_{p,\lambda} \|f\|_\infty \| \, |b_a|_a + |\beta|_a \|_{2p}^2 \tag{3.3}$$

and

$$\|v\|_2 \le C_{p,\lambda} \|f\|_2 + C_{p,\lambda} \|f\|_\infty \| \, |b_a|_a + |\beta|_a \|_{2p}^p \tag{3.4}$$

Remark. Note that (3.2) is the claim of Theorem 2.1. We will use (3.3) and (3.4) to prove Theorem 2.2.

Proof of Lemma 3.1. It follows from (3.1') that

$$\lambda \|v\|_p^p - \langle (\beta + \nabla, A_P \nabla v)_0, v|v|^{p-2} \rangle = \langle (b_a - \beta, A \nabla v)_0, v|v|^{p-2} \rangle + \langle f, v|v|^{p-2} \rangle.$$

Integration by parts gives

$$\lambda \|v\|_p^p + \frac{4}{pp'} \| \, |\nabla v |v|^{\frac{p-2}{2}} |_a \|_2^2 = \frac{2}{p} \langle (b_a - \beta, \nabla v |v|^{\frac{p-2}{2}})_a, v|v|^{\frac{p-2}{2}} \rangle + \langle f, v|v|^{p-2} \rangle. \tag{3.5}$$

By the Hölder and Young inequalities and the maximum principle, there exists $C_{p,\lambda} > 0$ which depends on p and λ only, such that

$$|\langle f, v|v|^{p-2} \rangle| \le (\lambda/3) \|v\|_p^p + C_{p,\lambda} \|f\|_p^p$$

and

$$\begin{aligned} \left| \frac{2}{p} \langle (b_a - \beta, A \nabla v |v|^{\frac{p-2}{2}})_0, v|v|^{\frac{p-2}{2}} \rangle \right| &\le \frac{2}{p} \| \, |\nabla v |v|^{\frac{p-2}{2}} |_a \|_2 \|v\|_p^{\frac{p-1}{2}} \|v\|_\infty^{\frac{1}{2}} \| \, |b_a - \beta|_a \|_{2p} \\ &\le \frac{2}{pp'} \| \, |\nabla v |v|^{\frac{p-2}{2}} |_a \|_2^2 + \frac{\lambda}{3} \|v\|_p^p + C_{p,\lambda} \|f\|_\infty^p \| \, |b_a - \beta|_a \|_{2p}^{2p} \end{aligned}$$

Apply these two estimates to (3.5) to get (3.3) and, with $p = 2$, (3.2). To get (3.4), put $p = 2$ in (3.5) and use the estimate

$$\begin{aligned} |\langle (b_a - \beta, A \nabla v)_0, v \rangle| &\le \langle |v|^{\frac{1}{p}} |b_a - \beta|_a, |\nabla v|_a, |v|^{\frac{1}{p'}} \rangle \\ &\le \| \, |\nabla v|_a \|_2^2 + \frac{\lambda}{3} \|v\|_2^2 + C_{p,\lambda} \|f\|_\infty^2 \| \, |b_a - \beta|_a \|_{2p}^{2p}. \end{aligned}$$

□

Proof of Theorem 2.2 for case $p \geq 2$. It follows from (3.1') and the Schwartz inequality that

$$\big\|\chi\Delta_{a_P} v\big\|_2 \leq \lambda\|\chi v\|_2 + \|\chi f\|_2 + \big\|\chi|\nabla v|_a|b_a|_a\big\|_2. \tag{3.6}$$

Combining (A.3) with (3.6) and applying the Hölder inequality, one has

$$\big\|\,|\nabla v|_a\big\|_{2p}^{2p} \leq C_{p,\lambda}\|f\|_\infty^2\big\|\,|\nabla v|_a\big\|_{2p}^{2p-4}\Big(\|v\|_p^2 + \|f\|_p^2 + \big\|\,|\nabla v|_a\big\|_{2p}^2 M_p^2.\Big)$$

Hence, by the Young inequality,

$$\big\|\,|\nabla v|_a\big\|_{2p}^4 \leq \frac{1}{2}\big\|\,|\nabla v|_a\big\|_{2p}^4 + C_{p,\lambda}\|f\|_\infty^2\big(\|v\|_p^2 + \|f\|_p^2\big) + C_{p,\lambda}\|f\|_\infty^4 M_p^4.$$

Thus (2.3) follows from (3.3).

Let now $p = 2$ so $\chi = 1$. Combine (A.4) with (3.6) (observe that $A_P^{\boldsymbol{\nabla}} = PA^{\boldsymbol{\nabla}}P$) and apply the Hölder inequality:

$$\big\||D^2vA|_{\sigma_a^2}\big\|_2^2 \leq C_\lambda\Big(\|v\|_2^2 + \|f\|_2^2 + \big\|\,|\nabla v|_a\big\|_4^2 M_p^2.\Big)$$

Substitute (2.3) and (3.3) with $p = 2$. $\square$

Proof of Theorem 2.2 for case $p < 2$. From (3.1'), the Schwartz inequality and the fact that $\chi \leq \varepsilon^{p-2}$ we get

$$\big\|\chi\Delta_{a_P} v\big\|_2 \leq \varepsilon^{p-2}\big(\lambda\|v\|_2 + \|f\|_2\big) + \big\|\chi|\nabla v|_a|b_a|_a\big\|_2. \tag{3.7}$$

Put $\varepsilon = \|f\|_\infty$ and combine (A.3) and (A.5) ($A_P^{\boldsymbol{\nabla}} = PA^{\boldsymbol{\nabla}}P$) with (3.7), (3.4) and the fact that $\|f\|_p^p \geq \|f\|_2^2\|f\|_\infty^{p-2}$ since $p < 2$ to obtain

$$\begin{aligned}\big\|\,|\nabla v|_{a,\varepsilon}^{p-1}|\nabla v|_a\big\|_2^2 \leq& C_{p,\lambda}\|f\|_\infty^p\|f\|_p^p + C_{p,\lambda}\|f\|_\infty^{2p}\big\||\beta|_a + |b_a|_a\big\|_2^2 \\ &+ C_{p,\lambda}\|f\|_\infty^2\big\|\chi|\nabla v|_a(|\beta|_a + |b_a|_a + [PA^{\boldsymbol{\nabla}}P]_a)\big\|_2^2\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\big\||D^2vA|_{\sigma_a^2}\big\|_p^p \leq& C_{p,\lambda}\|f\|_p^p + C_{p,\lambda}\|f\|_\infty^p\big\||\beta|_a + |b_a|_a\big\|_2^2 + C_p\|f\|_\infty^p\mu(X) \\ &+ C_{p,\lambda}\|f\|_\infty^{2-p}\big\|\chi|\nabla v|_a(|\beta|_a + |b_a|_a + [PA^{\boldsymbol{\nabla}}P]_a)\big\|_2^2.\end{aligned} \tag{3.9}$$

Observe that $\big(\chi|\nabla v|_a\big)^{p'} \leq |\nabla v|_{a,\varepsilon}^{p-1}|\nabla v|_a$ since $p < 2$. Hence, by the Hölder inequality,

$$\big\|\chi|\nabla v|_a\phi\big\|_2^2 \leq \big\|\,|\nabla v|_{a,\varepsilon}^{p-1}|\nabla v|_a\big\|_2^{\frac{2}{p'}}\|\phi\|_{2p}^2 \quad \forall\phi \in L^{2p}. \tag{3.10}$$

Apply (3.10) and the Young inequality to (3.8) to get

$$\big\|\,|\nabla v|_{a,\varepsilon}^{p-1}|\nabla v|_a\big\|_2^2 \leq C_{p,\lambda}\|f\|_\infty^p\|f\|_p^p + C_{p,\lambda}\|f\|_\infty^{2p}M_p^{2p}. \tag{3.11}$$

Finally, apply (3.10) and (3.11) to (3.9). $\square$

Appendix : Inequalities

Let $w \in \mathcal{F}C_b^2(E)$, P be such that (1.3) holds and $w \equiv w \circ P$. Let $A_P(x)$ be the self-adjoint operator on H_0 associated with (the closure of) $a(x)\big[P\cdot\big]$, A_P^{∇} be defined in the same way as A^{∇}, $\Delta_{a_P} w := (\nabla, A_P, \nabla w)_0$. Observe $|\nabla w|_a^2 = (\nabla w, A_P \nabla w)_0$ and $|D^2 w A|_{\sigma_a^2} = |D^2 w A_P|_{\sigma_a^2}$. For $p \geq 1$, let $|\nabla w|_{a,\varepsilon}^2 := |\nabla w|_a^2 + \varepsilon^2$ with $\varepsilon = 0$ for $p \geq 2$ and $\varepsilon > 0$ otherwise. Set $\chi = |\nabla w|_{a,\varepsilon}^{p-2}$. Then the following inequalities hold.

$$\begin{aligned} \| \, |\nabla w|_{a,\varepsilon}^{p-1} |\nabla w|_a \|_2^2 \leq \varepsilon^{2p-2} \|w\|_2^2 + 2|p-1\|^2 \|w\|_\infty^2 \big\| \, |\nabla w|_{a,\varepsilon}^{p-3} \big|\nabla |\nabla w|_a^2 \big|_a \big\|_2^2 \\ +2(\varepsilon^2 + \|w\|_\infty^2)(\|\chi \Delta_{a_P} w\|_2^2 + \|\chi |\nabla w|_a |\beta|_a \|_2^2). \end{aligned} \tag{A.1}$$

$$\begin{aligned} \Big| \big\|\chi |D^2 w A|_{\sigma_a^2} \big\|_2^2 + \big((p/2)-1\big) \big\| \, |\nabla w|_{a,\varepsilon}^{p-3} \big|\nabla |\nabla w|_a^2 \big|_a \big\|_2^2 - \big\|\chi \Delta_{a_P} w \big\|_2^2 \Big| \\ \leq 2 \big\|\chi |D^2 w A|_{\sigma_a^2} \big\|_2 \big\|\chi |\nabla w|_a (|\beta|_a + [A_P^{\nabla}]_a) \big\|_2 \\ +3\,|p-2|\, \big\|\chi \big|\nabla |\nabla w|_a \big|_a \big\|_2 \big(\big\|\chi |\nabla w|_a [A_P^{\nabla}]_a \big\|_2 + \big\|\chi \Delta_{a_P} w \big\|_2 \big) \\ + \big\|\chi \Delta_{a_P} w \big\|_2 \big\|\chi |\nabla w|_a |\beta|_a \big\|_2 + 2 \big\|\chi |\nabla w|_a (|\beta|_a + [A_P^{\nabla}]_a) \big\|_2^2 \end{aligned} \tag{A.2}$$

$$\begin{aligned} \| \, |\nabla w|_{a,\varepsilon}^{p-1} |\nabla w|_a \|_2^2 \leq \varepsilon^{2p-2} \|w\|_2^2 + C_p(\varepsilon^2 + \|w\|_\infty^2) \|\chi \Delta_{a_P} w\|_2^2 \\ + C_p(\varepsilon^2 + \|w\|_\infty^2) \|\chi |\nabla w|_a (|\beta|_a + [A_P^{\nabla}]_a)\|_2^2, \ p > 3/2. \end{aligned} \tag{A.3}$$

$$\big\| |D^2 w A|_{\sigma_a^2} \big\|_2^2 \leq C_2 \big\|\Delta_{a_P} w \big\|_2^2 + C_2 \big\| \ |\nabla w|_a (|\beta|_a + [A_P^{\nabla}]_a) \big\|_2^2 \tag{A.4}$$

$$\begin{aligned} \big\| |D^2 w A|_{\sigma_a^2} \big\|_p^p \leq \varepsilon^{p-2} \|w\|_2^2 + C_p \varepsilon^p \mu(X) + C_p \varepsilon^{-p} (\varepsilon^2 + \|w\|_\infty^2) \|\chi \Delta_{a_P} w\|_2^2 \\ + C_p \varepsilon^{-p} (\varepsilon^2 + \|w\|_\infty^2) \|\chi |\nabla w|_a (|\beta|_a + [A_P^{\nabla}]_a)\|_2^2, \ 3/2 < p < 2. \end{aligned} \tag{A.5}$$

Proof of A.1. Integration by parts gives

$$\begin{aligned} \langle (\nabla w, A_P \nabla w)_0, |\nabla w|_{a,\varepsilon}^{2p-2} \rangle = & - \langle w |\nabla w|_{a,\varepsilon}^{2p-2}, \Delta_{a_P} w + (\beta, \nabla w)_a \rangle \\ & - (p-1) \langle w, |\nabla w|_{a,\varepsilon}^{2p-4} (\nabla |\nabla w|_{a,\varepsilon}^2, \nabla w)_a \rangle. \end{aligned} \tag{A.6}$$

Observe that $\nabla |\nabla w|_{a,\varepsilon}^2 = \nabla |\nabla w|_a^2$. So $|(\nabla |\nabla w|_{a,\varepsilon}^2, A_P \nabla w)_0| \leq |\nabla w|_a \big|\nabla |\nabla w|_a^2 \big|_a$, due to the Schwartz inequality. Thus (the absolute value of) the last term in the right hand side of (A.6) is estimated with

$$\begin{aligned} |p-1|\, \|w\|_\infty \langle |\nabla w|_{a,\varepsilon}^{p-1} |\nabla w|_a, |\nabla w|_{a,\varepsilon}^{p-3} \big|\nabla |\nabla w|_a^2 \big|_a \rangle \\ \leq \frac{1}{4} \| \, |\nabla w|_{a,\varepsilon}^{p-1} |\nabla w|_a \|_2^2 + |p-1|^2 \|w\|_\infty^2 \big\| \, |\nabla w|_{a,\varepsilon}^{p-3} \big|\nabla |\nabla w|_a^2 \big|_a \big\|_2^2, \end{aligned} \tag{A.7}$$

by the Schwartz inequality.

Consider the first term in the r.h.s. of (A.6). $|\nabla w|^2_{a,\varepsilon} = |\nabla w|^2_a + \varepsilon^2$, by the definition, and $|(\beta, \nabla w)_a| \le |\nabla w|_a |\beta|_a$, by the Schwartz inequality. Therefore (the absolute value of) the term does not exceed

$$\begin{aligned} &\langle \varepsilon |w| \chi,\, \varepsilon\chi(|\Delta_{a_P} w| + |\nabla w|_a |\beta|_a)\rangle \\ &\quad + \big\langle \chi |\nabla w|^2_a,\, \|w\|_\infty \chi(|\Delta_{a_P} w| + |\nabla w|_a|\beta|_a)|\big\rangle \le (1/4)\|\chi |\nabla w|^2_a\|^2_2 \\ &\quad + (\varepsilon^2/4)\|\chi w\|^2_2 + (\|w\|^2_\infty + \varepsilon^2)(\|\chi \Delta_{a_P} w\|^2_2 + \|\chi |\nabla w|_a |\beta|_a)\|^2_2), \end{aligned} \tag{A.8}$$

by the Young inequality. Thus, it follows from (A.6-8) that

$$\begin{aligned} &\|\,|\nabla w|^{p-1}_{a,\varepsilon} |\nabla w|_a\|^2_2 \le \frac{\varepsilon^2}{4}\|\chi w\|^2_2 + \frac{1}{4}\|\chi|\nabla w|^2_a\|^2_2 + \frac{1}{4}\|\,|\nabla w|^{p-1}_{a,\varepsilon}|\nabla w|_a\|^2_2 \\ &+ |2p-2|^2 \|w\|^2_\infty \Big\|\,|\nabla w|^{p-3}_{a,\varepsilon} \big|\nabla |\nabla w|^2_a\big|_a\Big\|^2_2 + (\varepsilon^2 + \|w\|^2_\infty)(\|\chi\Delta_{a_P} w\|^2_2 + \|\chi|\nabla w|_a|\beta|_a\|^2_2). \end{aligned}$$

Now observe that $\varepsilon^2\chi^2 \le \varepsilon^{2p-2}$. Both values vanish when $p \ge 2$ since $\varepsilon = 0$ and $\chi \le \varepsilon^{p-2}$ when $p < 2$. So $\varepsilon^2\|\chi w\|^2_2 \le \varepsilon^{2p-2}\|w\|^2_2$. Further, it is obvious that $\chi|\nabla w|^2_a \le |\nabla w|^{p-1}_{a,\varepsilon}|\nabla w|_a$ since $p \ge 1$. Therefore A.1 follows. □

Proof of (A.2). First we give a brief proof for the case when the background space is finite dimensional, $\mu(dx) = dx$ and $a[\cdot,\cdot]$ coincides with $(\cdot,\cdot)_0$. Consider $\langle \mathrm{div}(\chi^2 \nabla w),\, \Delta w\rangle$. After a twice integration by parts it takes the form

$$\big\|\chi|D^2 w|_{\sigma^2}\big\|^2_2 + \big((p/2) - 1\big)\big\|\,|\nabla w|^{p-3}_\varepsilon \big|\nabla|\nabla w|^2\big|\big\|^2_2$$

On the other hand, remove the brackets in the original expression to get

$$\langle \mathrm{div}(\chi^2\nabla w),\, \Delta w\rangle = \big\|\chi\Delta w\big\|^2_2 + (2p-4)\langle(\nabla|\nabla w|, \nabla w)|\nabla w|^{p-4}_\varepsilon, \chi\Delta w\rangle.$$

So

$$\begin{aligned} &\Big|\big\|\chi|D^2 w|_{\sigma^2}\big\|^2_2 + \big((p/2)-1\big)\big\|\,|\nabla w|^{p-3}_\varepsilon\big|\nabla|\nabla w|^2\big|\big\|^2_2 - \big\|\chi\Delta w\big\|^2_2\Big| \\ &\qquad \le \Big|(2p-4)\langle(\nabla|\nabla w|, \nabla w)|\nabla w|^{p-4}_\varepsilon, \chi\Delta w\rangle\Big| \\ &\qquad \le |2p-4|\,\big\|\chi\big|\nabla|\nabla w|\big|\big\|_2\big\|\chi\Delta w\big\|_2, \end{aligned}$$

due to the Schwartz and Hölder' inequalities.

In the general case $\Delta_{a_P} w$ and $\nabla + \beta$ replace Δw and div, respectively. Consider the expression

$$\langle \Delta_{a_P} w,\, (\nabla + \beta, \chi^2 A_P \nabla w)_0\rangle. \tag{A.9}$$

For simplicity, let us choose a basis $\{e_k\}$ in Range P orthonormal in H_0 and denote $a^{ij} := a(x)[e_i, e_j]$, $w_{e_j} := (e_j, \nabla w)_0$ and $\beta_{e_j} := (e_j, \beta)_0$, $i, j = 1 \ldots rank\, P$. Then (A.9) equals to

$$\begin{aligned} &\langle \nabla_{e_i}(a^{ij}\, w_{e_j}), (\nabla_{e_k} + \beta_{e_k})(\chi^2\, a^{km}\, w_{e_m})\rangle \\ &\qquad = -\langle \nabla_{e_k}\nabla_{e_i}(a^{ij}\, w_{e_j}),\, \chi^2\, a^{km}\, w_{e_m}\rangle \\ &\qquad = \langle \nabla_{e_k}(a^{ij}\, w_{e_j}),\, (\nabla_{e_i} + \beta_{e_i})(\chi^2\, a^{km}\, w_{e_m})\rangle. \end{aligned} \tag{A.10}$$

(We use the summation convention.) Here we should rather have a and ∇w twice differentiable in order to integrate by parts. However, one can approximate them, together with the first derivatives, by smooth functions, do the integration by parts and then pass to the limit in the approximation.

(A.10) equals to

$$\begin{aligned}\langle \mathrm{tr}_0\big\{\boldsymbol{\nabla}[A_P\nabla w]\boldsymbol{\nabla}[\chi^2 A_P\nabla w]\big\}\rangle &+ \langle \chi^2, (A_P\nabla w, \boldsymbol{\nabla}[A_P\nabla w]\beta)_0\rangle \\ = \langle \chi^2 \mathrm{tr}_a\big\{\big(\boldsymbol{\nabla}[A_P\nabla w]\big)^2\big\}\rangle &+ \langle (\nabla w, \boldsymbol{\nabla}[A_P\nabla w]\nabla\chi^2)_a\rangle \\ &+ \langle \chi^2, (\nabla w, \boldsymbol{\nabla}[A_P\nabla w]\beta)_a\rangle.\end{aligned} \tag{A.11}$$

In the last equality we used the facts that $\mathrm{tr}_0\{G\} = \mathrm{tr}_a\{G\}$ for a finite rank G, $(A_P g, h)_0 = (g,h)_a$ if $g = Pg$ and $h = Ph$ and that $\boldsymbol{\nabla}[\phi\xi] = \phi\,\boldsymbol{\nabla}[\xi] + \nabla\phi\otimes\xi$, for (cylinder differentiable) E-valued vector field ξ and function ϕ. Observe also that $A_P\nabla w$ is a cylindric differentiable E-valued function. So $\boldsymbol{\nabla}[A_P\nabla w] : \hat{E}\to E$ and hence $\boldsymbol{\nabla}[A_P\nabla w]\beta$ is well-defined.

The first two terms of the right hand side of (A.11) contain $\big\|\chi|D^2wA|_{\sigma_a^2}\big\|_2^2$ and $\big((p/2)-1\big)\big\|\,|\nabla w|_{a,\varepsilon}^{p-3}\big|\nabla|\nabla w|_a^2\big|_a\big\|_2^2$, respectively. Namely,

$$\begin{aligned}&\Big|\langle \chi^2\mathrm{tr}_a\big\{\big(\boldsymbol{\nabla}[A_P\nabla w]\big)^2\big\}\rangle - \big\|\chi|D^2wA|_{\sigma_a^2}\big\|_2^2\Big| \\ &\qquad\le 2\big\|\chi|D^2wA|_{\sigma_a^2}\big\|_2\big\|\chi|\nabla w|_a[A_P^{\boldsymbol{\nabla}}]_a\big\|_2 + \big\|\chi|\nabla w|_a[A_P^{\boldsymbol{\nabla}}]_a\big\|_2^2\end{aligned} \tag{A.12}$$

and

$$\begin{aligned}&\Big|\langle (\nabla w, \boldsymbol{\nabla}[A_P\nabla w]\nabla\chi^2)_a\rangle - \big((p/2)-1\big)\big\|\,|\nabla w|_{a,\varepsilon}^{p-3}\big|\nabla|\nabla w|_a^2\big|_a\big\|_2^2\Big| \\ &\qquad\le 3|p-2|\,\big\|\chi\big|\nabla|\nabla w|_a\big|_a\big\|_2\big\|\chi|\nabla w|_a[A_P^{\boldsymbol{\nabla}}]_a\big\|_2\end{aligned} \tag{A.13}$$

We delegate the proofs of (A.12-13) to separate sections .

Consider now the last term in the right hand side of (A.11). We have

$$\begin{aligned}|(\nabla w, \boldsymbol{\nabla}[A_P\nabla w]\beta)_a| &\le \big|\boldsymbol{\nabla}[A_P\nabla w]\big|_{\sigma_a^2}|\nabla w|_a|\beta|_a \\ &\le |D^2wA_P|_{\sigma_a^2}|\nabla w|_a|\beta|_a + [A_P^{\boldsymbol{\nabla}}]_a|\nabla w|_a^2|\beta|_a.\end{aligned} \tag{A.14}$$

In the last inequality we used the fact that $\boldsymbol{\nabla}[A_P\nabla w] = D^2wA_P + A_P^{\boldsymbol{\nabla}}[\nabla w]$ implying

$$\big|\boldsymbol{\nabla}[A_P\nabla w]\big|_{\sigma_a^2} \le |D^2wA_P|_{\sigma_a^2} + |\nabla w|_a[A_P^{\boldsymbol{\nabla}}]_a. \tag{A.15}$$

Apply the Hölder inequality and (A.14) to the last summand of (A.11). Then

$$\begin{aligned}|\langle\chi^2, (\nabla w, \boldsymbol{\nabla}[A_P\nabla w]\beta)_a\rangle| &\le \big\|\chi|D^2wA|_{\sigma_a^2}\big\|_2\big\|\chi|\nabla w|_a|\beta|_a\big\|_2 \\ &\quad+\frac{1}{2}\big\|\chi|\nabla w|_a(|\beta|_a + [A_P^{\boldsymbol{\nabla}}]_a)\big\|_2^2.\end{aligned} \tag{A.16}$$

Now come back to (A.9) and proceed to $(\nabla+\beta,\chi^2 A_P\nabla w)_0$. It is easy to compute that

$$(\nabla+\beta,\chi^2 A_P\nabla w)_0 = \chi^2\Delta_{a_P} w + (2p-4)\chi|\nabla w|_{a,\varepsilon}^{p-4}|\nabla w|_a(\nabla|\nabla w|_a,\nabla w)_a + \chi^2(\beta,\nabla w)_a.$$

By the Schwartz inequality, $\big|(\nabla|\nabla w|_a,\nabla w)_a\big| \le |\nabla w|_a\big|\nabla|\nabla w|_a\big|_a$ and $|(\beta,\nabla w)_a| \le |\nabla w|_a|\beta|_a$. Observe that $|\nabla w|_a \le |\nabla w|_{a,\varepsilon}$, so $|\nabla w|_{a,\varepsilon}^{p-4}|\nabla w|_a^2 \le \chi$. Hence, by the Hölder inequality, we obtain

$$\begin{aligned}&\Big|\langle\Delta_{a_P} w,\,(\nabla+\beta,\chi^2 A\nabla w)_0\rangle - \big\|\chi\Delta_{a_P} w\big\|_2^2\Big|\\ &\le \big\|\chi\Delta_{a_P} w\big\|_2\big\|\chi|\nabla w|_a|\beta|_a\big\|_2^2 + |2p-4|\,\big\|\chi\Delta_{a_P} w\big\|_2\big\|\chi\big|\nabla|\nabla w|_a\big|_a\big\|_2.\end{aligned} \tag{A.17}$$

Now combining (A.11-13) and (A.16-17) we get (A.2). $\square$

Proof of (A.12). Since $\boldsymbol{\nabla}[A_P\nabla w] = A_P^{\boldsymbol{\nabla}}[\nabla w] + D^2 w A_P$,

$$\begin{aligned}\mathrm{tr}_a\big\{\big(\boldsymbol{\nabla}[A_P\nabla w]\big)^2\big\} &= \mathrm{tr}_a\big\{(A_P^{\boldsymbol{\nabla}}[\nabla w])^2\big\}\\ &\quad + 2\mathrm{tr}_a\big\{D^2 w A_P\, A_P^{\boldsymbol{\nabla}}[\nabla w]\big\} + \mathrm{tr}_a\big\{D^2 w A D^2 w A\big\}.\end{aligned}$$

The last term above equals to $|D^2 wA|_{\sigma_a^2}^2$. Use the Schwartz inequality to estimate the remaining two terms in the r.h.s.:

$$\big|\mathrm{tr}_a\big\{(A_P^{\boldsymbol{\nabla}}[\nabla w])^2\big\}\big| \le \big|A_P^{\boldsymbol{\nabla}}[\nabla w]\big|_{\sigma_a^2}^2 \le [A_P^{\boldsymbol{\nabla}}]_a^2|\nabla w|_a^2$$

and

$$\big|\mathrm{tr}_a\big\{D^2 w A_P A_P^{\boldsymbol{\nabla}}[\nabla w]\big\}\big| \le |D^2 w A_P|_{\sigma_a^2}\big|A_P^{\boldsymbol{\nabla}}[\nabla w]\big|_{\sigma_a^2} \le |D^2 wA|_{\sigma_a^2}[A_P^{\boldsymbol{\nabla}}]_a|\nabla w|_a.$$

The rest is straightforward. $\square$

Proof of (A.13). Since $\nabla|\nabla w|_a^2 = \nabla|\nabla w|_{a,\varepsilon}^2$, one has

$$\begin{aligned}(\nabla w,\boldsymbol{\nabla}[A_P\nabla w]\nabla|\nabla w|_{a,\varepsilon}^{2p-4})_a &= (p-2)|\nabla w|_{a,\varepsilon}^{2p-6}(\nabla w, A_P^{\boldsymbol{\nabla}}[\nabla w]\nabla|\nabla w|_a^2)_a\\ &\quad + (p-2)|\nabla w|_{a,\varepsilon}^{2p-6}(D^2 wA\nabla w,\nabla|\nabla w|_a^2)_a.\end{aligned}$$

Observe that

$$\nabla|\nabla w|_a^2 = 2D^2 wA\nabla w + A_P^{\boldsymbol{\nabla}}[\nabla w]\nabla w. \tag{A.18}$$

Hence,

$$\begin{aligned}(\nabla w, A_P^{\boldsymbol{\nabla}}[\nabla w]\nabla|\nabla w|_a^2)_a + (D^2 wA\nabla w,\nabla|\nabla w|_a^2)_a &= \frac{1}{2}\big|\nabla|\nabla w|_a^2\big|_a^2 +\\ &+ (A_P^{\boldsymbol{\nabla}}[\nabla w]\nabla|\nabla w|_a^2,\nabla w)_a - \frac{1}{2}(A_P^{\boldsymbol{\nabla}}[\nabla w]\nabla w,\nabla|\nabla w|_a^2)_a.\end{aligned}$$

By the Cauchy inequality,

$$|(A_P^{\nabla}[\nabla w]\nabla|\nabla w|_a^2, \nabla w)_a| \leq |A_P^{\nabla}[\nabla w]|_{\sigma_a^2}|\nabla w|_a \big|\nabla|\nabla w|_a^2\big|_a \leq 2[A_P^{\nabla}]_a \big|\nabla|\nabla w|_a\big|_a |\nabla w|_a^3$$

and $(A_P^{\nabla}[\nabla w]\nabla w, \nabla|\nabla w|_a^2)_a \leq 2[A_P^{\nabla}]_a \big|\nabla|\nabla w|_a\big|_a |\nabla w|_a^3$. The rest is straightforward. □

Proof of (A.3-5). Due to (A.18), we have

$$\big|\nabla|\nabla w|_a^2\big|_a \leq |\nabla w|_a(2|D^2 w A_P|_{\sigma_a^2} + |\nabla w|_a[A_P^{\nabla}]_a).$$

So for every $\delta > 1$ there exists $C_\delta > 0$ such that

$$\begin{aligned} \big\| \, |\nabla w|_{a,\varepsilon}^{p-3} \big|\nabla|\nabla w|_a^2\big|_a \big\|_2^2 &\leq 4\big\|\chi\big|\nabla|\nabla w|_a\big|_a\big\|_2^2 \\ &\leq 4\delta\|\chi|D^2 w A_P|_{\sigma_a^2}\|_2^2 + C_\delta\|\chi|\nabla w|_a[A_P^{\nabla}]_a|_2^2 \end{aligned} \tag{A.19}$$

Put $o_p = (p-2)\wedge 0$. Then we have, due to (A.2), that, for every $\delta > 1$ there exists $C_{p,\delta} > 0$ such that

$$\begin{aligned} \big(1 + 2\delta o_p\big)\big\|\chi|D^2 w A|_{\sigma_a^2}\big\|_2^2 &\leq C_{p,\delta}\big\|\chi\Delta_{a_P} w\big\|_2^2 + C_{p,\delta}\big\|\chi|\nabla w|_a(|\beta|_a + [A_P^{\nabla}]_a)\big\|_2^2 \\ &+ C_{p,\delta}\big\|\chi|D^2 w A|_{\sigma_a^2}\big\|_2\big(\big\|\chi\Delta_{a_P} w\big\|_2 + \big\|\chi|\nabla w|_a(|\beta|_a + [A_P^{\nabla}]_a)\big\|_2^2\big). \end{aligned}$$

If $p > 3/2$ the we can pick up $\delta > 1$ such that $1 + 2\delta o_p = \nu > 0$. Apply the Cauchy inequality to get that, for every $p > 3/2$, there exists $C_p > 0$ such that

$$\big\|\chi|D^2 w A_P|_{\sigma_a^2}\big\|_2^2 \leq C_p\big\|\chi\Delta_{a_P} w\big\|_2^2 + C_p\big\|\chi|\nabla w|_a(|\beta|_a + [A_P^{\nabla}]_a)\big\|_2^2. \tag{A.20}$$

(A.20) with $p = 2$ gives (A.4). Substitute (A.20) in (A.19) and the result to (A.1) to get (A.3). To obtain (A.5), observe that, for $1 \leq p < 2$,

$$\| \, |\nabla w|_{a,\varepsilon}\|_{2p}^{2p} = \| \, |\nabla w|_{a,\varepsilon}^{p-1}|\nabla w|_a\|_2^2 + \varepsilon^2 \| \, |\nabla w|_{a,\varepsilon}^{p-1}\|_2^2 \leq \| \, |\nabla w|_{a,\varepsilon}^{p-1}|\nabla w|_a\|_2^2 + \varepsilon^{2p}\mu(X).$$

Substituting (A.3) into the above inequality we arrive at

$$\begin{aligned} \| \, |\nabla w|_{a,\varepsilon}\|_{2p}^{2p} \leq& \varepsilon^{2p-2}\|w\|_2^2 + \varepsilon^{2p}\mu(X) \\ &+ C_p(\varepsilon^2 + \|w\|_2^2)\big(\big\|\chi\Delta_{a_P} w\big\|_2^2 + \big\|\chi|\nabla w|_a(|\beta|_a + [A_P^{\nabla}]_a)\big\|_2^2\big). \end{aligned} \tag{A.21}$$

By the Young inequality, for $p < 2$, only, such that

$$\||D^2 w A_P|_{\sigma_a^2}\|_p^p \leq \varepsilon^{-p}\| \, |\nabla w|_{a,\varepsilon}\|_{2p}^{2p} + C_p\varepsilon^{2-p}\|\chi|D^2 w A_P|_{\sigma_a^2}\|_2^2.$$

Finally, substitute (A.20-21) into the above inequality to get (A.5). □

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