A category of pseudo-tangles with classifying space $\Omega^\infty S^\infty$ and applications *

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Abstract

By using Quillen's $S^{-1}S$ construction, we construct a category $\mathcal{G}$ with classifying space $\Omega^\infty S^\infty$, which has a pictorial description by pseudo-tangles. This leads to an elementary interpretation of the generators of the stable stems $\pi_n^S = \pi_n(\Omega^\infty S^\infty)$ for $n = 0, 1, 2$. In addition, we investigate the homotopy fibre of the inclusion $\mathcal{G} \hookrightarrow \mathcal{F}_{\mathrm{P,D}}$, where $\mathcal{F}_{\mathrm{P,D}}$ denotes the free permutative category with duality, and compute the homotopy groups of $\mathcal{F}_{\mathrm{P,D}}$ for $n = 0, 1, 2$.

Key words: pseudo-tangles, category, classifying space, $S^{-1}S$ construction, stable homotopy of spheres, duality

0 Introduction

For this paper a monoidal category means a strict monoidal category in the usual sense, i.e. it consists of a category $\mathcal{C}$, a functor $\oplus : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ (called tensor product) and an object $0 \in \mathcal{C}$ (called unit) such that $\oplus$ is strictly associative and unital on objects and morphisms.

A monoidal category is called permutative if there exists a family of natural isomorphisms

$$c_{V,W} : V \oplus W \longrightarrow W \oplus V$$

(indexed by all pairs of objects (V, W) in $\mathcal{C}$) which again satisfy certain relations and in particular $c_{W,V} \circ c_{V,W} = id$.

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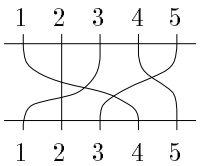
The forgetful functor from the category of permutative categories to the category of all categories has a left adjoint, the value of which on the trivial category is $\mathcal{F}_{\mathrm{P}} = \coprod_{k \geq o} \Sigma_k$, the "free permutative category generated by one object". It has an elementary interpretation by pictures called "pseudo-braids". (By convention the direction of the tracks is from the bottom to the top.)

$$\sigma = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ 3\ 2\ 5\ 1\ 4 \end{pmatrix} \quad \hat{=} \quad$$



If a monoidal category $\mathcal{S}$ satisfies certain conditions, one obtains a new category $\mathcal{S}^{-1}\mathcal{S}$ and a canonical functor $\mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{S}$ which yields a group completion $B\mathcal{S} \longrightarrow B(\mathcal{S}^{-1}\mathcal{S})$ on the level of classifying spaces by Quillen's "$S^{-1}S$ construction" [1].
In this paper we show that in the case $\mathcal{S} = \mathcal{F}_{\mathrm{P}}$ the $S^{-1}S$ construction has a pictorial interpretation, i.e. there is a pictorially described category $\mathcal{G}$ which is equivalent to $\mathcal{F}_{\mathrm{P}}^{-1}\mathcal{F}_{\mathrm{P}}$. By the famous theorem of Priddy and Barrat [6] it is known that the group completion of $B\mathcal{F}_{\mathrm{P}}$ is $\Omega^\infty S^\infty$, i.e. stable homotopy of the 0-sphere. Hence the category $\mathcal{G}$ is a combinatorial model for this space, and we obtain combinatorial proofs of $\pi_0(\Omega^\infty S^\infty) = \mathbb{Z}$ and $\pi_1(\Omega^\infty S^\infty) = \mathbb{Z}/2$. Further, we show that $\pi_2(\Omega^\infty S^\infty)$ has a non-trivial element of order 2.

A duality structure in a monoidal category $\mathcal{C}$ assigns to each object V another object V^* and two morphisms

$$b_V : 0 \longrightarrow V \oplus V^*, \qquad d_V : V^* \oplus V \longrightarrow 0$$

with $(id \oplus d)(b \oplus id) = id_V$ and $(d \oplus id)(id \oplus b) = id_{V^*}$ for all objects V.
In this paper the "free permutative category with duality generated by one object" is investigated. It is denoted by $\mathcal{F}_{\mathrm{P,D}}$ and has again a pictorial description.
$\mathcal{F}_{\mathrm{P}}$ is a subcategory of $\mathcal{F}_{\mathrm{P,D}}$, and $\pi_0 B\mathcal{F}_{\mathrm{P,D}}$ is the algebraic group completion of the monoid $\pi_0 B\mathcal{F}_{\mathrm{P}}$. Hence Zig Fiedorowicz suggested the question whether the process of adding a free duality structure to $\mathcal{F}_{\mathrm{P}}$ gives a topological group completion on classifying spaces.
Unfortunately, this question has a negative answer as will be shown in this paper by the non-contractibility of the homotopy fibre of the inclusion $\mathcal{G} \hookrightarrow \mathcal{F}_{\mathrm{P,D}}$. In any case we obtain:

Theorem: The canonical map $\Omega^\infty S^\infty \to B\mathcal{F}_{\mathrm{P,D}}$ is 1-connected but not 2-connected.

This leads to the homotopy groups $\pi_i(B\mathcal{F}_{\mathrm{P,D}})$ for $i = 0, 1, 2$, and it is possible to give generators for these groups.

1 Basic definitions and a short review of the $S^{-1}S$ construction

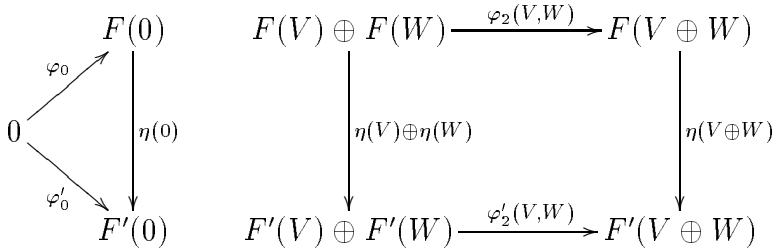
1.1 Definition: A monoidal category is a category $\mathcal{C}$ equipped with a strictly associative functor $\oplus : \mathcal{C} \times C \longrightarrow C$ (called tensor product) and a strictly unit object $0 \in ob\,\mathcal{C}$ for this tensor product.

1.2 Definition: (a) Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$ where $F : \mathcal{C} \longrightarrow D$ is a functor, φ_0 is an isomorphism from 0 to $F(0)$ and

$$\varphi_2(V, W) : F(V) \oplus F(W) \longrightarrow F(V \oplus W)$$

is a family of natural isomorphisms indexed by all pairs (V, W) of objects of $\mathcal{C}$ satisfying the usual associative and unital properties (cf. [3]).
A monoidal functor is called strict if φ_0 and all φ_2 are identities.

(b) A natural tensor transformation $\eta : (F, \varphi_0, \varphi_2) \longrightarrow (F', \varphi_0', \varphi_2')$ between monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ is a natural transformation $\eta : F \longrightarrow F'$ such that the following diagrams commute for each pair (V, W) of objects in $\mathcal{C}$:



A natural tensor isomorphism is a natural tensor transformation that is also a natural isomorphism.

(c) A tensor equivalence between monoidal categories is a monoidal functor $F : \mathcal{C} \longrightarrow D$ such that there exists a monoidal functor $F' : \mathcal{D} \longrightarrow C$ and natural tensor isomorphisms $\eta : id_{\mathcal{D}} \longrightarrow F \circ F'$ and $\eta' : F' \circ F \longrightarrow id_{\mathcal{C}}$.

1.3 Definition: A permutative category is a monoidal category $\mathcal{C}$ equipped with a family of natural isomorphisms $c_{V,W} : V \oplus W \longrightarrow W \oplus V$ indexed by all pairs (V, W) of objects of $\mathcal{C}$ with the following properties:

$$\begin{aligned} c_{W,V} \circ c_{V,W} &= id_{V \oplus W} \\ c_{U \oplus V, W} &= (c_{U,W} \oplus id_V) \circ (id_U \oplus c_{V,W}) \\ c_{U, V \oplus W} &= (id_V \oplus c_{U,W}) \circ (c_{U,V} \oplus id_W) \end{aligned}$$

1.4 Definition: A monoidal functor $F : \mathcal{C} \longrightarrow D$ between permutative categories $\mathcal{C}$ and $\mathcal{D}$ is said to be permutative when the following diagram commutes

for all pairs (V, W) of $\mathcal{C}$:

$$\begin{array}{ccc} F(V) \oplus F(W) & \xrightarrow{\varphi_2(V,W)} & F(V \oplus W) \\ {\scriptstyle c_{F(V),F(W)}}\downarrow & & \downarrow{\scriptstyle F(c_{V,W})} \\ F(W) \oplus F(V) & \xrightarrow{\varphi_2(W,V)} & F(W \oplus V) \end{array}$$

1.5 Review of the $S^{-1}S$ construction: (See [1]). Suppose S is an abelian monoid which acts on a set X. The monoid S is said to act invertibly on X if for each $s \in S$ the translation $X \longrightarrow X, \quad x \longmapsto s \cdot x$ is a bijection. Define $S^{-1}X$ to be the quotient $S \times X/S$, where S acts on each factor of the product. Let S act on $S^{-1}X$ by $t \cdot (s, x) := (s, t \cdot x)$, and define $X \longrightarrow S^{-1}X$ to be $x \longmapsto (0, x)$. Then the translation $(s, x) \longmapsto (s, t \cdot x)$ has an inverse given by $(s, x) \longmapsto (t + s, x)$, so S acts invertibly on $S^{-1}X$. The arrow $X \longrightarrow S^{-1}X$ respects the S-action and is a universal arrow from X to a set upon which S acts invertibly.

Now let $\mathcal{S}$ be a monoidal category. Define a category $\mathcal{S}^{-1}\mathcal{S}$ in the following way: $\mathcal{S}^{-1}\mathcal{S}$ has the same objects as $\mathcal{S} \times S$, and a morphism from (V_-, V_+) to (W_-, W_+) is an isomorphism class of triples

$$(K, \quad \alpha_- : K \oplus V_- \longrightarrow W_-, \quad \alpha_+ : K \oplus V_+ \longrightarrow W_+)$$

where K is an object and α_-, α_+ are morphisms of $\mathcal{S}$.
An isomorphism between triples (K, α_-, α_+) and $(K', \alpha'_-, \alpha'_+)$ is an isomorphism $K \xrightarrow{\cong} K'$ in $\mathcal{S}$ which makes the following diagrams commute:

$$\begin{array}{ccc} K \oplus V_- & \xrightarrow{\cong} & K' \oplus V_- \\ & {\scriptstyle \alpha_-}\searrow \quad \swarrow{\scriptstyle \alpha'_-} & \\ & W_- & \end{array} \qquad \begin{array}{ccc} K \oplus V_+ & \xrightarrow{\cong} & K' \oplus V_+ \\ & {\scriptstyle \alpha_+}\searrow \quad \swarrow{\scriptstyle \alpha'_+} & \\ & W_+ & \end{array}$$

The isomorphism class of a triple (K, α_-, α_+) is denoted by $[K, \alpha_-, \alpha_+]$.

The composition for morphisms $F = [K, \alpha_-, \alpha_+]$ and $G = [L, \beta_-, \beta_+]$ is

$$G \circ F = [L \oplus K, \beta_- \circ (id_L \oplus \alpha_-), \beta_+ \circ (id_L \oplus \alpha_+)].$$

$\mathcal{S}^{-1}\mathcal{S}$ has a canonical monoidal structure: On objects the tensor product is given by $(V_-, V_+) \oplus (W_-, W_+) := (V_- \oplus W_-, V_+ \oplus W_+)$, and for morphisms $F = [K, \alpha_-, \alpha_+]$, $G = [L, \beta_-, \beta_+]$ we have

$$F \oplus G = [K \oplus L, \ (\alpha_- \oplus \beta_-) \circ (id \oplus \tau \oplus id), \ (\alpha_+ \oplus \beta_+) \circ (id \oplus \tau \oplus id)],$$

where τ denotes the switch map. The canonical functor $\mathcal{S} \longrightarrow \mathcal{S}^{-1}\mathcal{S}$, given by $V \longmapsto (0, V)$, strictly preserves the monoidal structure.

If $\mathcal{S}$ is a permutative category, then $\mathcal{S}^{-1}\mathcal{S}$ is also permutative by

$$c_{(V_-,V_+),(W_-,W_+)} = [0, c_{V_-,W_-}, c_{V_+,W_+}].$$

In this situation the canonical functor is permutative, and we obtain:

Theorem: Suppose $(\mathcal{S}, \oplus)$ is a permutative category such that all morphisms of $\mathcal{S}$ are isomorphisms and all translations $\mathcal{S} \longrightarrow \mathcal{S},\ V \longmapsto W \oplus V$ are faithful. Then the canonical functor induces a group completion $B\mathcal{S} \longrightarrow B\mathcal{S}^{-1}\mathcal{S}$.

This means: The monoid $S = \pi_0(B\mathcal{S})$ acts on $H_p(B\mathcal{S})$, hence there is a universal arrow $H_p(B\mathcal{S}) \longrightarrow (\pi_0(B\mathcal{S}))^{-1}H_p(B\mathcal{S})$. S acts invertibly on $H_p(B\mathcal{S}^{-1}\mathcal{S})$, and the canonical functor gives a map $H_p(B\mathcal{S}) \longrightarrow H_p(B\mathcal{S}^{-1}\mathcal{S})$ which respects the S-action. So we get an induced map

$$(\pi_0(B\mathcal{S}))^{-1}H_p(B\mathcal{S}) \longrightarrow H_p(B\mathcal{S}^{-1}\mathcal{S})$$

which is an isomorphism for all p.

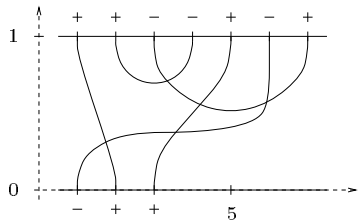
2 The category $\mathcal{G}$

2.1 Construction: Let $\mathcal{G}$ be the following category: The set of objects is the free monoid on the alphabet $\{+, -\}$. We think of an object w with k letters as the subset $\{1, \ldots, k\} \subset \mathbb{R}$ with an assigned $+$ or $-$ to each i.
A morphism of a word w_1 with k_1 letters to another word w_2 with k_2 letters is represented by an unordered set $\{f_\lambda\}_{\lambda\in\Lambda}$ of $\frac{1}{2}(k_1 + k_2)$ paths $f_\lambda : [0, 1] \longrightarrow \mathbb{R} \times [0, 1]$ subject to the following conditions:

(1) Each f_λ connects two different points of the subset $\{1, \ldots, k_1\} \times \{0\} \cup \{1, \ldots, k_2\} \times \{1\}$.
(2) Each point of $\{1, \ldots, k_1\} \times \{0\} \cup \{1, \ldots, k_2\} \times \{1\}$ is in the image of exactly one f_λ.
(3) Each f_λ connects either an element from $\{1, \ldots, k_1\} \times \{0\}$ with an element from $\{1, \ldots, k_2\} \times \{1\}$ having the same sign (called "vertical strings") or two elements from $\{1, \ldots, k_2\} \times \{1\}$ having different signs (called "arcs").

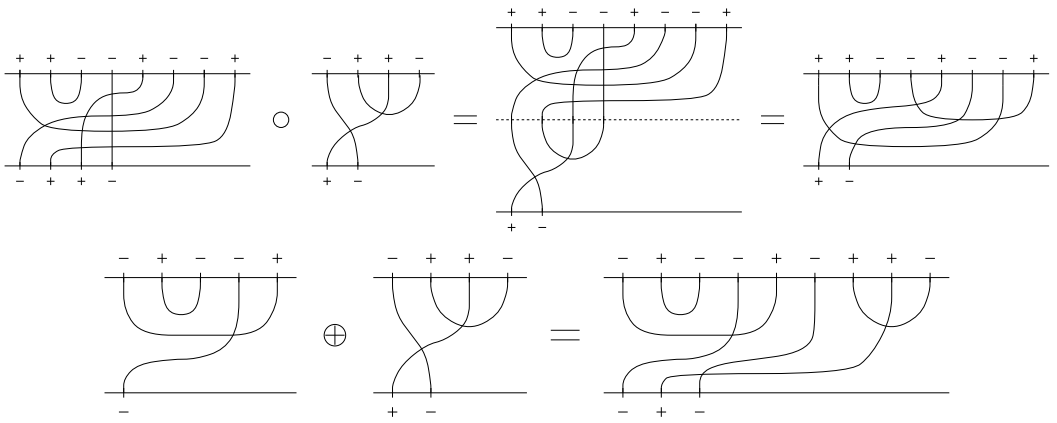
The term "string" is used for both kinds of paths.
Two such systems $\{f_\lambda\}_{\lambda\in\Lambda}$ and $\{f'_\lambda\}_{\lambda\in\Lambda}$ represent the same morphism if there is a family of homotopies $\{H_\lambda : f_\lambda \simeq f'_\lambda \quad \text{or} \quad -f_\lambda \simeq f'_\lambda\}_{\lambda\in\Lambda}$ which keep boundary points fixed. ($-f_\lambda$ denotes the reversed parametrization of f_λ.)

Morphisms can be illustrated by pictures like this:



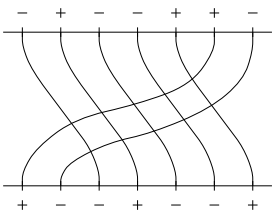
We will call such a picture a "pseudo-tangle" because it looks like a tangle with unconsidered crossings.

$\mathcal{G}$ is a permutative category with composition by stacking pseudo-tangles and tensor product by putting words and pseudo-tangles side by side (the empty word gives the unit object):



The symmetry isomorphisms $c_{v,w} : v \oplus w \longrightarrow w \oplus v$ are given by pictures of the following pattern:

$c_{+-,-+--+}$ =



The object $++\ldots++$ of $\mathcal{G}$ which consists of k letters "+" is sometimes simply denoted by k, and $-k = --\ldots--$ denotes the word of k letters "−". Using this notation we have, for example, $2 \oplus -1 \oplus 1 \oplus -3 = ++-+---$.

2.2 Definition: Let $\mathcal{F}_{\mathrm{P}}$ be the full permutative subcategory of $\mathcal{G}$ whose set of objects is $\{k | k \in \mathbb{N}\}$. Its morphisms will be called "pseudo-braids".

In short we have $mor_{\mathcal{F}_{\mathrm{P}}}(k,k) = \Sigma_k$ (the permutation group of the set $\{1,\ldots,k\}$)

and $mor_{\mathcal{F}_\mathrm{P}}(l,k) = \emptyset$ for $l \neq k$.

2.3 Theorem: $\mathcal{F}_\mathrm{P}$ is the free permutative category, generated by one object.

This is well known. The following theorem is the main result of this section:

2.4 Theorem: There is a permutative functor $\mathcal{F}_\mathrm{P}^{-1}\mathcal{F}_\mathrm{P} \longrightarrow \mathcal{G}$ which induces a homotopy equivalence on classifying spaces.

2.5 Corollary: $B\mathcal{G}$ has the homotopy type of $\Omega^\infty S^\infty$ as an infinite loop space.

PROOF. It is well known that the group completion of the classifying space of a permutative category is an infinite loop space and that permutative functors induce infinite loop maps.
$B\mathcal{F}_\mathrm{P}^{-1}\mathcal{F}_\mathrm{P}$ and $B\mathcal{G}$ are grouplike (i.e. $\pi_0(B\mathcal{F}_\mathrm{P}^{-1}\mathcal{F}_\mathrm{P})$ and $\pi_0(B\mathcal{G})$ are groups under the induced multiplication) because of the properties of the $S^{-1}S$ construction and 3.1 below.
Further, the canonical map $B\mathcal{F}_\mathrm{P} \longrightarrow B\mathcal{F}_\mathrm{P}^{-1}\mathcal{F}_\mathrm{P}$ is a group completion ($\mathcal{F}_\mathrm{P}$ satisfies the sufficient conditions given in the $S^{-1}S$ construction). The rest follows from $B\mathcal{F}_\mathrm{P} = \coprod_{n\geq 0} B\Sigma_n$ and the following well-known theorem:

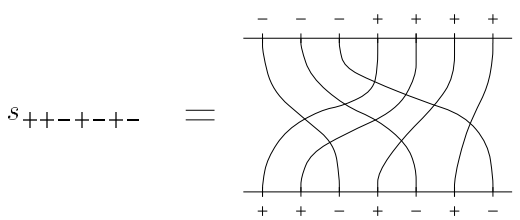
2.6 Theorem: The canonical map $\coprod_{n\geq 0} B\Sigma_n \longrightarrow \Omega^\infty S^\infty$ is a group completion. (See [6] or [9]).

(Here the multiplication on $\Omega^\infty S^\infty$ is the wedge product of maps $S^k \to S^k$).

PROOF OF 2.4. The required functor $\mathcal{F}_\mathrm{P}^{-1}\mathcal{F}_\mathrm{P} \longrightarrow \mathcal{G}$ is given by a composition $\mathcal{F}_\mathrm{P}^{-1}\mathcal{F}_\mathrm{P} \xrightarrow{\Phi} \mathcal{RG} \xrightarrow{J} \mathcal{G}$ where $\mathcal{RG}$ is a subcategory of $\mathcal{G}$, the inclusion J an equivalence of permutative categories and Φ an isomorphism.

An object of $\mathcal{G}$ is called ordered if it is of the pattern $w = --\cdots-++\cdots+$. Let $\mathcal{RG}$ be the full subcategory of $\mathcal{G}$ which consists of all ordered words, and let $J : \mathcal{RG} \longrightarrow G$ be the inclusion.

There is a retraction functor $R : \mathcal{G} \longrightarrow \mathcal{RG}$ given on objects by ordering the letters, i.e. $R(++-+-) = --+++$ etc. Define $\hat{w} := R(w)$, and let $s_w : w \longrightarrow \hat{w}$ be the isomorphism which orders the letters in such a way that the order among the letters "+" and "−" is preserved separately.



Define R on morphisms $\alpha : w_1 \longrightarrow w_2$ by $R(\alpha) := s_{w_2} \circ \alpha \circ s_{w_1}^{-1}$.
R is well-defined, and the functorial properties can easily be checked.

2.7 Lemma: $\mathcal{RG}$ is a permutative category with tensor product $\hat{\oplus} := R \circ \oplus$ (the empty word as a unit) and symmetry isomorphisms $\hat{c}_{v,w} := R(c_{v,w})$. It is equivalent as a permutative category to $\mathcal{G}$ by the functors J and R. In particular, the classifying spaces of $\mathcal{RG}$ and $\mathcal{G}$ are homotopy equivalent as infinite loop spaces.

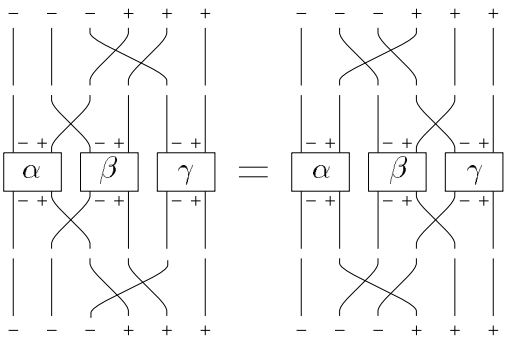
PROOF. The following identities are immediate for any objects u, v, w:

$$(1)\quad s_{(u\hat{\oplus}v)\oplus w} \circ (s_{u\oplus v} \oplus id_w) = s_{u\oplus(v\hat{\oplus}w)} \circ (id_u \oplus s_{v\oplus w}),$$

$$(2)\qquad\qquad s_{v\oplus w} = s_{\hat{v}\oplus\hat{w}} \circ (s_v \oplus s_w).$$

The associativity on objects is clear. The associativity on morphisms can be shown algebraically by using (1) or is to be seen from the following self-explanatory picture:

$$(\alpha\hat{\oplus}\beta)\hat{\oplus}\gamma \;=\; \alpha\hat{\oplus}(\beta\hat{\oplus}\gamma)$$



R strictly preserves the monoidal structure:
On objects the equation $R(v \oplus w) = R(v)\hat{\oplus}R(w)$ is evident.
Given any two morphisms $v_1 \xrightarrow{\alpha} v_2$ and $w_1 \xrightarrow{\beta} w_2$ one uses (2):

$$\begin{aligned} R(\alpha \oplus \beta) &= s_{v_2\oplus w_2}(\alpha \oplus \beta)s_{v_1\oplus w_1}^{-1} \\ &= s_{\hat{v_2}\oplus\hat{w_2}}(s_{v_2} \oplus s_{w_2})(\alpha \oplus \beta)(s_{v_1} \oplus s_{w_1})^{-1}s_{\hat{v_1}\oplus\hat{w_1}}^{-1} \\ &= s_{\hat{v_2}\oplus\hat{w_2}}(R(\alpha) \oplus R(\beta))s_{\hat{v_1}\oplus\hat{w_1}}^{-1} \\ &= R(\alpha)\hat{\oplus}R(\beta). \end{aligned}$$

The inclusion J is monoidal but not strictly. The necessary isomorphisms $\varphi_2(v, w) : J(v) \oplus J(w) \longrightarrow J(v\hat{\oplus}w)$ are in fact given by $s_{v\oplus w} : v \oplus w \longrightarrow v\hat{\oplus}w$, and the coherence condition turns out to be formula (1) (cf. 1.2 (a)).

The identity $R \circ J = Id_{\mathcal{RG}}$ is obvious, and a natural tensor isomorphism $Id_{\mathcal{G}} \longrightarrow J \circ R$ is given by the isomorphisms s_w:

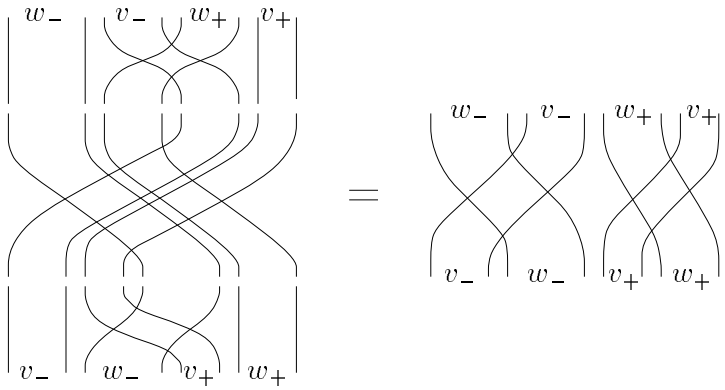
$$\begin{array}{ccc} Id(w_1) & \xrightarrow{s_{w_1}} & J \circ R(w_1) \\ {\scriptstyle Id(\alpha)}\downarrow & & \downarrow{\scriptstyle J \circ R(\alpha)} \\ Id(w_2) & \xrightarrow{s_{w_2}} & J \circ R(w_2) \end{array} \quad = \quad \begin{array}{ccc} w_1 & \xrightarrow{s_{w_1}} & \hat{w}_1 \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle s_{w_2} \circ \alpha \circ s_{w_1}^{-1}} \\ w_2 & \xrightarrow{s_{w_2}} & \hat{w}_2 \end{array}$$

In this case the appropriate coherence condition (cf. 1.2 (b)) is formula (2).

Finally, we have to check the $\hat{c}_{v,w}$ define a permutative structure which is compatible with J and R.
For any object w in $\mathcal{G}$ let w_- and w_+ denote the negative and positive part of $\hat{w}$, i.e. for $w = + + - + -$ we have $w_- = - -, \quad w_+ = + + +$ etc.
With this notation we obtain $\hat{w} = w_- \oplus w_+$ and for ordered v, w:

$$\hat{c}_{v,w} = R(c_{v,w}) = s_{w \oplus v} \circ c_{v,w} \circ s_{v \oplus w}^{-1} = c_{v_-,w_-} \oplus c_{v_+,w_+}\,.$$

The latter equation follows from the picture below.



The naturality of the isomorphisms $\hat{c}_{v,w}$ is derived from the naturality of $c_{v,w}$ and we have

$$\begin{aligned} \hat{c}_{u\hat{\oplus}v,w} &= (\hat{c}_{u,w} \hat{\oplus} id_v) \circ (id_u \hat{\oplus} \hat{c}_{v,w}), \\ \hat{c}_{u,v\hat{\oplus}w} &= (id_v \hat{\oplus} \hat{c}_{u,w}) \circ (\hat{c}_{u,v} \hat{\oplus} id_w), \\ \hat{c}_{w,v} \circ \hat{c}_{v,w} &= id_{v\hat{\oplus}w}. \end{aligned}$$

Using the obvious equations $(u\hat{\oplus}v)_- = u_- \oplus v_-$ and $(u\hat{\oplus}v)_+ = u_+ \oplus v_+$, we

obtain

$$\hat{c}_{u\hat{\oplus}v,w} = R(c_{u\hat{\oplus}v,w}) = c_{u_-\oplus v_-,w_-} \oplus c_{u_+\oplus v_+,w_+},$$

$$\hat{c}_{u,w}\hat{\oplus}id_v = R(\hat{c}_{u,w} \oplus id_v) = c_{u_-,w_-} \oplus id_{v_-} \oplus c_{u_+,w_+} \oplus id_{v_+},$$

$$id_u\hat{\oplus}\hat{c}_{v,w} = R(id_u \oplus \hat{c}_{v,w}) = id_{u_-} \oplus c_{v_-,w_-} \oplus id_{u_+} \oplus c_{v_+,w_+}.$$

In this way the first two of the above equations follow from the corresponding equations for $c_{v,w}$, and the third equation is obvious.

For R to be a permutative functor, we have to show $R(c_{v,w}) = \hat{c}_{\hat{v},\hat{w}}$. This is clear for ordered words v, w by definition, but it has to be true for all words in $\mathcal{G}$. The reader may convince himself that the equation

$$R(c_{v,w}) = c_{v_-,w_-} \oplus c_{v_+,w_+}$$

is true for arbitrary words v, w, which immediately leads to the assertion.

Finally, we check the analogous condition for the inclusion J. This is not a strict monoidal functor; it has coherence isomorphisms $\varphi_2(v, w) = s_{v\oplus w}$.
Hence 1.4 leads to the following diagram which commutes because of $\hat{c}_{v,w} = R(c_{v,w}) = s_{w\oplus v} \circ c_{v,w} \circ s^{-1}_{v\oplus w}$:

$$\begin{array}{ccc} v \oplus w & \xrightarrow{s_{v\oplus w}} & v\hat{\oplus}w \\ {\scriptstyle c_{v,w}}\downarrow & & \downarrow{\scriptstyle \hat{c}_{v,w}} \\ w \oplus v & \xrightarrow{s_{w\oplus v}} & w\hat{\oplus}v \end{array}$$

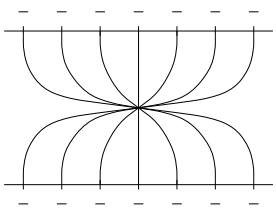
□

2.8 Remark: The proof of 2.7 in fact shows that $B\mathcal{RG}$ is a strong deformation retract of $B\mathcal{G}$. Hence all questions about the homotopy type of $\mathcal{G}$ can be treated equivalently in $\mathcal{RG}$, which will be undertaken later.

2.9 Lemma:
There is an isomorphism of permutative categories $\Phi : \mathcal{F}_{\mathrm{P}}^{-1}\mathcal{F}_{\mathrm{P}} \longrightarrow \mathcal{RG}$.
Hence $B\mathcal{F}_{\mathrm{P}}^{-1}\mathcal{F}_{\mathrm{P}}$ and $B\mathcal{RG}$ are homeomorphic as infinite loop spaces.

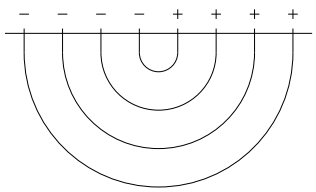
PROOF. We introduce some special morphisms of $\mathcal{G}$.
Let $t_{-k} : -k \longrightarrow -k$ be the morphism of $\mathcal{G}$, the picture of which is given by the permutation $i \longmapsto k - i + 1$ of Σ_k. We will depict it as follows:

t_{-7} =



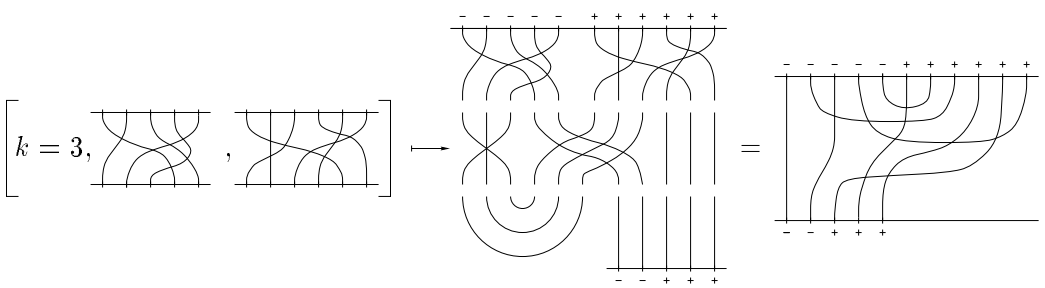
Let $b_{-k} : 0 \longrightarrow -k \oplus k$ denote the morphism which is represented by k arcs without any intersection among them.

b_{-4} =



Finally, recall the symmetry isomorphism $c_{k,-m} : k \oplus -m \longrightarrow -m \oplus k$ of $\mathcal{G}$, and note that we have two canonical embeddings $i_+, i_- : \mathcal{F}_{\mathrm{P}} \longrightarrow \mathcal{G}$ (i_+ = inclusion, i_- given by $k \longmapsto -k$ and copying the pseudo-braid).
Now define $\Phi(m,n) := -m \oplus n$ on objects, and if we are given a morphism $\alpha : (m,n) \longrightarrow (k+m, k+n)$, represented by some triple (k, α_-, α_+), let

$$\Phi(\alpha) := (i_-(\alpha_-) \oplus i_+(\alpha_+)) \circ (t_{-k} \oplus c_{k,-m} \oplus id_n) \circ (b_{-k} \oplus id_{-m \oplus n}).$$

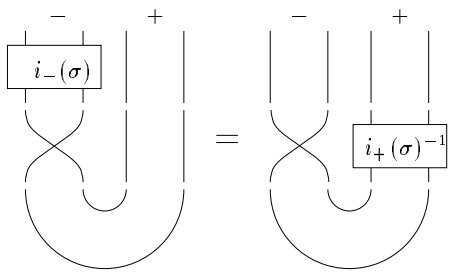


Note: $\Phi(\alpha)$ is indeed a morphism in $\mathcal{RG}$ although its constituents are morphisms in $\mathcal{G}$, and the number of arcs in $\Phi(\alpha)$ is equal to k.
The definition of Φ does not depend on the representing triple (k, α_-, α_+): Let $(k', \alpha'_-, \alpha'_+)$ be another triple which represents the same morphism. Then we have $k = k'$, and there exists an isomorphism σ in $\mathcal{F}_{\mathrm{P}}$ (i.e. $\sigma \in \Sigma_k$) with

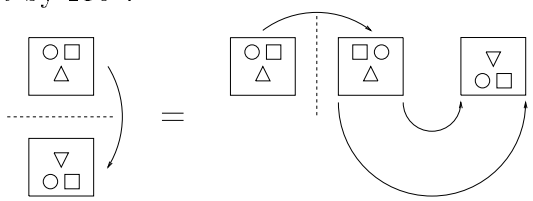
$$\alpha_- = \alpha'_- \circ (\sigma \oplus id_m) \quad \text{and} \quad \alpha_+ = \alpha'_+ \circ (\sigma \oplus id_n).$$

The crucial step is the following observation:

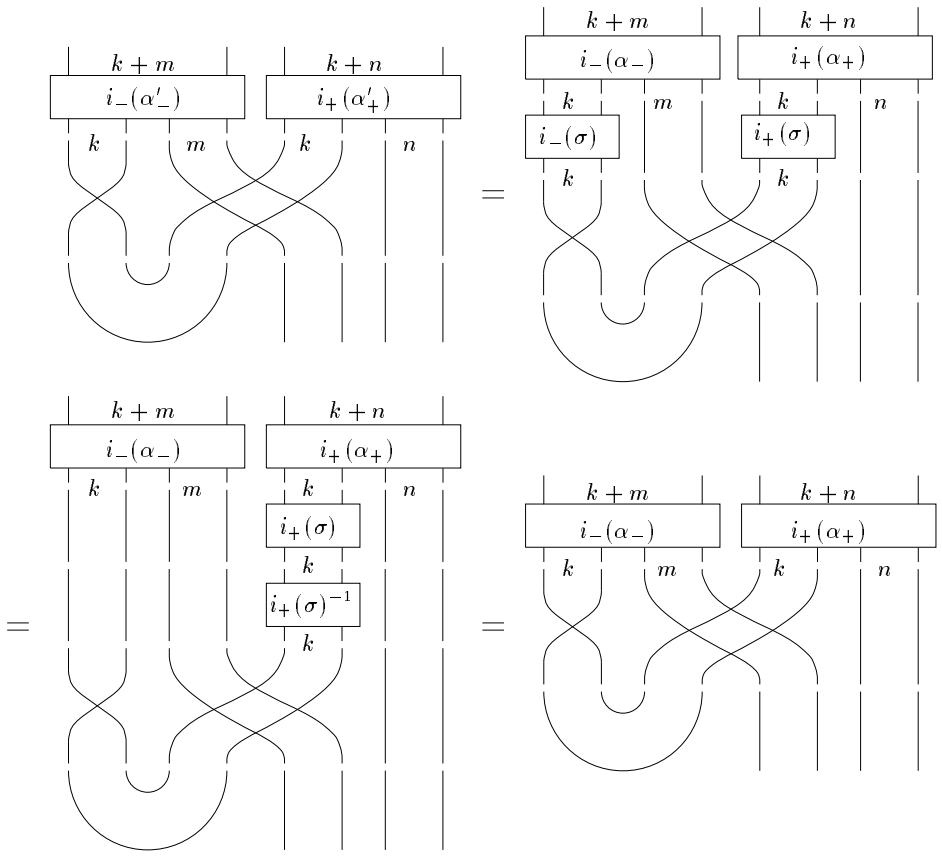
$$((i_-(\sigma) \circ t_{-k}) \oplus id_k) \circ b_{-k} = (t_{-k} \oplus i_+(\sigma)^{-1}) \circ b_{-k} \qquad \text{for all } \sigma \in \Sigma_k.$$



It is clear that the inverse of a given pseudo-braid is obtained by reflecting it at its bottom base. But it also can be built by reflecting it at a vertical axis and then rotating it by 180°:



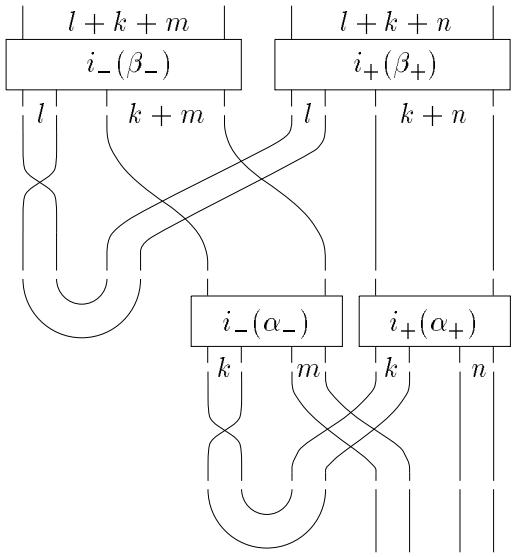
This is exactly what is done by pushing $i_-(\sigma)$ through t_{-k} and b_{-k}; hence the desired relation is true. Using this we have:



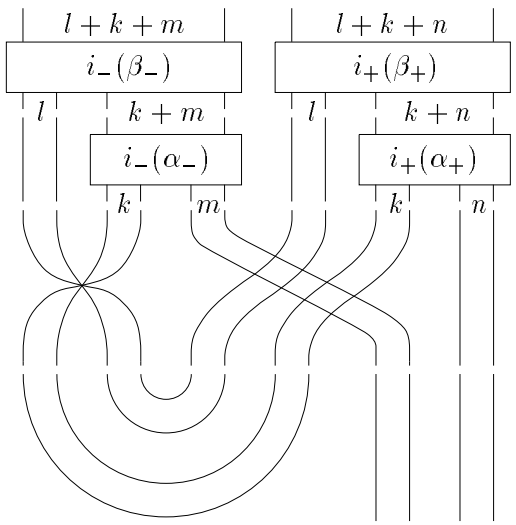
This shows the well-definition of Φ.

The functoriality of Φ is checked as follows: The identity of some object (m,n) in $\mathcal{F}_{\mathrm{P}}^{-1}\mathcal{F}_{\mathrm{P}}$ only has the representative $(0, id_m, id_n)$. In this case the morphisms t, c and b are identities, and the equation $\Phi(id_{m,n}) = id_{-_{m\oplus n}}$ becomes obvious. For any given composition $(m,n) \xrightarrow{\alpha} (k+m, k+n) \xrightarrow{\beta} (l+k+m, l+k+n)$ the relation $\Phi(\beta \circ \alpha) = \Phi(\beta) \circ \Phi(\alpha)$ is deduced from the following pictures:

$\Phi(\beta) \circ \Phi(\alpha) \quad =$



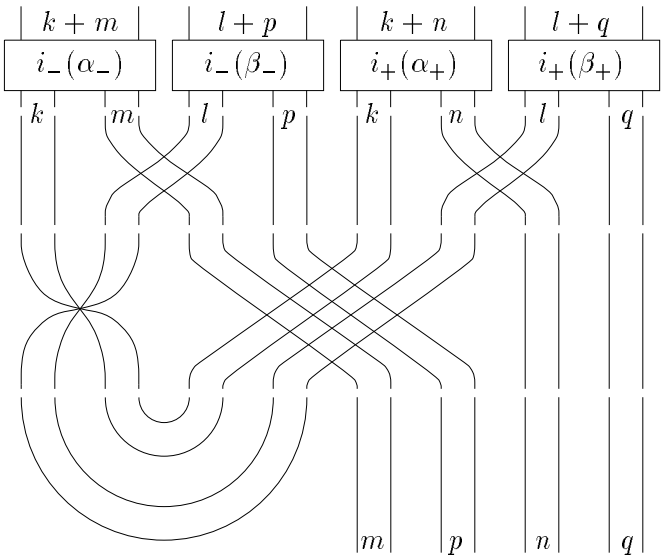
$=$



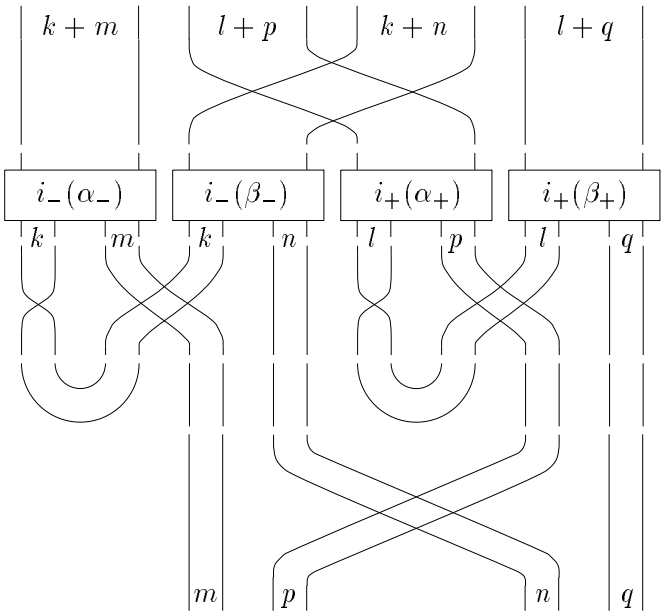
$= \quad \Phi(\beta \circ \alpha)$

Next we show the preservation of the monoidal structure by Φ, i.e. the equation $\Phi(\alpha \oplus \beta) = \Phi(\alpha) \hat{\oplus} \Phi(\beta)$ which is directly clear for all two objects. So let $(k, \alpha_-, \alpha_+), \quad (l, \beta_-, \beta_+)$ be representatives for α and β, respectively. Then the desired equation is proven by the following pictures:

$\Phi(\alpha \oplus \beta)$ $=$



$=$



$= \quad \Phi(\alpha)\hat{\oplus}\Phi(\beta)$

Following 1.5, the permutative structure of $\mathcal{F}_{\mathrm{P}}^{-1}\mathcal{F}_{\mathrm{P}}$ is given by

$$c_{(m_1,n_1),(m_2,n_2)} = [0, c_{m_1,m_2}, c_{n_1,n_2}].$$

If Φ is applied to this triple, the morphisms t, c and b are again identities, which leads to

$$\begin{aligned}\Phi(c_{(m_1,n_1),(m_2,n_2)}) &= \Phi([0, c_{m_1,m_2}, c_{n_1,n_2}]) \\ &= i_-(c_{m_1,m_2}) \oplus i_+(c_{n_1,n_2})\end{aligned}$$

$$= R(c_{-m_1 \oplus n_1, -m_2 \oplus n_2})$$
$$= \hat{c}_{-m_1 \oplus n_1, -m_2 \oplus n_2}$$
$$= \hat{c}_{\Phi(m_1,n_1),\Phi(m_2,n_2)}.$$

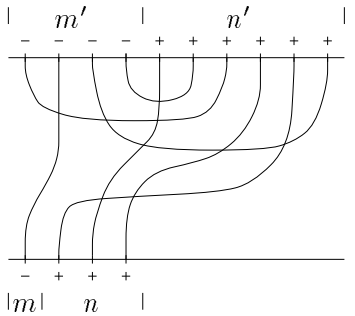
(For the third equality compare the proof of 2.7).
This shows that Φ strictly preserves the permutative structure.

Finally, we have to show that Φ is an isomorphism of categories.
The functor Φ is an isomorphism if it is fully faithful, since it is obviously bijective on objects. Suppose we are given a pseudo-tangle

$$\varphi : -m \oplus n \longrightarrow -m' \oplus n'$$

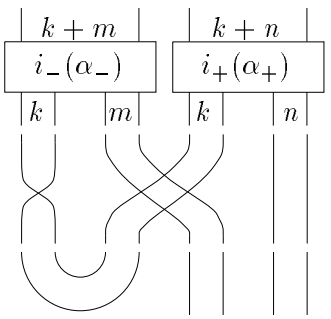
and we are asking for triples (k, α_-, α_+) representing morphisms in $\mathcal{F}_{\mathrm{P}}^{-1}\mathcal{F}_{\mathrm{P}}$ which map to φ.

φ $=$



Firstly, we observe that $m' = k+m, \quad n' = k+n$ where $k \in \mathbb{N}$ is the number of arcs in φ, because every arc adds two letters with different signs to the source object. This number k is inevitably the object needed in the representing triple, and α_- and α_+ are morphisms $k+m \longrightarrow k+m$ and $k+n \longrightarrow k+n$, respectively, in $\mathcal{F}_{\mathrm{P}}$.

As we have already seen, Φ applies the following machinery to α_- and α_+:



So, those strings of α_- (resp. α_+) which have their lower end among the last m (resp. n) strings have already been completely determined. The other strings

contribute to arcs and are not unique. But every concrete choice for the remaining k strings of α_- determines the last k strings of α_+ and vice versa. There are $k!$ such choices which correspond by the simultaneous action of Σ_k on the first k strings of the lower bound of α_- and α_+, respectively. But this leads precisely to the notion of isomorphic triples in the construction of $\mathcal{F}_{\mathrm{P}}^{-1}\mathcal{F}_{\mathrm{P}}$. Hence all representing triples are isomorphic, and there is exactly one morphism in $\mathcal{F}_{\mathrm{P}}^{-1}\mathcal{F}_{\mathrm{P}}$ which maps to φ, i.e. Φ is in fact an isomorphism of categories.

This completes the proof of this lemma and of Theorem 2.4. $\square$

3 Generators for the lower homotopy groups of $\mathcal{G}$

For any choice of basepoint, the first three homotopy groups of $\Omega^\infty S^\infty$ are well known as

$$\pi_0(\Omega^\infty S^\infty) = \mathbb{Z}, \qquad \pi_1(\Omega^\infty S^\infty) = \mathbb{Z}/2, \qquad \pi_2(\Omega^\infty S^\infty) = \mathbb{Z}/2.$$

The first two of these can be computed directly from the model $\mathcal{G}$. We further show that $\pi_2(B\mathcal{G})$ has at least a non-trivial element of order 2 by giving an explicit representative.

3.1 Theorem: $\pi_0(B\mathcal{G}) = \mathbb{Z}$ as a group. A representative for the component which corresponds to $k \in \mathbb{Z}$ is given by the object k of $\mathcal{G}$.

The proof is left to the reader as an easy exercise.

To compute $\pi_1(B\mathcal{G})$ we work in the subcategory $\mathcal{RG}$ of ordered words (cf. 2.8). $\mathcal{RG}^{(0)}$ is the full category of $\mathcal{G}$, which consists of the objects

$$0,\ -+,\ --++,\ ---+++,\ \ldots$$

and these words are denoted by 0, (1), (2), (3), ... according to the number of letters "$-$". Any loop in $B\mathcal{RG}$ is homotopic to a loop represented by a finite combination of morphisms in $\mathcal{RG}$ of the form

$$\omega: \quad 0 \xrightarrow{\alpha_1} (n_1) \xleftarrow{\alpha_2} (n_2) \xrightarrow{\alpha_3} \cdots \xleftarrow{\alpha_{k-2}} (n_{k-2}) \xrightarrow{\alpha_{k-1}} (n_{k-1}) \xleftarrow{\alpha_k} 0.$$

(Note: There is no morphism with target 0 apart from the identity). Every pseudo-tangle α can be represented in a generic way (i.e. no triple points,

no self-intersections, no overlaps, etc. of the strings). Define the intersection number $\#\alpha$ to be the number of crossing points modulo 2.
Note that this leads to the equation $\#(\beta \circ \alpha) = \#\beta + \#\alpha$ for any given composable pair of morphisms.
Call α odd (resp. even) if $\#\alpha = 1$ (resp. $\#\alpha = 0$). A path is called odd (resp. even) if the sum of the intersection numbers of its defining morphisms is odd (resp. even).

We fix three essential properties of $\mathcal{RG}$ in the following lemma:

3.2 Lemma:
(a) To any two morphisms $\alpha, \alpha' : (k) \longrightarrow (n)$ there exits a morphism $\sigma : (n) \longrightarrow (n)$ with $\sigma \circ \alpha' = \alpha$.
(b) To any two morphisms $(n') \xleftarrow{\alpha'} (k) \xrightarrow{\alpha} (n)$ with $n' \leq n$ there exists a morphism $(n') \xrightarrow{\beta} (n)$ with $\beta \circ \alpha' = \alpha$.
(c) The path defined by an even endomorphism $\sigma : (n) \longrightarrow (n)$ is based null homotopic.

PROOF.
(a) The group $End(n) = \Sigma_n \times \Sigma_n$ operates transitively on $mor((k),(n))$ for all $k \leq n$, since every two pseudo-tangles $(k) \longrightarrow (n)$ only differ by a suitable permutation of the strings at their upper bound.

(b) follows immediately from (a).

(c) For $n < 2$ there is nothing to show, $End(0)$ and $End(1)$ are trivial.
The symmetric group Σ_n is generated by simple transpositions, i.e. by pseudo-braids with exactly one crossing point. Pairs of such simple pseudo-braids form even morphisms of $End(n) = \Sigma_n \times \Sigma_n$ and these generate all even endomorphisms. This is proven algebraically as follows:
Let $(\sigma, \tau) \in \Sigma_n \times \Sigma_n$ be an even endomorphism. Decompose σ and τ into simple transpositions

$$\sigma = \sigma_p \circ \cdots \circ \sigma_1, \qquad \tau = \tau_q \circ \cdots \circ \tau_1.$$

Then $p + q$ is even. Because of symmetry we may assume $p \leq q$ and define

$$\gamma_i := (\sigma_i, \tau_i) \text{ for } 1 \leq i \leq p.$$

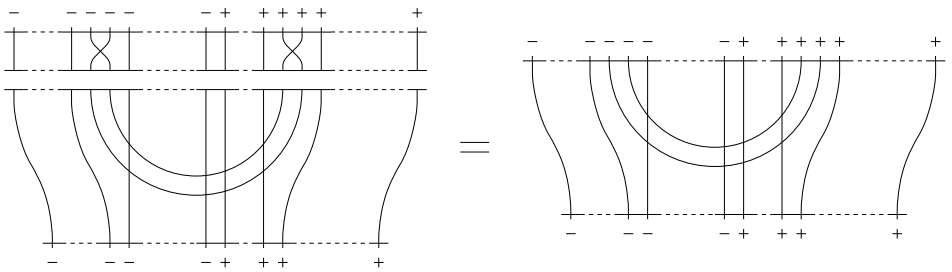
If $p = q$, we get $(\sigma, \tau) = \gamma_p \circ \cdots \circ \gamma_1$, and we have finished.
Otherwise, choose any simple transposition ε of Σ_n and define

$$\gamma_i := (\varepsilon, \tau_i) \quad \text{for} \quad p + 1 \leq i \leq q.$$

$q-p$ is even and $\varepsilon \circ \varepsilon = id$. It follows that

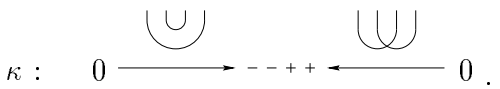
$$(\sigma, \tau) = \gamma_q \circ \cdots \circ \gamma_{p+1} \circ \gamma_p \circ \cdots \circ \gamma_1.$$

Now it is sufficient to prove the assertion of (c) for an endomorphism σ, which has exactly two crossing points, one of which is among the negative strings and the other among the positive strings. Define a morphism $\alpha : (n-2) \longrightarrow (n)$ by connecting the positions which belong to the crossing strings of σ with two arcs (there are two possibilities to do so, but it does not matter which one is chosen), and keep all $n-2$ vertical strings without any crossing among them. Then we obtain a morphism α with $\sigma \circ \alpha = \alpha$:

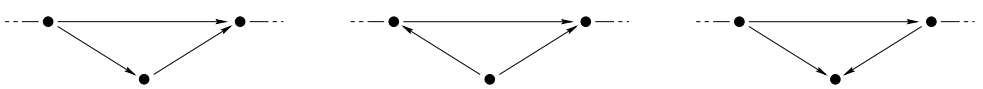


Therefore the path defined by $(n) \xrightarrow{\sigma} (n)$ is based homotopic to the path $(n) \xleftarrow{\alpha} (n-2) \xrightarrow{\alpha} (n)$, which is clearly based null homotopic. $\square$

3.3 Theorem: $\pi_1(B\mathcal{G}, 0) = \mathbb{Z}/2$, a generator is given by the path



PROOF.
Denote the left of these morphisms $0 \longrightarrow (2)$ by u and the other by t.
To investigate edge paths up to homotopy it suffices to consider those homotopies which are determined by 2-cells. These are given by composable pairs $(n) \xrightarrow{\alpha} (n') \xrightarrow{\beta} (n'')$ which form a triangle with edges α, β and $\beta \circ \alpha$. Hence there are three elementary manipulations to replace a morphism in the chain of ω up to homotopy by two others and vice versa:



From the equation $\#(\beta \circ \alpha) = \#\beta + \#\alpha$ it follows that ω does not change its intersection number under the above manipulations. Therefore an odd path cannot be homotopic to an even path, and the theorem reduces to the following two assertions:

(1) All even loops are null homotopic.

(2) All odd loops are homotopic to κ.

By 3.2 (b), every sequence $(n) \longleftarrow (k) \longrightarrow (n')$ of ω can be replaced by a single morphism $(n) \longrightarrow (n')$ or $(n) \longleftarrow (n')$ depending on whether $n \leq n'$ or $n \geq n'$. This single morphism can be composed with one of its neighbours and we obtain a representative of ω, which is shortened by two morphisms. Finite repetition leads to a representative of the form

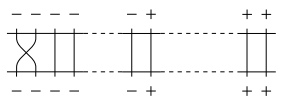
$$\omega : \quad 0 \xrightarrow{\alpha} (n) \xleftarrow{\alpha'} 0.$$

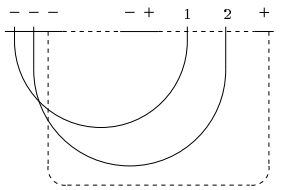
Now applying 3.2 (a) to $(n) \xleftarrow{\alpha'} 0 \xrightarrow{\alpha} (n)$ we find an endomorphism $\sigma : (n) \longrightarrow (n)$ with $\sigma \circ \alpha' = \alpha$, i.e. $\alpha' = \sigma^{-1} \circ \alpha$, and get the sequence

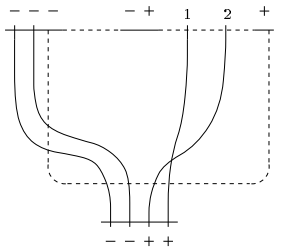
$$\omega : \quad 0 \xrightarrow{\alpha} (n) \xrightarrow{\sigma} (n) \xleftarrow{\alpha} 0.$$

σ has the same intersection number as ω. Hence if ω is even, σ is null homotopic because of 3.2 (c), and this leads to statement (1).

Now assume that ω is odd, i.e. σ is also odd.

$\varepsilon :=$ 

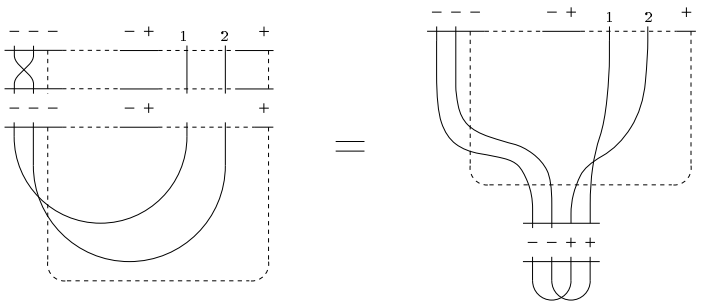
$\alpha =$ 

$\overline{\alpha} :=$ 

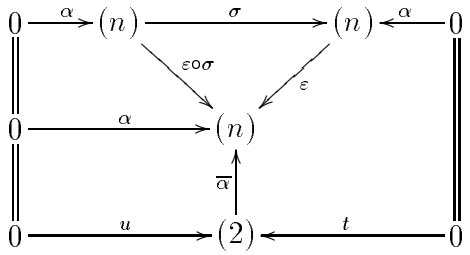
Define $\varepsilon : (n) \longrightarrow (n)$ to be the endomorphism, which has exactly one crossing point between the first two strings from the left side and no others.

Take α and define a new morphism $\overline{\alpha} : (2) \longrightarrow (n)$ by "opening" those two arcs which belong to the first two letters of (n) in the following way: Connect the first letter "$-$" of (n) with the first letter of (2) and the corresponding "$+$" of (n) with the last letter of (2). Similarly, connect the second "$-$" of (n) and the corresponding "$+$" with the second and third letters of (2). All other arcs of α remain the same.

This obviously gives the equation $\overline{\alpha} \circ u = \alpha$. But we also obtain $\varepsilon \circ \alpha = \overline{\alpha} \circ t$:



Consider the following diagram:



All parts of this diagram are commutative, except the upper left corner. But since σ and ε are odd, their composition $\varepsilon \circ \sigma$ is even and therefore null homotopic by 3.2 (c). Hence the upper left corner can be filled somehow by a commutative diagram, and the whole diagram yields a homotopy between ω and the path κ. $\square$

$B\mathcal{G}$ is an infinite loop space, hence all path components are homotopy equivalent and we have $\pi_1(B\mathcal{G}, w) = \mathbb{Z}/2$ for any chosen basepoint w. The generators of the other path components can be found as the images of the above generator under explicit functors, which induce homotopy equivalences between the various components:

3.4 Proposition: The functors $l_+,\ l_- : \mathcal{G} \longrightarrow \mathcal{G}$ which are defined by

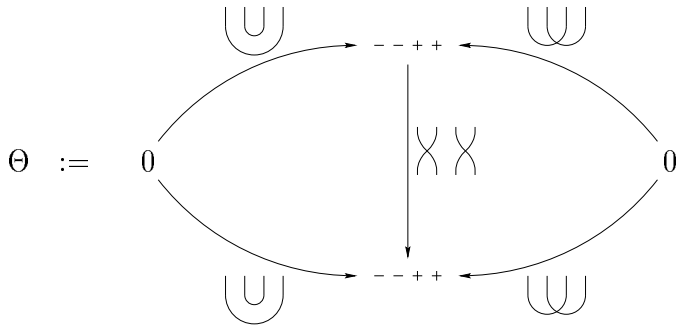
$$l_+(w) := + \oplus w,\ l_+(\alpha) := id_+ \oplus \alpha, \quad l_-(w) := - \oplus w,\ l_-(\alpha) := id_- \oplus \alpha,$$

map $\mathcal{G}^{(k)}$ into $\mathcal{G}^{(k+1)}$ and vice versa and induce homotopy inverse maps on classifying spaces. The same is true for $r_+, r_- : \mathcal{G} \longrightarrow \mathcal{G}$ given by

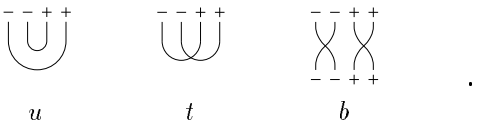
$$r_+(w) := w \oplus +,\ r_+(\alpha) := \alpha \oplus id_+, \quad r_-(w) := w \oplus -,\ r_-(\alpha) := \alpha \oplus id_-.$$

The proof is an easy exercise.

3.5 Theorem: $\pi_2(B\mathcal{G}, 0)$ has a non-trivial element of order 2, given by the realization of the following diagram:



PROOF. The given picture defines both a 2-sphere and a 2-cycle in $B\mathcal{G}$, and the Hurewicz homomorphism $\pi_2(B\mathcal{G}) \longrightarrow H_2(B\mathcal{G})$ leads from the first interpretation to the second. As a cycle Θ is defined by $(b,u)-(b,t)$, where u, t and b denote the morphisms



.

It will be shown that Θ represents a non-trivial $\mathbb{Z}/2$–element in $H_2(B\mathcal{G})$. Then by the Hurewicz map it has to be non-trivial in $\pi_2(B\mathcal{G})$ as well.

At first, we see that the order of Θ is at most 2, since 2Θ is the boundary of $(b,b,t)-(b,b,u)$ and therefore zero in $H_2(B\mathcal{G})$:

$$\begin{aligned}
\partial((b,b,t)-(b,b,u)) &= \partial(b,b,t)-\partial(b,b,u)\\
&= (b,b)-(b,b\circ t)+(b\circ b,t)-(b,t)\\
&\quad -(b,b)+(b,b\circ u)-(b\circ b,u)+(b,u)\\
&= (b,b)-(b,t)+0-(b,t)\\
&\quad -(b,b)+(b,u)-0+(b,u)\\
&= 2((b,u)-(b,t))\\
&= 2\Theta.
\end{aligned}$$

Note that the realization of $(b,b,t)-(b,b,u)$ actually gives a contraction of the sphere 2Θ in $B\mathcal{G}$.

The proof that Θ itself is not trivial is done in several steps:

Recall that $\mathcal{F}_{\mathrm{P}}^{(k)} = End(k) = \Sigma_k$, and there is the following presentation:

Let $t_1, \ldots, t_{k-1}$ be transpositions of Σ_k defined by

$$t_1 = (12), \quad t_2 = (23), \quad \ldots \quad t_{k-1} = (k-1,k).$$

Then we have

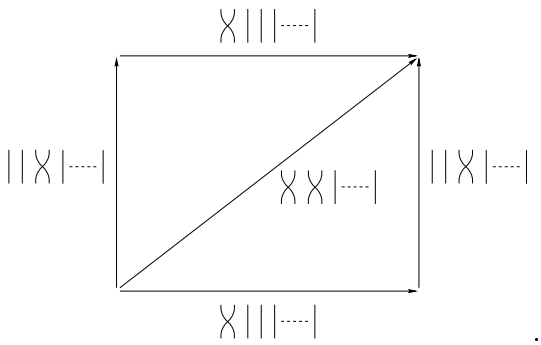
$$\Sigma_k = \langle t_1, \ldots, t_{k-1} \quad | \quad t_i^2 = (t_j t_{j+1})^3 = (t_m t_l)^2 = 1 \rangle$$

with $\quad i, m, l \in \{1, \ldots, k-1\}, \quad j \in \{1, \ldots, k-2\}, \quad |m-l| \geq 2.$
(See [2], Prop. 2.12.1).

Step 1: $$H_2(B\Sigma_k) = \begin{cases} 0 \quad \text{for } k \leq 3 \\ \mathbb{Z}/2 \text{ for } k \geq 4 \end{cases},$$

and a generator for $k \geq 4$ is represented by

$\Gamma_k := (t_1, t_3) - (t_3, t_1)$



.

The above picture gives the graphical interpretation of Γ_k. The upper and lower sides correspond to t_1, the left and right sides correspond to t_3.

An abstract computation of these homology groups is the combination of the theorems 2.7.3 and 2.12.3 of [2], and the assertion about the generator is contained in the proofs of those theorems.

Step 2: The inclusion $\Sigma_{k_0} = \mathcal{F}_{\mathrm{P}}^{(k_0)} \xrightarrow{i_+} \mathcal{G}^{(k_0)}$ (cf. 2.2) induces an isomorphism

$$(i_+)_* : H_2(B\Sigma_{k_0}) \xrightarrow{\cong} H_2(B\mathcal{G}^{(k_0)}) \quad \text{for} \quad k_0 \geq 4.$$

Recall the statements of the $S^{-1}S$ construction 1.5 and of Theorem 2.4.
The monoid $S = \pi_0(B\mathcal{F}_{\mathrm{P}}) = \mathbb{N}$ acts on

$$H_2(B\mathcal{F}_{\mathrm{P}}) = \bigoplus_{k \in \mathbb{N}} H_2(B\Sigma_k) \cong \bigoplus_{k \geq 4} \mathbb{Z}/2,$$

the k-th summand of which is generated by $\overline{\Gamma}_k$, i.e. the class of Γ_k. The action is translation, i.e. adding positive trivial strings. So for $n \in \mathbb{N} = \pi_0(B\mathcal{F}_{\mathrm{P}})$ we have

$$n \cdot \overline{\Gamma}_k = \overline{\Gamma}_{k+n}.$$

(By definition Γ_{k+n} is created from Γ_k by adding n trivial strings on the right side, while the translation from $\pi_0(B\mathcal{F}_{\mathrm{P}})$ is given from the left side. But obviously this does not matter, since an evident automorphism of Σ_{k+n} exchanges these different 2-cycles, which shows that both represent $\overline{\Gamma}_{k+n}$).

Invoking the theorem of 1.5, we find an isomorphism

$$\begin{aligned} H_2(B\mathcal{G}) &\cong (\pi_0(B\mathcal{F}_{\mathrm{P}}))^{-1} H_2(B\mathcal{F}_{\mathrm{P}}) \\ &= (\pi_0(B\mathcal{F}_{\mathrm{P}}) \times H_2(B\mathcal{F}_{\mathrm{P}}))/\pi_0(B\mathcal{F}_{\mathrm{P}}) \\ &\cong (\mathbb{N} \times \bigoplus_{k\geq 4} \mathbb{Z}/2)/\mathbb{N} \\ &\cong \bigoplus_{k\in\mathbb{Z}} \mathbb{Z}/2. \end{aligned}$$

The latter isomorphism follows from the definition of the action of $\mathbb{N}$ on $\mathbb{N} \times \bigoplus_{k\geq 4} \mathbb{Z}/2$, which, by using $n \cdot \overline{\Gamma}_k = \overline{\Gamma}_{k+n}$, leads to

$$\begin{aligned} (m, (x_4, x_5, x_6, \ldots)) &\sim n \cdot (m, (x_4, x_5, x_6, \ldots)) \\ &= (n+m, n \cdot (x_4, x_5, x_6, \ldots)) \\ &= (n+m, (\underbrace{0, \ldots, 0}_{n}, x_4, x_5, \ldots)). \end{aligned}$$

Now look at the following commutative diagram:

$$\begin{array}{ccc} H_2(B\Sigma_{k_0}) & \xrightarrow{(i_+)_*} & H_2(B\mathcal{G}^{(k_0)}) \\ \downarrow & & \downarrow \\ \bigoplus_{k\geq 4} H_2(B\mathcal{F}_{\mathrm{P}}^{(k)}) & \xrightarrow{(i_+)_*} & \bigoplus_{k\in\mathbb{Z}} H_2(B\mathcal{G}^{(k)}) \\ \| & & \| \\ H_2(B\mathcal{F}_{\mathrm{P}}) & \xrightarrow{(i_+)_*} & H_2(B\mathcal{G}) \\ \downarrow{\scriptstyle \xi} & \overset{\cong}{\nearrow}{\scriptstyle \Psi} & \\ (\pi_0(B\mathcal{F}_{\mathrm{P}}))^{-1} H_2(B\mathcal{F}_{\mathrm{P}}) & & \end{array}$$

The universal arrow is $\xi : \gamma \longmapsto (0, \gamma)$, and the induced map can be expressed as $\Psi : (n, \gamma) \longmapsto (l_-)_*^n (i_+)_*(\gamma)$ with l_- being the functor of 3.4.

This shows that the map $(i_+)_* : H_2(B\mathcal{F}_{\mathrm{P}}) \longrightarrow H_2(B\mathcal{G})$ corresponds to the inclusion $\bigoplus_{k\geq 4} \mathbb{Z}/2 \hookrightarrow \bigoplus_{k\in\mathbb{Z}} \mathbb{Z}/2$. Now the assertion follows immediately.

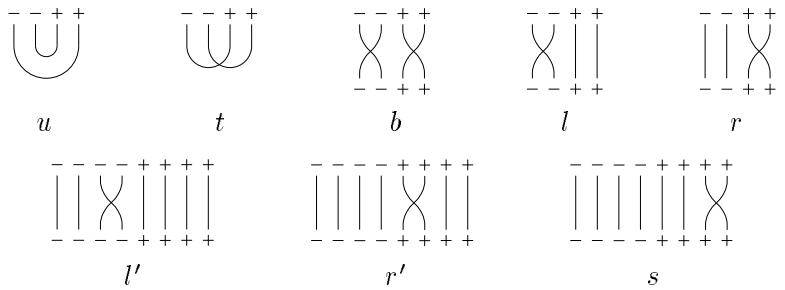
Step 3: Using the functor l_- of 3.4, a generator for $H_2(B\mathcal{G}^{(0)})$ is given by

$$\Gamma := (l_-)^4_*(i_+)_*(\Gamma_4).$$

This is obvious, since $(i_+)_*(\Gamma_4)$ is a generator of $H_2(B\mathcal{G}^{(4)})$ according to step 2, and l_- is a homotopy equivalence.

Step 4: Θ and Γ are homologous.

Consider the following morphisms of $\mathcal{G}^{(0)}$:



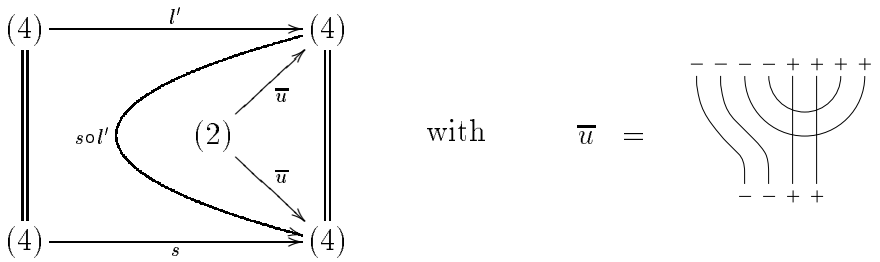
We have $\Theta = (b,u) - (b,t)$ and obviously $\Gamma = (r',s) - (s,r')$.
Compute the following boundary:

$$\begin{aligned}\partial((l,r,u) - (l,l,u) + (b,l,u) - (b,l,l)) = \quad & (l,r) - (l,t) + (b,u) - (r,u) \\ & -(l,l) + (l,t) - 0 + (l,u) \\ & +(b,l) - (b,t) + (r,u) - (l,u) \\ & -(b,l) + 0 - (r,l) + (l,l) \\ = \ & ((b,u) - (b,t)) - ((r,l) - (l,r)).\end{aligned}$$

This shows that Θ is homologous to $(r,l) - (l,r)$.

The latter cycle is homotopic (and hence homologous) to $(r',l') - (l',r')$, since it only has two additional positive and negative trivial strings (cf. 3.4).

The path defined by l' is based homotopic to the path defined by s, as the following commutative diagram shows:



This homotopy leads to a homotopy between the cycle $(r', l') - (l', r')$ and $(r', s) - (s, r')$, which is Γ.

This completes the proof of 3.5. $\square$

4 The free permutative category with duality

4.1 Definition: A duality in a monoidal category associates to every object V an object V^* and two morphisms

$$b_V : 0 \longrightarrow V \oplus V^*, \qquad d_V : V^* \oplus V \longrightarrow 0$$

subject to the following relations:

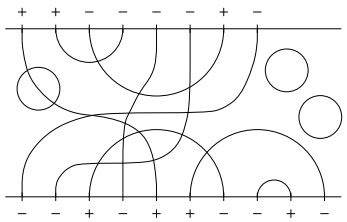
$$id_V \oplus d_V)(b_V \oplus id_V) = id_V, \qquad (d_V \oplus id_{V^*})(id_{V^*} \oplus b_V) = id_{V^*}$$

4.2 Construction: Let $\mathcal{F}_{\mathrm{P,D}}$ be the following category: The set of objects is the free monoid on the alphabet $\{+, -\}$. We think of an object with k letters as the subset $\{1, \ldots, k\} \subset \mathbb{R}$ with an assigned $+$ or $-$ to each i.
A morphism of a word w_1 with k_1 letters to another word w_2 with k_2 letters is represented by an unordered finite set $\{f_\lambda\}_{\lambda \in \Lambda}$ of paths $f_\lambda : [0,1] \longrightarrow \mathbb{R} \times [0,1]$ subject to the following conditions:

(1) Each f_λ connects two different points of the subset $\{1, \ldots, k_1\} \times \{0\} \cup \{1, \ldots, k_2\} \times \{1\}$, or it is a closed path, i.e. $f_\lambda(0) = f_\lambda(1)$.
(2) Each point of $\{1, \ldots, k_1\} \times \{0\} \cup \{1, \ldots, k_2\} \times \{1\}$ is in the image of exactly one f_λ, which is not closed.
(3) Each f_λ which is not closed connects either an element from $\{1, \ldots, k_1\} \times \{0\}$ with an element from $\{1, \ldots, k_2\} \times \{1\}$ having the same sign (called "vertical strings")
or two elements from $\{1, \ldots, k_2\} \times \{1\}$ having different signs (called "upper arcs")
or two elements from $\{1, \ldots, k_1\} \times \{0\}$ having different signs (called "lower arcs").

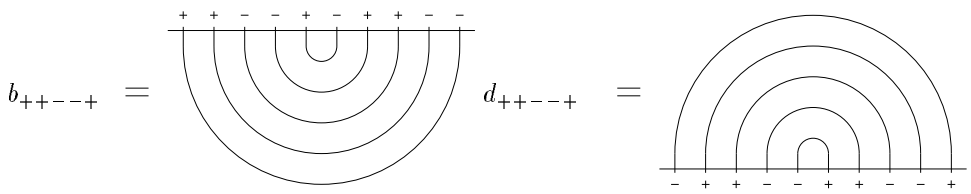
The closed paths are called "cycles". The term "string" is used for all kinds of paths. Two such systems $\{f_\lambda\}_{\lambda\in\Lambda}$ and $\{f'_\lambda\}_{\lambda\in\Lambda}$ represent the same morphism if there is a family of homotopies $\{H_\lambda : f_\lambda \simeq f'_\lambda \quad \text{or} \quad -f_\lambda \simeq f'_\lambda\}_{\lambda\in\Lambda}$ which keep the boundary points of the non-closed paths fixed.

Morphisms can be illustrated by pictures like this:



Again, such a picture is called a "pseudo-tangle".
Composition, tensor product and the permutative structure are defined in the same way as in $\mathcal{G}$.

$\mathcal{F}_{\mathrm{P,D}}$ has a duality structure. One obtains the dual w^* of a word w by reading it from right to left and replacing "+" by "−" and vice versa. Example: $(++--+)^* = -++--$.
The morphisms b_w and d_w are given by the unique pseudo-tangles, which have no cycles and no crossing point among the strings:



4.3 Theorem:
(a) $\mathcal{F}_{\mathrm{P,D}}$ is isomorphic to the free permutative category with duality which is generated by one object.

(b) Under tensor product and composition, $\mathcal{F}_{\mathrm{P,D}}$ is generated by the duality morphisms b_w, d_w and the symmetry isomorphisms $c_{v,w}$, where v, w run through all objects of $\mathcal{F}_{\mathrm{P,D}}$.

PROOF. This is derived from Proposition 4.1, Theorem 8.2 and §9 of [4] by using the methods of [7]. $\mathcal{F}_{\mathrm{P,D}}$ is isomorphic to the category $F''\mathcal{A}$ of [4, §9], where $\mathcal{A}$ is the trivial category. So in [4, Prop. 4.1] the morphisms a, a^{-1}, r, r^{-1} and f are identities. □

Define $\mathcal{RF}_{\mathrm{P,D}}$ to be the full subcategory of $\mathcal{F}_{\mathrm{P,D}}$ which consists of all ordered words, and define functors $R : \mathcal{F}_{\mathrm{P,D}} \longrightarrow \mathcal{RF}_{\mathrm{P,D}}$ and $J : \mathcal{RF}_{\mathrm{P,D}} \longrightarrow \mathcal{F}_{\mathrm{P,D}}$ as on page 7. The following lemma has the same proof as 2.7.

4.4 Lemma: $\mathcal{RF}_{\mathrm{P,D}}$ is a permutative category with tensor product $\hat{\oplus} := R\circ\oplus$ and symmetry isomorphisms $\hat{c}_{v,w} := R(c_{v,w})$. It is equivalent as a permutative category to $\mathcal{F}_{\mathrm{P,D}}$ by the functors J and R. In particular, the classifying spaces of $\mathcal{RF}_{\mathrm{P,D}}$ and $\mathcal{F}_{\mathrm{P,D}}$ are homotopy equivalent as infinite loop spaces.

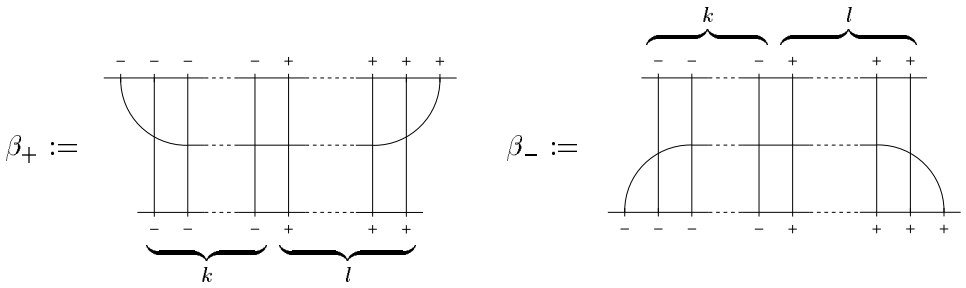
4.5 Remarks:
(a) $B\mathcal{RF}_{\mathrm{P,D}}$ is a strong deformation retract of $B\mathcal{F}_{\mathrm{P,D}}$.

(b) $\mathcal{G}$ is a subcategory of $\mathcal{F}_{\mathrm{P,D}}$ and the retraction R commutes with the inclusion, i.e. the inclusion $(B\mathcal{RF}_{\mathrm{P,D}}, B\mathcal{RG}) \hookrightarrow (B\mathcal{F}_{\mathrm{P,D}}, B\mathcal{G})$ is a homotopy equivalence of pairs.

PROOF. For (a) compare 2.8, and (b) is a trivial consequence. □

Directly from 4.3 we obtain

4.6 Proposition: Under composition, $\mathcal{F}_{\mathrm{P,D}}$ is generated by the symmetry isomorphisms $c_{v,w}$ and the following morphisms $\beta_- = \beta_-^{(k,l)}$, $\beta_+ = \beta_+^{(k,l)}$



where the numbers k and l vary in $\mathbb{N}$.

5 The homotopy fibre of the inclusion $\mathcal{G} \hookrightarrow \mathcal{F}_{\mathrm{P,D}}$

Let $I : \mathcal{G} \hookrightarrow \mathcal{F}_{\mathrm{P,D}}$ denote the inclusion functor.

5.1 Observation:
$\pi_0(B\mathcal{F}_{\mathrm{P,D}}) = \mathbb{Z}$ as a group. In particular, $B\mathcal{F}_{\mathrm{P,D}}$ is an infinite loop space. $\pi_0 BI : \pi_0(B\mathcal{G}) \longrightarrow \pi_0(B\mathcal{F}_{\mathrm{P,D}})$ is an isomorphism of groups.

PROOF. The proof is essentially the same as for 3.1. □

5.2 Theorem: (Quillen's Theorem B (dual version), [8]) Let $F : \mathcal{C} \longrightarrow C'$ be a functor such that for every arrow $f : Y \longrightarrow Y'$ in $\mathcal{C}'$ the induced functor $f_* : F/Y \longrightarrow F/Y'$ gives a homotopy equivalence on classifying spaces. Then

for any object Y of $\mathcal{C}'$ the cartesian square of categories

$$\begin{array}{ccc} F/Y & \xrightarrow{j} & \mathcal{C} \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{C}'/Y & \xrightarrow{j'} & \mathcal{C}' \end{array} \qquad \begin{array}{l} j(X,v) = X \\ F'(X,v) = (F(X),v) \\ j'(Y',v) = Y' \end{array}$$

is homotopy-cartesian (i.e. homotopy-cartesian after applying the functor B). Consequently, for any X in $F^{-1}(Y)$ we have an exact sequence

$$\cdots \to \pi_{i+1}(B\mathcal{C}', Y) \to \pi_i(B(F/Y), \overline{X}) \xrightarrow{j_*} \pi_i(B\mathcal{C}, X) \xrightarrow{F_*} \pi_i(B\mathcal{C}', Y) \to \cdots$$

where $\overline{X} = (X, id_Y)$.

(Recall: $ob\, F/Y$ = pairs (X,u) with $X \in ob\, \mathcal{C}, u : F(X) \longrightarrow Y$. A morphism from (X,u) to (X',u') is a map $v : X \longrightarrow X'$ such that $u = u' \circ F(v)$. Every arrow $f : Y \longrightarrow Y'$ in $\mathcal{C}'$ induces a functor $f_* : F/Y \longrightarrow F/Y'$ by

$$f_*(X,u) = (X, f \circ u), \qquad f_*(v) = v.)$$

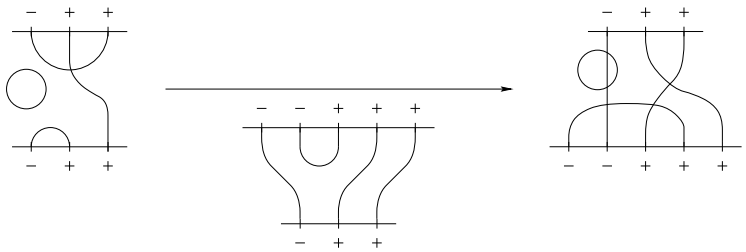
The spaces $B\mathcal{G}$ and $B\mathcal{F}_{\mathrm{P,D}}$ are infinite loop spaces. So, all their path components are homotopy equivalent, and for homotopy questions it is sufficient to concentrate on a single chosen component. Fortunately, Theorem B can be applied after restricting $I : \mathcal{G} \hookrightarrow \mathcal{F}_{\mathrm{P,D}}$ to the component of the object $+$. Again, the appropriate subcategories are denoted $\mathcal{G}^{(1)}$ and $\mathcal{F}^{(1)}_{\mathrm{P,D}}$, respectively.

5.3 Theorem: The inclusion functor $I : \mathcal{G}^{(1)} \hookrightarrow \mathcal{F}^{(1)}_{\mathrm{P,D}}$ satisfies the condition of Theorem B. Hence $I/+$ is a category whose classifying space describes the homotopy fibre of BI.

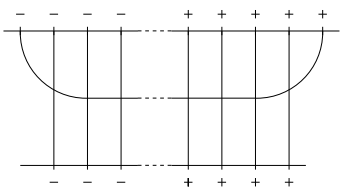
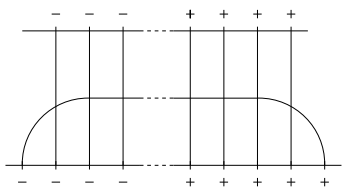
PROOF. It suffices to show that the inclusion $I : \mathcal{R}G^{(1)} \hookrightarrow \mathcal{R}\mathcal{F}^{(1)}_{\mathrm{P,D}}$ satisfies the condition of Theorem B. Note that $\mathcal{R}G^{(1)}$ and $\mathcal{R}\mathcal{F}^{(1)}_{\mathrm{P,D}}$ are full subcategories of $\mathcal{G}$ and $\mathcal{F}_{\mathrm{P,D}}$, respectively.

The set of objects for both $\mathcal{R}G^{(1)}$ and $\mathcal{R}\mathcal{F}^{(1)}_{\mathrm{P,D}}$ is $\{+, -++, --+++, \ldots\}$ and these words will be denoted $w_0, w_1, w_2, \ldots$ according to the number of letters "$-$".
An object of I/w_k is a pair (w, α) with $w \in ob\, \mathcal{R}G^{(1)}$ and $\alpha : I(w) = w \longrightarrow w_k$ being a morphism in $\mathcal{F}_{\mathrm{P,D}}$. Obviously, w is determined by α and can be dropped from the notation. But for a better distinction, objects are sometimes denoted by $(w \uparrow_\alpha w_k)$. A morphism from $(w \uparrow_\alpha w_k)$ to $(w' \uparrow_{\alpha'} w_k)$ is a morphism $w \xrightarrow{\varphi} w'$ in $\mathcal{G}$ with $\alpha = \alpha' \circ I(\varphi)$, i.e. $\alpha = \alpha' \circ \varphi$.

For example, a typical morphism in I/w_1 can be pictured as follows:



Every arrow $\beta : w_k \longrightarrow w_l$ in $\mathcal{RF}^{(1)}_{\mathrm{P,D}}$ induces a functor $\beta_* : I/w_k \longrightarrow I/w_l$ by $\beta_*(w \uparrow_\alpha w_k) = (w \uparrow_{\beta\circ\alpha} w_l)$ on objects and $\beta_*(\varphi) = \varphi$ on morphisms, which has to be a homotopy equivalence on classifying spaces.
Because of 4.6 it suffices to prove the assertion for the morphisms $\beta_+ : w_k \longrightarrow w_{k+1}$ and $\beta_- : w_{k+1} \longrightarrow w_k$:

$\beta_+ \;:=$  $\beta_- \;:=$ 

We will prove the following claim:

β_+ and β_- induce homotopy inverse maps between $B(I/w_k)$ and $B(I/w_{k+1})$.

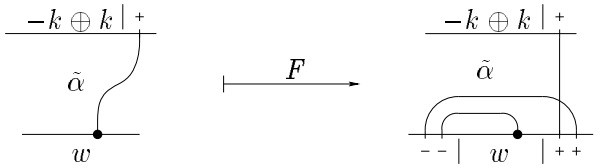
The value of the functor $(\beta_- \circ \beta_+)_* : I/w_k \longrightarrow I/w_k$ is given on an object by placing $\beta_- \circ \beta_+$ on top of it, which gives the identity plus one cycle. So $(\beta_- \circ \beta_+)_*$ simply adds one cycle to every object and there is no natural transformation between Id_{I/w_k} and this functor, since the morphisms of $\mathcal{G}$ contain no cycles. But there is a certain functor $F : I/w_k \longrightarrow I/w_k$ and there are natural transformations

$$Id_{I/w_k} \xrightarrow{\eta_t} F \xleftarrow{\eta_u} (\beta_- \circ \beta_+)_*$$

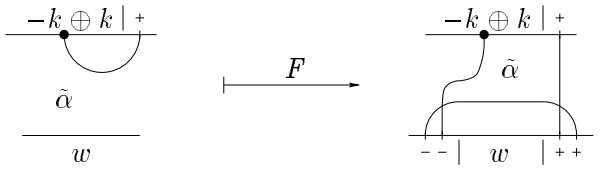
which together yield the desired homotopy.

The functor F associates to an object $(w \uparrow_\alpha w_k)$ a slightly modified object $((-2 \oplus w \oplus 2) \uparrow w_k)$ and the concrete definition depends on whether the final letter "+" of w_k belongs to a vertical string or to an upper arc (then the object α is said to be of type (S) or type (A), respectively).

(S)



(A)

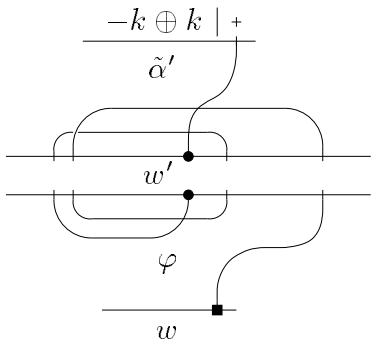


These pictures are to be understood as follows: The only important string of α is that which belongs to the final letter "+" of $w_k = -k \oplus k \oplus +$, called the "final string". The black dot marks the position where this string ends. For the definition of $F(w \uparrow_\alpha w_k)$ the final string is changed with the help of the additional letters at the bottom base and two further strings. All other strings of $F(w \uparrow_\alpha w_k)$ are given by the remaining strings of α (indicated by $\tilde{\alpha}$).

On morphisms $\alpha \xrightarrow{\varphi} \alpha'$ the functor F is given by $F(\varphi) := id_{-2} \oplus \varphi \oplus id_2$. It is clear that F satisfies the identities $F(id) = id,\ F(\varphi \circ \varphi') = F(\varphi) \circ F(\varphi')$. It has only to be shown that F is well-defined, i.e. that $F(\varphi)$ is a morphism from $F(w \uparrow_\alpha w_k)$ to $F(w' \uparrow_{\alpha'} w_k)$. The following cases, depending on the type of α and α', have to be distinguished:

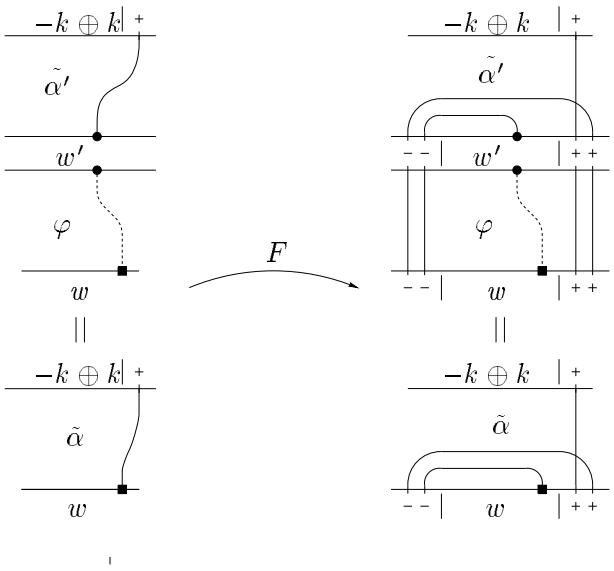
Case 1: α and α' are both of type (S).

After composition with φ, the black dot of α' is somehow connected with the distinguished letter at the bottom base which belongs to the final string of α (indicated by the black square in the pictures below). This may happen by a simple vertical string, but it is also possible that several upper arcs of φ and lower arcs of α' are involved.

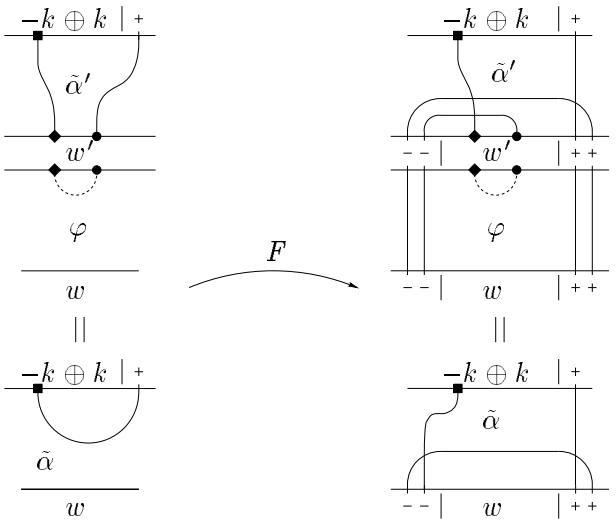


In any case, this does not matter, because F preserves all these possibly involved arcs, and the assertion can be seen from the following picture, in which

the connection between the distinguished letters is drawn by a dashed line.



Case 2: α is of type (A) and α' is of type (S).



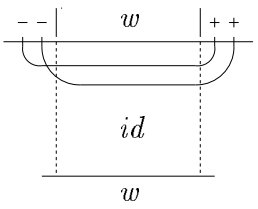
Again, there is some connection between the black dot and the black square, but the square is now located at the top of α' and is connected with another distinguished letter (black rhombus) at the bottom of α'. Hence there is a link between the dot and the rhombus which is given by a simple upper arc of φ or by a similar detour as in the preceding case.

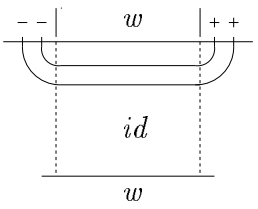
Case 3: α and α' are both of type (A).

Here φ has no effect on the relevant arc of α and the situation is trivial.

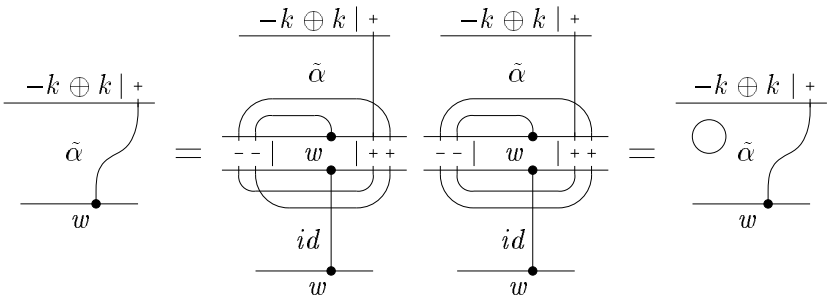
Obviously, the fourth case (α being of type (S), α' of type (A)) does not exist. Hence the well-definition of F is proved.

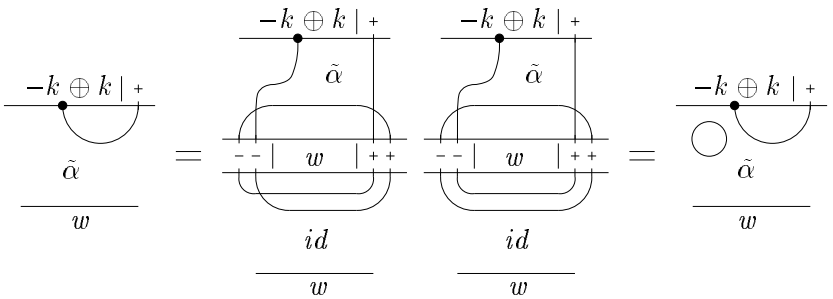
Now we give the stated natural transformations $Id_{I/w_k} \xrightarrow{\eta_t} F \xleftarrow{\eta_u} (\beta_- \circ \beta_+)_*$. For a given object $(w \uparrow_\alpha w_k)$ of I/w_k the transformations are defined by

$\eta_t(w \uparrow_\alpha w_k) :=$ 

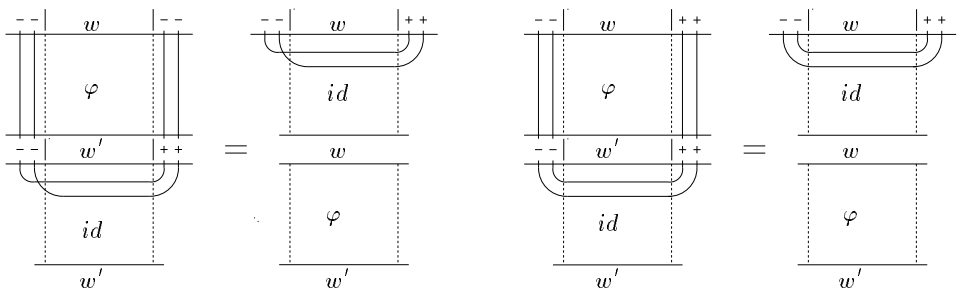
$\eta_u(w \uparrow_\alpha w_k) :=$ 

The following pictures show that $\eta_t(\alpha)$ and $\eta_u(\alpha)$ are actually morphisms from $Id(w \uparrow_\alpha w_k)$ and $(\beta_- \circ \beta_+)_*(w \uparrow_\alpha w_k)$, respectively, to $F(w \uparrow_\alpha w_k)$.

(S) 

(A) 

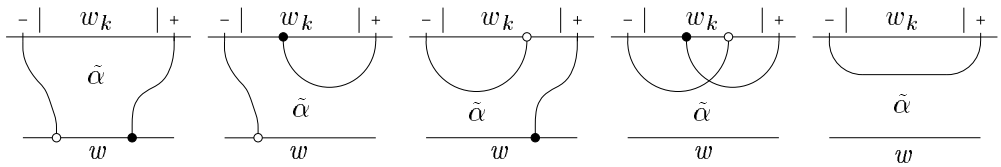
Finally, the equation $F(\varphi) \circ \eta(\alpha) = \eta(\alpha') \circ \varphi$ has to be checked for $\eta = \eta_t, \eta_u$. (Note that $Id(\varphi) = (\beta_- \circ \beta_+)_*(\varphi) = \varphi$). This is done by the next picture:



This completes the proof of $B(\beta_- \circ \beta_+)_* \simeq id_{B(I/w_k)}$. This also covers the case $k = 0$, when the object type (A) cannot occur; this only reduces the number

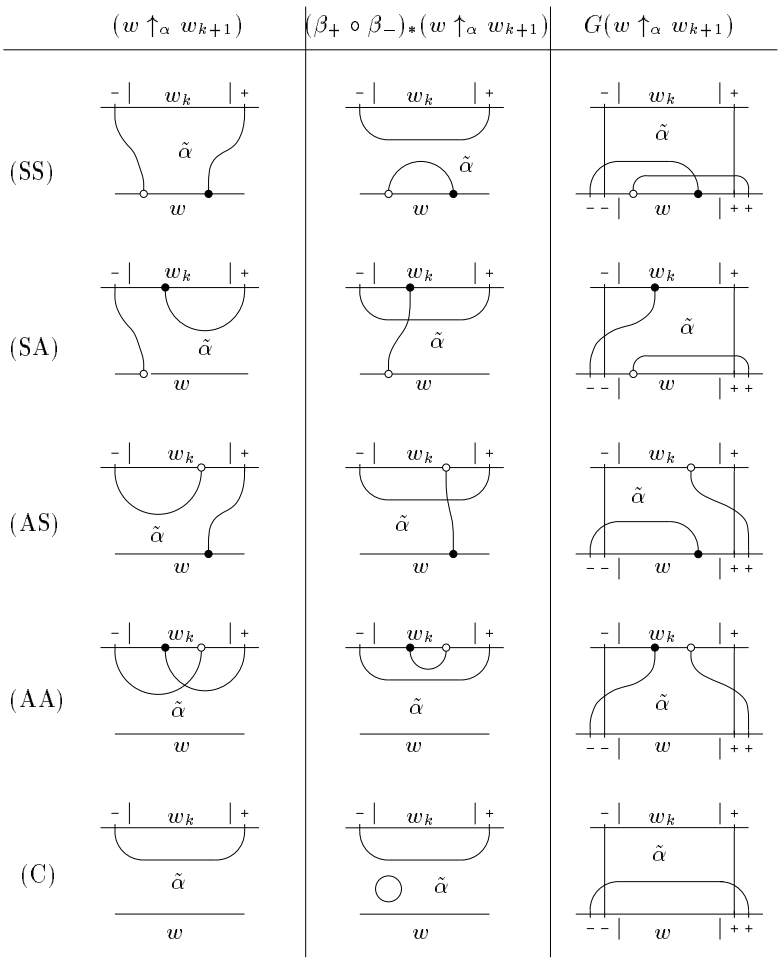
of cases which have to be checked.

The way to prove $B(\beta_+ \circ \beta_-)_* \simeq id_{B(I/w_{k+1})}$ is very similar: one gives a functor $G : I/w_{k+1} \longrightarrow I/w_{k+1}$ and natural transformations $Id \xrightarrow{\eta_t} G \xleftarrow{\eta_u} (\beta_+ \circ \beta_-)_*$, but there are more cases which have to be distinguished. We have five types of objects $(w \uparrow_\alpha w_{k+1})$ in I/w_{k+1}, depending on the shape of the strings belonging to the initial and to the final letter of $w_{k+1} = - \oplus w_k \oplus +$:



(SS) (SA) (AS) (AA) (C)

The values $(\beta_+ \circ \beta_-)_*(w \uparrow_\alpha w_{k+1})$ are computed by placing $\beta_+ \circ \beta_-$ on top of these objects. The following table shows the different types in the first column, the appropriate values under the functor $(\beta_+ \circ \beta_-)_*$ in the second column and the definition of G on these different types in the final column:



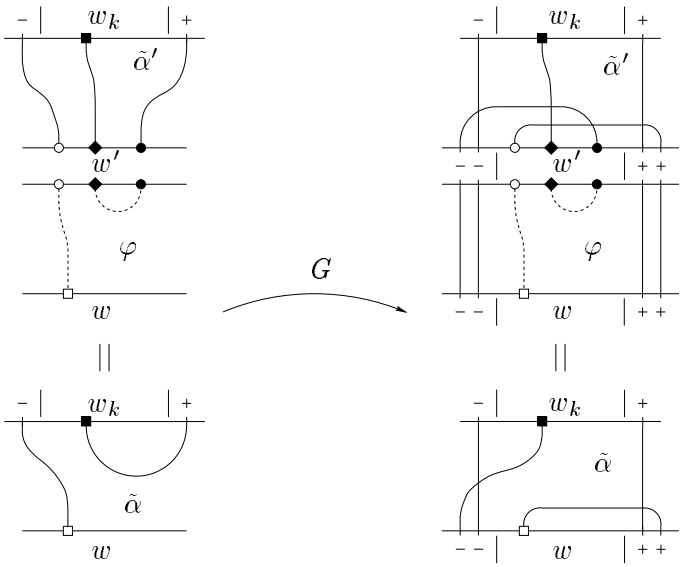
Again, G is defined by $G(\varphi) = id_{-2} \oplus \varphi \oplus id_2$ on morphisms.
For the well-definition of G, eleven transitions have to be checked:

Case	1	2	3	4	5
Type of α	(SS)	(SA)	(AS)	(AA)	(C)
Type of α'	(SS)	(SS)	(SS)	(SS)	(SS)

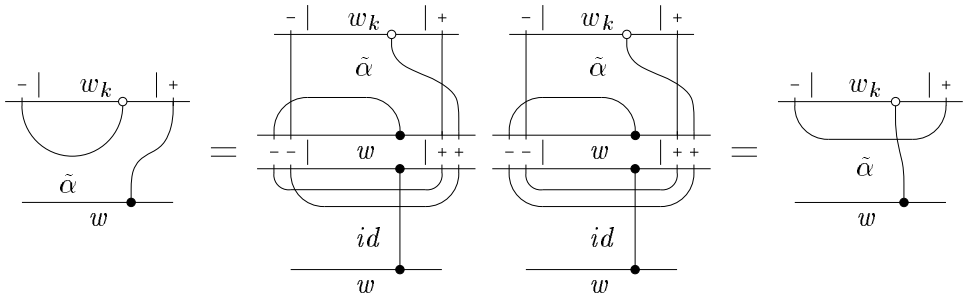
6	7	8	9	10	11
(SA)	(AA)	(AS)	(AA)	(AA)	(C)
(SA)	(SA)	(AS)	(AS)	(AA)	(C)

As an example, we give the picture for the second case (transition from (SA) to (SS)) and invite the reader to check the other cases by himself:

Case 2: α is of type (SA) and α' is of type (SS).



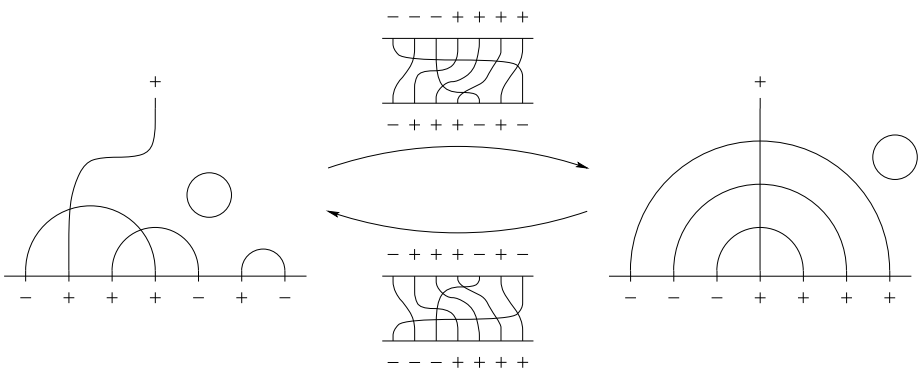
The natural transformations $Id \xrightarrow{\eta_t} G \xleftarrow{\eta_u} (\beta_+ \circ \beta_-)_*$ are defined exactly in the same way as before. Hence the only thing which remains to be shown is that $\eta_t(\alpha)$ and $\eta_u(\alpha)$ are actually morphisms from $Id(\alpha)$ and $(\beta_+ \circ \beta_-)_*(\alpha)$, respectively, to $G(\alpha)$. Of the five cases we prove the case (AS):

(AS) 

For $k = 0$, i.e. $w_{k+1} = - + +$, the object types (SA) and (AA) do not occur. Again, this only reduces the number of cases which have to be checked. This completes the proof of 5.3. $\square$

5.4 Remarks:
The inclusion $I : \mathcal{G}^{(0)} \longrightarrow \mathcal{F}^{(0)}_{\mathrm{P,D}}$ does not fullfil the condition of Quillen's Theorem B, because $I/0$ is not connected, while all other I/w are.
We have worked with the dual formulation of Theorem B. It is not possible to use the original formulation, since all categories $w\backslash I$ already have completely different sets of path components.

Theorem B can be applied to the inclusion $I : \mathcal{G}^{(1)} \longrightarrow \mathcal{F}^{(1)}_{\mathrm{P,D}}$, and now we take the simplest model for its homotopy fibre, which is $I/+$, because every object $(w \uparrow_\alpha +)$ consists of only lower arcs, cycles and exactly one vertical string.
A further step of simplification is the passage to an equivalent subcategory of $I/+$ by choosing a skeleton, i.e. a full subcategory such that any object of $I/+$ is isomorphic to exactly one object of this subcategory (cf. [5]).
The isomorphisms of $I/+$ are defined by isomorphisms in $\mathcal{G}$. Hence any two objects $(w \uparrow_\alpha +)$, $(w' \uparrow_{\alpha'} +)$ are isomorphic if and only if the numbers of lower arcs and cycles coincide. We define $\mathcal{H}$ to be the full subcategory of $I/+$ which consists of objects $(w \uparrow_\alpha +)$ in which w is ordered, the vertical string links the upper "+" with the "+" in the middle of w and all lower arcs have no intersections among them. This category $\mathcal{H}$ is a skeleton of $I/+$. The next picture shows the link between an arbitrary object of $I/+$ and the corresponding object from $\mathcal{H}$ by an isomorphism (which is not unique):



We have the following corollary from 5.3:

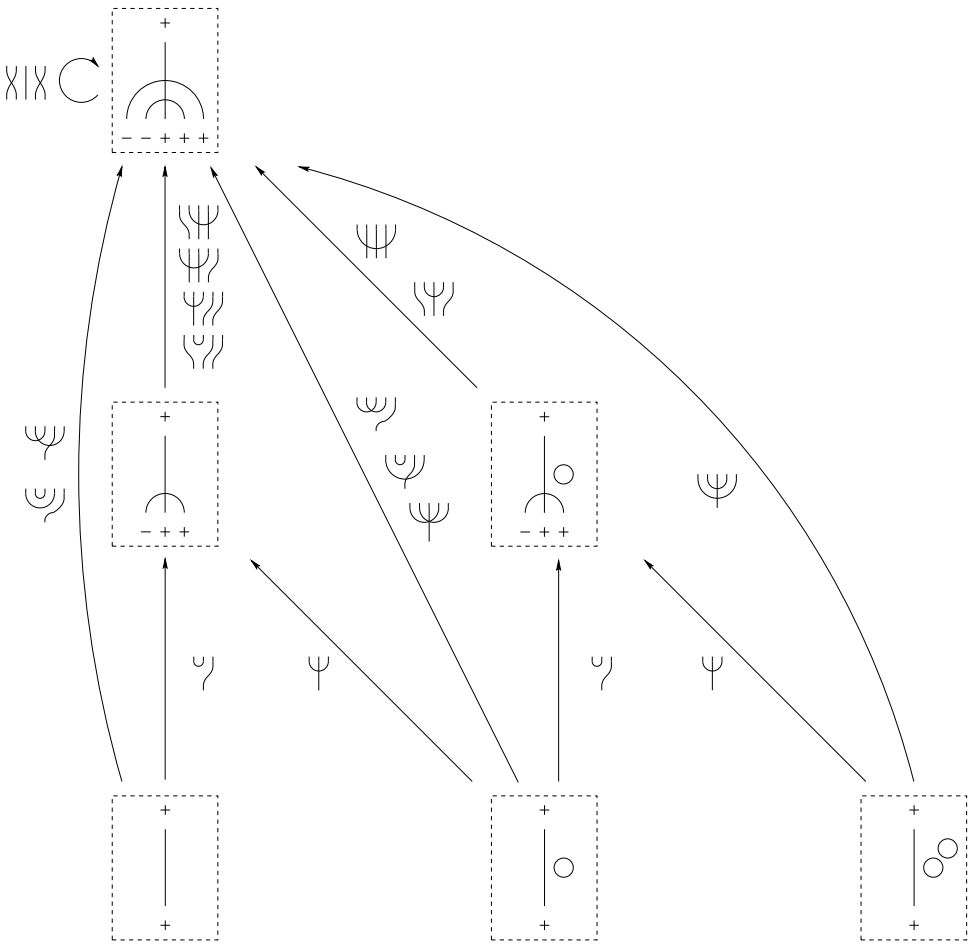
5.5 Corollary: For any choice of basepoint in $\mathcal{H}$ there is an exact sequence

$$\cdots \xrightarrow{I_*} \pi_{i+1} B\mathcal{F}^{(1)}_{\mathrm{P,D}} \longrightarrow \pi_i B\mathcal{H} \xrightarrow{p_*} \pi_i B\mathcal{G}^{(1)} \xrightarrow{I_*} \pi_i B\mathcal{F}^{(1)}_{\mathrm{P,D}} \longrightarrow \cdots$$

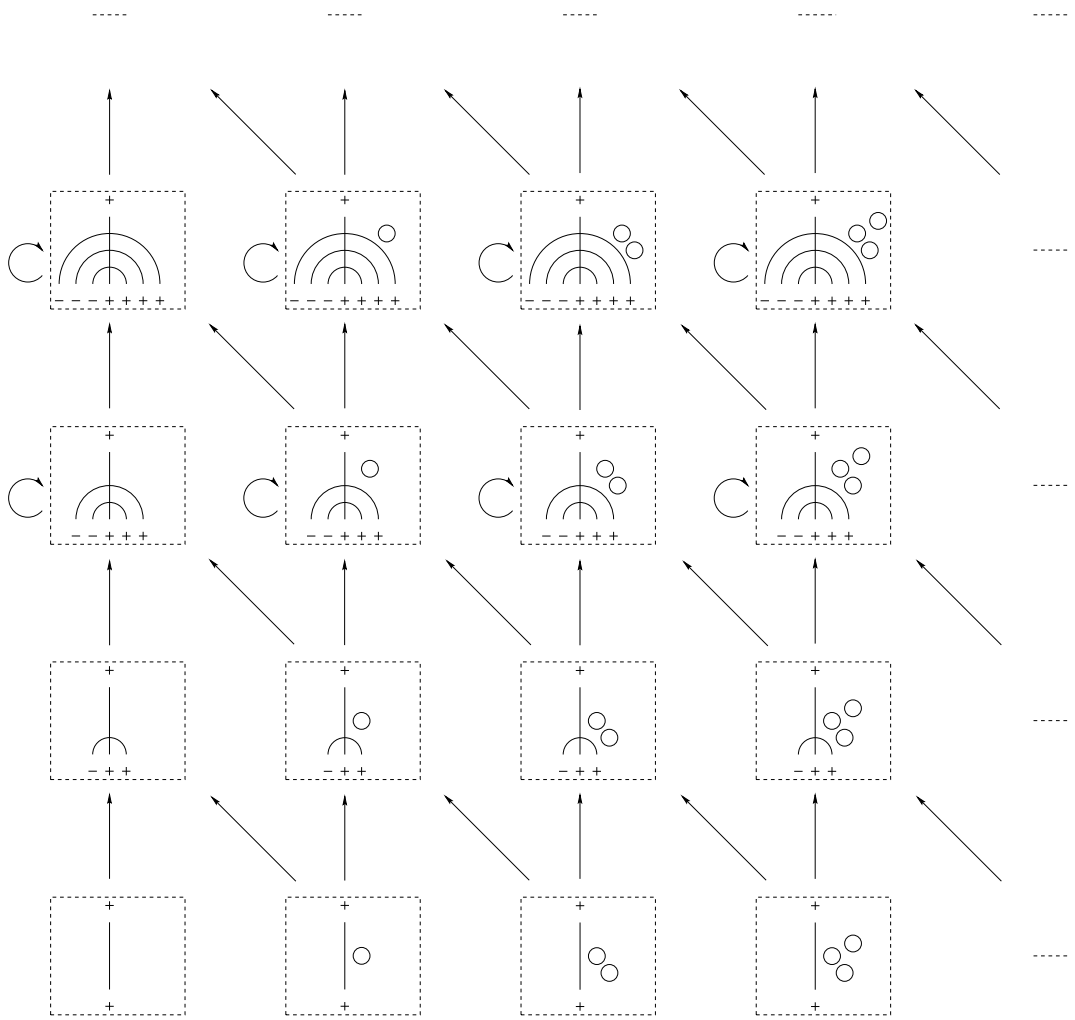
where $p : \mathcal{H} \longrightarrow \mathcal{G}^{(1)}$ is the functor given by

$$p(w \uparrow_\alpha +) = w, \qquad p(\varphi) = \varphi$$

We start with the analysis of $\mathcal{H}$. To get used to the structure of $\mathcal{H}$, the following picture displays some objects and all non-identity morphisms between them:



This leads to the following scheme (each arrow stands for a collection of morphisms):



Obviously, the scheme contains all objects. The arrows indicate generating morphisms, hence there is a morphism from one object to another if and only if there is a path along some arrows in the scheme.
The diagonal arrows stand for morphisms which open cycles to lower arcs. The vertical arrows indicate morphisms which generate a lower arc without decreasing the number of cycles by decomposing the vertical string or another lower arc in a suitable way. Finally, the circular arrows stand for non-trivial automorphisms by permuting the lower arcs. We arrive at a first observation:

5.6 Observation: $\mathcal{H}$ is path-connected, i.e. $\pi_0(B\mathcal{H}) = 0$.

The following result, which is our next goal, is more complicated:

5.7 Theorem: $\mathcal{H}$ is simply-connected, i.e. $\pi_1(B\mathcal{H}) = 0$.

Before starting with the proof of the theorem, we record some crucial properties of $\mathcal{H}$ in the following lemma. Let (c, a) denote the object of $\mathcal{H}$ with

c cycles and a arcs, and Δ_n the full subcategory of $\mathcal{H}$ which consists of all objects (c,a) with $c+a=n$.

5.8 Lemma:

(a) Δ_n is contractible.

(b) Each morphism $(c,a) \longrightarrow (0,n)$ in $\mathcal{H}$ can be factored (non-uniquely) as $(c,a) \longrightarrow (0,c+a) \longrightarrow (0,n)$.

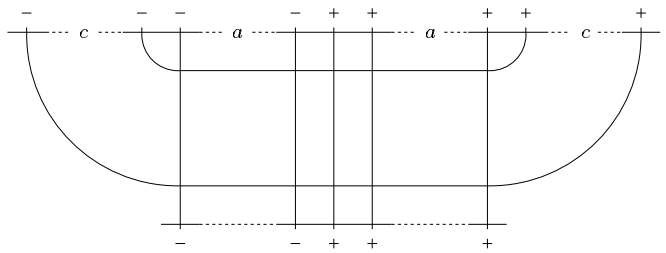
(c) Each morphism $(0,n') \longrightarrow (0,n)$ in $\mathcal{H}$ can be factored (non-uniquely) as $(0,n') \longrightarrow (0,k) \longrightarrow (0,n)$, where $(0,k)$ is any chosen object with $n' \leq k \leq n$.

(d) To each pair $\varphi_1, \varphi_2 : (0,0) \longrightarrow (0,n)$ in $\mathcal{H}$ there exists an automorphism $\sigma : (0,n) \longrightarrow (0,n)$ with $\sigma \circ \varphi_1 = \varphi_2$.

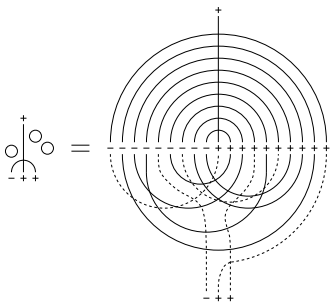
PROOF. (a) is trivial since $(n,0)$ is an initial object in Δ_n.

(b) The factorization mentioned in the lemma is not unique at all, and we give only one construction, which of course contains various choices.
Let $\varphi : (c,a) \longrightarrow (0,n)$ be any given morphism and $\psi : (c,a) \longrightarrow (0,c+a)$ the morphism, which is defined by the following picture.

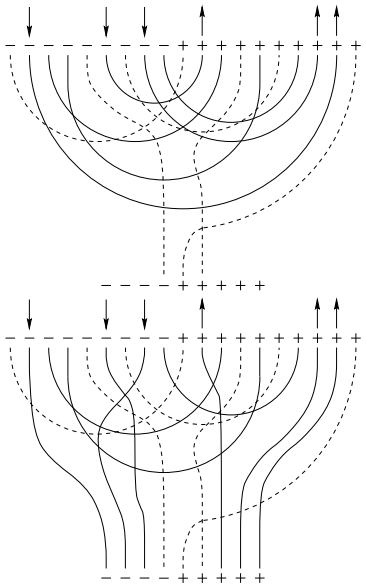
ψ :=



We will construct a morphism $\tilde{\varphi} : (0,c+a) \longrightarrow (0,n)$ with $\varphi = \tilde{\varphi} \circ \psi$.
Recall that $(0,n)$ and (c,a) can be interpreted as morphisms in $\mathcal{F}_{\mathrm{P,D}}$ with $(c,a) = (0,n) \circ \varphi$. The strings of φ can be divided into two classes: those upper arcs which contribute to cycles when composing φ with $(0,n)$ (called "cyclic arcs"), and all other strings which contribute to the vertical string or lower arcs of (c,a), respectively (called the "remainder"). The definition of $\tilde{\varphi}$ preserves the structure of the remainder and several cyclic arcs, while certain cyclic arcs are "opened" to become closed again by ψ. A more exact description is as follows:



Divide φ into cyclic arcs and the remainder. (The remainder is represented by dashed lines in the picture.)

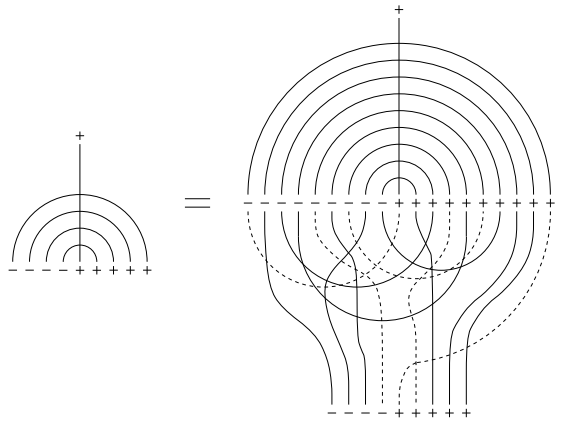


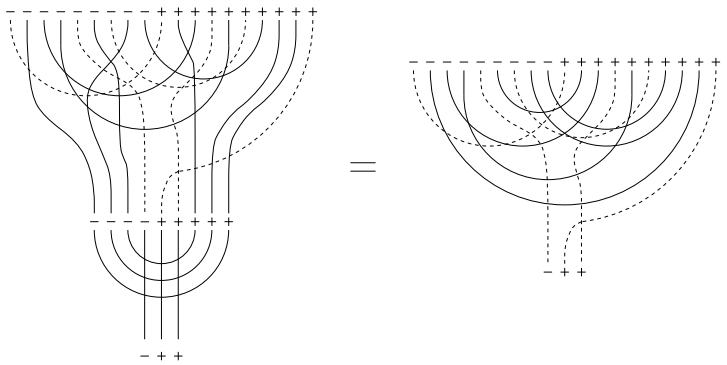
Choose c cyclic arcs by taking one from each set of arcs which belong to the same cycle after composing with $(0,n)$, and add c letters "$-$" and c letters "$+$" to the bottom base.

"Open" the marked arcs by connecting the arrowed positions with the additional letters in the bottom base such that the ith letter "$-$" from the left and the ith letter "$+$" from the right belong to the same opened arc. Preserve the rest of the cyclic arcs and the remainder.

The resulting pseudo-tangle is the desired $\tilde{\varphi}$ because it is indeed a morphism from $(0, c+a)$ to $(0,n)$ and it satisfies the equation $\tilde{\varphi} \circ \psi = \varphi$.

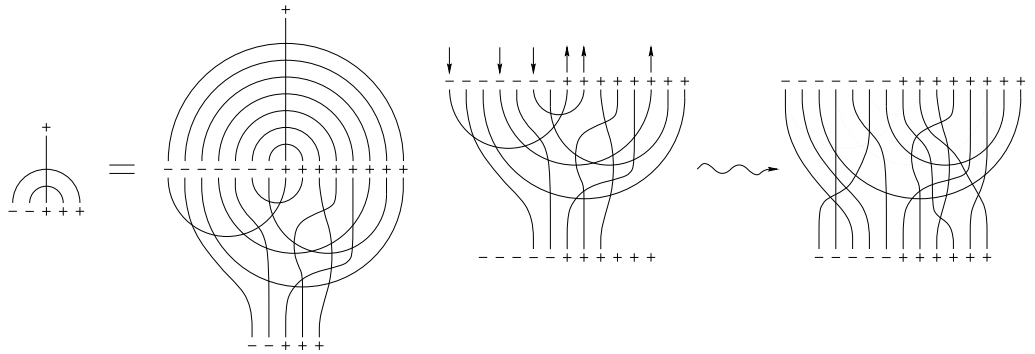
The following pictures clarify these properties in our concrete example.





(c) Let $\varphi : (0,n') \longrightarrow (0,n)$ be any given morphism and $(0,k)$ another object with $n' \leq k \leq n$. Create a pseudo-tangle $\bar{\varphi}$ as follows:

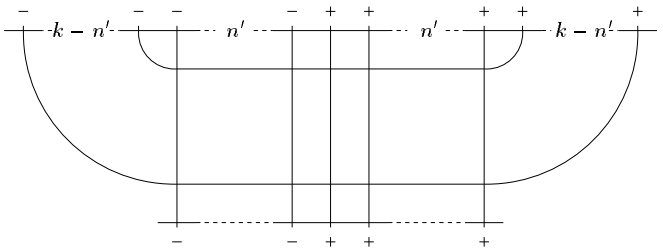
Add $k - n'$ letters "$-$" and "$+$", respectively, to the bottom base of φ.
Choose $k - n'$ upper arcs and open them in the sense of the preceding proof.



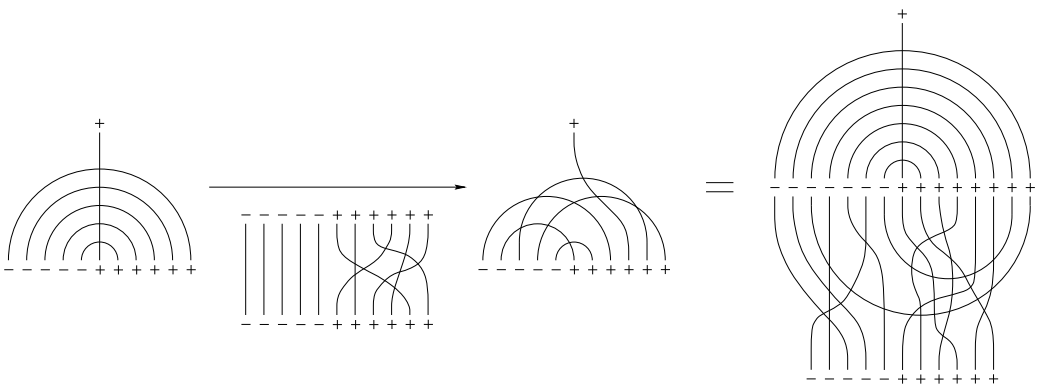
(In this example we have $n = 7, \quad n' = 2, \quad k = 5$.)

Obviously, if $\bar{\varphi}$ is composed with $\bar{\psi}$, where $\bar{\psi}$ is defined by

$\bar{\psi} :=$



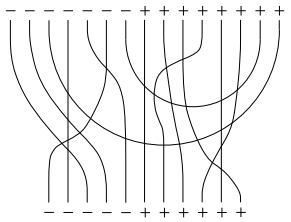
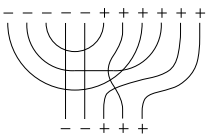
then we arrive at φ again. One problem remains: If $\bar{\varphi}$ is interpreted as a morphism with target $(0, n)$, then the source object is not necessarily $(0, k)$. But it is an object $((-k \oplus k + 1) \uparrow_\alpha +)$ in $I/+$ which is isomorphic to $(0, k)$, since it contains k lower arcs and no cycles.



Hence if we choose an isomorphism $(0, k) \xrightarrow{\sigma} (-k \oplus (k + 1) \uparrow_\alpha +)$ in $I/+$ (which has already been done in the upper picture) and define

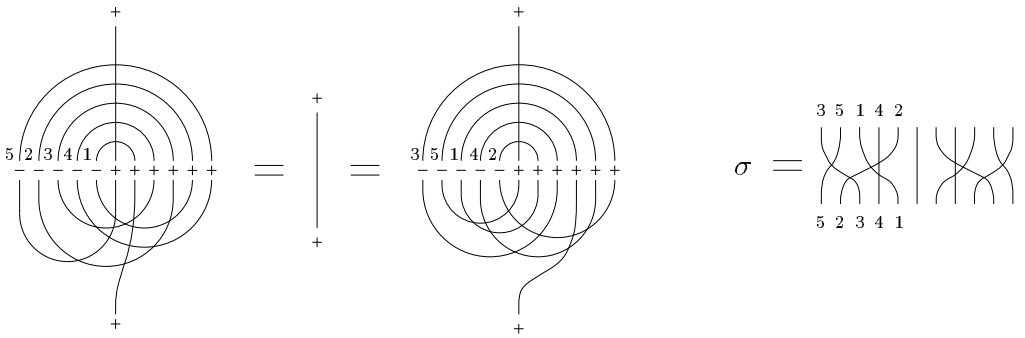
$$\tilde{\varphi} := \bar{\varphi} \circ \sigma, \qquad \psi := \sigma^{-1} \circ \bar{\psi},$$

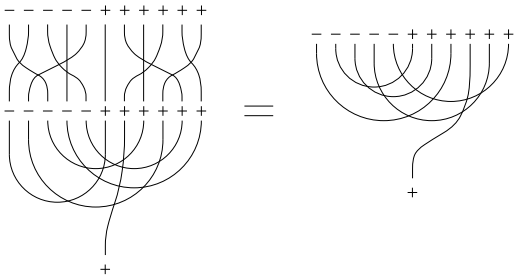
then we obtain $(0,n') \xrightarrow{\psi} (0,k) \xrightarrow{\tilde{\varphi}} (0,n)$ with $\tilde{\varphi} \circ \psi = \varphi$.

$\tilde{\varphi} =$  $\psi =$ 

The reader is invited to check all properties in the given example.

(d) A morphism $(0,0) \longrightarrow (0,n)$ links all lower arcs of $(0,n)$ to a single vertical string. Hence any morphism determines and is determined by the order of the lower arcs of $(0,n)$, which is obtained by running through this single string. Obviously, this leads to $n!$ different morphisms.
The endomorphisms of $(0,n)$ are permutations of the lower arcs of $(0,n)$. Hence all endomorphisms are invertible and we get $End(0,n) \cong \Sigma_n$ with again $n!$ morphisms.
Two morphisms $\varphi_1, \varphi_2 : (0,0) \longrightarrow (0,n)$ differ only by the order they determine on the lower arcs of $(0,n)$; the permutation $\sigma : (0,n) \longrightarrow (0,n)$, which leads from the first order to the second, satisfies $\sigma \circ \varphi_1 = \varphi_2$.





The lemma is proven. □

PROOF OF 5.7. $\mathcal{H}$ has a canonical filtration $\{\mathcal{H}_n\}_{n\in\mathbb{N}}$ by its "lower left corners", i.e. $\mathcal{H}_n$ is the full subcategory of objects (c,a) with $c+a\leq n$. Since

$$\pi_1 B\mathcal{H} = \varinjlim \pi_1 B\mathcal{H}_n$$

it is sufficient to prove $\pi_1 B\mathcal{H}_n = 0$. This is trivial for $\mathcal{H}_0$. We assume by induction this statement to be true for $\mathcal{H}_{n-1}$. As a basepoint in $B\mathcal{H}_n$ we choose the point corresponding to the object $(0,n)$.
Any loop in $B\mathcal{H}_n$ may be represented by

$$\omega: \quad (0,n) \xleftarrow{\varphi_1} (c_1,a_1) \xrightarrow{\varphi_2} (c_2,a_2) \xleftarrow{\varphi_3} \cdots \xleftarrow{\varphi_{k-1}} (c_{k-1},a_{k-1}) \xrightarrow{\varphi_k} (0,n),$$

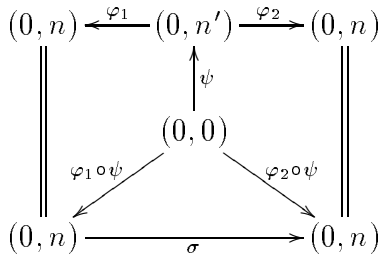
where no arrow is an endomorphism, with one small exception which is treated in the following step 1. (Note: All morphisms with source $(0,n)$ are isomorphisms.)

Step 1: ω of the form $(0,n) \xrightarrow{\varphi} (0,n)$ is null homotopic.

There is nothing to show, since ω is a loop in Δ_n, which is contractible.

Step 2: ω of the form $(0,n) \xleftarrow{\varphi_1} (0,n') \xrightarrow{\varphi_2} (0,n)$ is null homotopic.

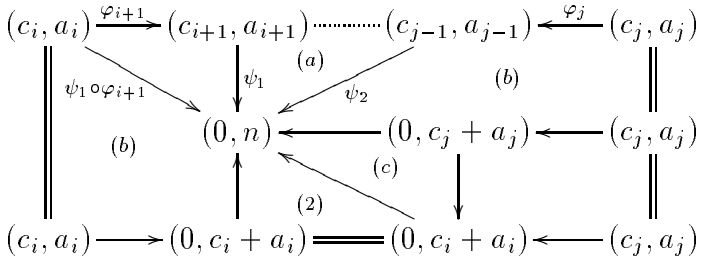
Choose any $\psi : (0,0) \longrightarrow (0,n')$ and an automorphism $\sigma : (0,n) \longrightarrow (0,n)$ with $\sigma \circ (\varphi_1 \circ \psi) = \varphi_2 \circ \psi$ according to 5.8 (d). Then the following diagram gives a homotopy to a loop treated in step 1.



Step 3: Every path $\omega' : (c_i,a_i) \xrightarrow{\varphi_{i+1}} (c_{i+1},a_{i+1}) \xleftarrow{\varphi_{i+2}} \cdots \xleftarrow{\varphi_j} (c_j,a_j)$ with $(c_i,a_i),(c_j,a_j) \in \mathcal{H}_{n-1}$ is homotopic rel boundary points to a loop in $\mathcal{H}_{n-1}$.

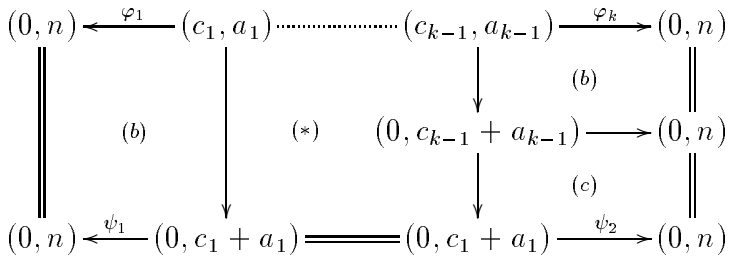
We can assume $(c_k,a_k) \in \Delta_n$ for all $i<k<j$, since ω' can be decomposed into such paths and paths which already belong completely to $\mathcal{H}_{n-1}$. Furthermore, we can assume $c_i+a_i \geq c_j+a_j$ because of symmetry. Choose morphisms $\psi_1 : (c_{i+1},a_{i+1}) \longrightarrow (0,n)$, $\psi_2 : (c_{j-1},a_{j-1}) \longrightarrow (0,n)$. All further arrows in the following diagram are obtained from composition and application of 5.8 (b) and (c). The two triangles, which do not necessarily commute, can somehow

be filled by the contractibility of Δ_n (labelled with (a)) and by the assertion of step 2 (indicated by (2)), respectively.



Step 4: ω of the form $(0,n) \xleftarrow{\varphi_1} (c_1,a_1)\cdots(c_{k-1},a_{k-1}) \xrightarrow{\varphi_k} (0,n)$ with $(c_1,a_1), (c_{k-1},a_{k-1}) \in \mathcal{H}_{n-1}$ is null homotopic.

Because of symmetry we may assume $c_1 + a_1 \geq c_{k-1} + a_{k-1}$. Then look at the following diagram in which again (b) and (c) mark the corresponding statements in 5.8, and (∗) marks that part of the diagram which can be filled by the assertion of step 3 and the induction hypothesis:



This is a reduction to step 2.

Step 5: An arbitrary $\omega : (0,n) \xleftarrow{\varphi_1} (c_1,a_1) \xrightarrow{\varphi_2} (c_2,a_2)\cdots(c_{k-1},a_{k-1}) \xrightarrow{\varphi_k} (0,n)$ is null homotopic.

Assume $c_1 + a_1 = n$, i.e. φ_1 is a morphism of Δ_n. Then φ_2 is also a morphism and (c_2, a_2) an object of Δ_n. Choose any morphism $(0,n) \xleftarrow{\psi} (c_2,a_2)$. Since Δ_n is contractible, the path defined by $(0,n) \xleftarrow{\varphi_1} (c_1,a_1) \xrightarrow{\varphi_2} (c_2,a_2)$ is homotopic to ψ, and ω is homotopic to $(0,n) \xleftarrow{\psi\circ\varphi_3} (c_3,a_3)\cdots(c_{k-1},a_{k-1}) \xrightarrow{\varphi_k} (0,n)$. In this way ω can be shortened gradually until the first remaining object (c_i, a_i) does not belong to Δ_n, and the same can be done with the other end of ω. This leads to step 4. If all (c_i, a_i) belong to Δ_n, we have immediately finished, because of its contractibility.

This completes the proof of 5.7. □

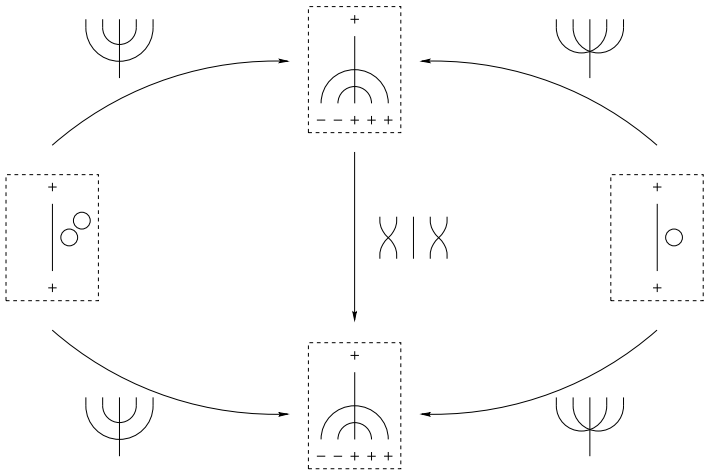
$B\mathcal{H}$ is not 2-connected, as the following result shows:

5.9 Proposition: The homomorphism

$$p_* : \pi_2(B\mathcal{H}, (+ \uparrow_{|\circ} +)) \longrightarrow \pi_2(B\mathcal{G}, +),$$

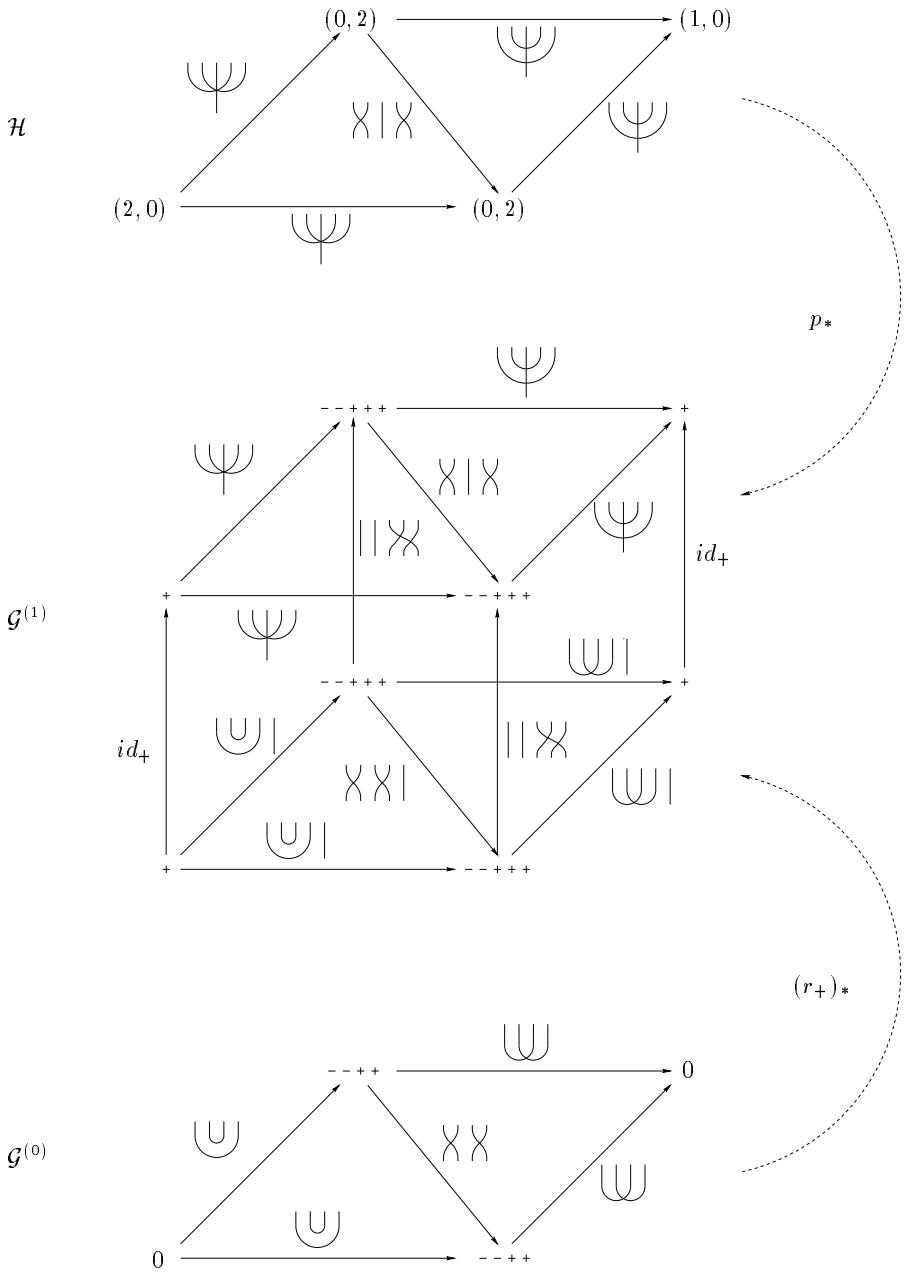
which is induced by the functor $p : \mathcal{H} \longrightarrow \mathcal{G}, \quad p(w \uparrow_\alpha +) = w$ (cf. 5.5), is surjective.

PROOF. From 2.4 we have $\pi_2(B\mathcal{G}, 0) = \pi_2(\Omega^\infty S^\infty, *)$, which is well known to be $\mathbb{Z}/2$. The 2-sphere Θ of 3.5 represents the non-trivial element, and the non-trivial element of $\pi_2(B\mathcal{G}, +)$ is represented by $(r_+)_*(\Theta)$ (cf. 3.4). Hence it suffices to lift $(r_+)_*(\Theta)$ up to homotopy into $B\mathcal{H}$.
This is done by the realization of the following picture which defines a 2-sphere Θ' in $B\mathcal{H}$ (compare the picture on page 36):



□

The picture on the next page clarifies the connection between Θ' and Θ. After realization the cube in the middle defines a homotopy between $p_*(\Theta')$ and $(r_+)_*(\Theta)$.



In this way we obtain $\pi_2(B\mathcal{H}) \neq 0$ and conclude that $B\mathcal{G}$ and $B\mathcal{F}_{\mathrm{P,D}}$ are not of the same homotopy type, i.e. the addition of a free duality structure to $\mathcal{F}_{\mathrm{P}}$ does not give a topological group completion on the level of classifying spaces.

6 The lower homotopy groups of $\mathcal{F}_{\mathrm{P,D}}$ and the generator for $\pi_1(B\mathcal{F}_{\mathrm{P,D}}, 0)$

6.1 Proposition: The functors $l_+,\ l_- : \mathcal{F}_{\mathrm{P,D}} \longrightarrow \mathcal{F}_{\mathrm{P,D}}$, defined by

$$l_+(w) := + \oplus w,\ l_+(\alpha) := id_+ \oplus \alpha, \quad l_-(w) := - \oplus w,\ l_-(\alpha) := id_- \oplus \alpha,$$

map $\mathcal{F}_{\mathrm{P,D}}^{(k)}$ into $\mathcal{F}_{\mathrm{P,D}}^{(k+1)}$ and vice versa and induce homotopy inverse maps on classifying spaces. The same is true for $r_+, r_- : \mathcal{F}_{\mathrm{P,D}} \longrightarrow \mathcal{F}_{\mathrm{P,D}}$ given by

$$r_+(w) := w \oplus +,\ r_+(\alpha) := \alpha \oplus id_+,\ \ r_-(w) := w \oplus -,\ r_-(\alpha) := \alpha \oplus id_-.$$

This is proven in exactly the same way as 3.4.

Hence the following diagram commutes up to natural transformation so that the inclusion $\mathcal{G}^{(k)} \hookrightarrow \mathcal{F}_{\mathrm{P,D}}^{(k)}$ has the same homotopy theoretic properties as $\mathcal{G}^{(1)} \hookrightarrow \mathcal{F}_{\mathrm{P,D}}^{(1)}$.

$$\begin{array}{ccc} \mathcal{G}^{(k+1)} & \xrightarrow{I} & \mathcal{F}_{\mathrm{P,D}}^{(k+1)} \\ {\scriptstyle l_+}\uparrow & & \downarrow{\scriptstyle l_-} \\ \mathcal{G}^{(k)} & \xrightarrow{I} & \mathcal{F}_{\mathrm{P,D}}^{(k)} \end{array}$$

Invoking 5.6 and 5.7, the exact sequence of 5.5 ends with

$$\cdots \pi_2 B\mathcal{H} \xrightarrow{p_*} \pi_2 B\mathcal{G}^{(1)} \xrightarrow{I_*} \pi_2 B\mathcal{F}_{\mathrm{P,D}}^{(1)} \longrightarrow 0 \xrightarrow{p_*} \pi_1 B\mathcal{G}^{(1)} \xrightarrow{I_*} \pi_1 B\mathcal{F}_{\mathrm{P,D}}^{(1)} \longrightarrow 0$$

and by 5.9 the homomorphism $I_* : \pi_2 B\mathcal{G}^{(1)} \longrightarrow \pi_2 B\mathcal{F}_{\mathrm{P,D}}^{(1)}$ turns out to be the zero map. Hence we can summarize (using also 5.1):

6.2 Theorem:
(a) For any choice of basepoint in $\mathcal{G}$ the inclusion $I : \mathcal{G} \longrightarrow \mathcal{F}_{\mathrm{P,D}}$ is 1-connected but not 2-connected, i.e. the induced homomorphism

$$I_* : \pi_i(B\mathcal{G}, w) \longrightarrow \pi_i(B\mathcal{F}_{\mathrm{P,D}}, w)$$

is an isomorphism for $i = 0, 1$ and a surjection for $i = 2$.

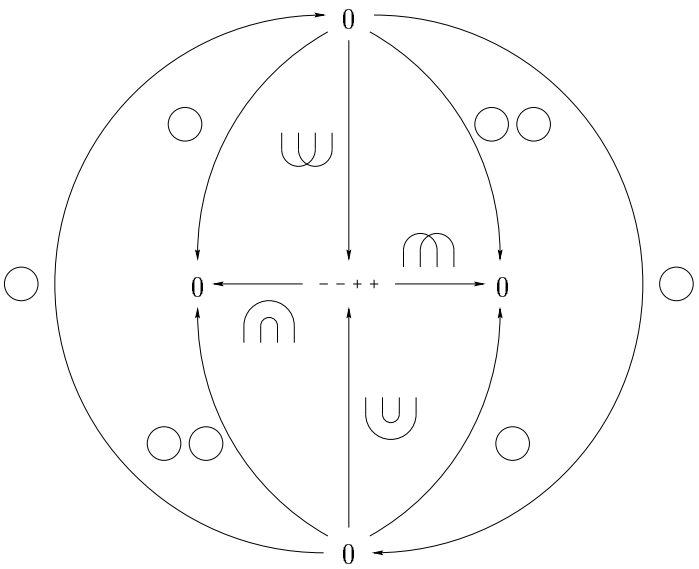
(b) The homotopy groups for $i = 0, 1, 2$ of $\mathcal{F}_{\mathrm{P,D}}$ are

$$\pi_0(B\mathcal{F}_{\mathrm{P,D}}, w) = \mathbb{Z},$$
$$\pi_1(B\mathcal{F}_{\mathrm{P,D}}, w) = \mathbb{Z}/2,$$
$$\pi_2(B\mathcal{F}_{\mathrm{P,D}}, w) = 0.$$

By the above theorem, the generator of $\pi_1(B\mathcal{G}, 0)$ generates $\pi_1(B\mathcal{F}_{\mathrm{P,D}}, 0)$ as well, but there is another very nice representative:

6.3 Proposition: The endomorphism $0 \longrightarrow 0$ of $\mathcal{F}_{\mathrm{P,D}}$, which is depicted by a single circle, represents the generator of $\pi_1(B\mathcal{F}_{\mathrm{P,D}}, 0)$.

PROOF. The picture below displays a homotopy between the path κ of 3.3 and the single circle. It also gives a direct proof that the path defined by two circles is null homotopic.



□

References

[1] D. Grayson, Higher algebraic K-Theory: II, in: *Lecture Notes in Mathematics* **551**, (Springer-Verlag, 1976) 217–240.

[2] G. Karpilovsky, The Schur Multiplier *London Mathematical Society Monographs New Series* **2**, (Oxford University Press, 1987).

[3] C. Kassel, Quantum Groups *Graduate Texts in Mathematics* **155**, (Springer-Verlag, 1994).

[4] G.M. Kelly and M.L. Laplaza, Coherence for Compact Closed Categories, *Journal of Pure and Applied Algebra* **19** (1980) 193–213.

[5] S. Mac Lane, Categories for the working mathematician *Graduate Texts in Mathematics* **5**, (Springer-Verlag, 1971).

[6] S. Priddy, On $\Omega^\infty S^\infty$ and the infinite symmetric group, in: *Proceedings of Symposia in Pure Mathematics* **22**, (Amer. Math. Soc., 1971) 217–220.

[7] M.C. Shum, Tortile tensor categories, *Journal of Pure and Applied Algebra* **93** (1994) 57–110.

[8] D. Quillen, Higher algebraic K-Theory: I, in: *Lecture Notes in Mathematics* **341**, (Springer-Verlag, 1973) 85–147.

[9] G. Segal, Categories and cohomology theories *Topology* **13** (1974) 293–312.