Improved upper bounds for the real part of nonmaximal eigenvalues of nonnegative matrices

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Abstract

In this note we improve an upper bound for the real part of nonmaximal eigenvalues of nonnegative irreducible matrices. We also improve an upper bound for the spectral radius of principal submatrices of nonnegative matrices.

Key words: nonnegative matrices, nonmaximal eigenvalues, Cheeger-type bounds

1 Introduction

Let $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ be a nonnegative irreducible matrix with positive left and right eigenvectors u and v. We arrange the eigenvalues of A as

$$\rho(A) = \lambda_n(A) > Re(\lambda_{n-1}(A)) \geq \ldots \geq Re(\lambda_1(A)).$$

In many applications of nonnegative matrices, such as finite Markov chains, algebraic connectivity of graphs and convergence rates of iterative methods one needs bounds for $\lambda_{n-1}(A)$, $Re(\lambda_{n-1}(A))$, or the so-called spectral gap $|\rho(A) - \lambda_{n-1}(A)|$.

It is proved by Friedland and Gurvits in [FG] that

$$Re(\lambda_{n-1}(A)) \quad \leq \quad \rho(A) - \frac{1}{2}(\rho(A) - \max_i a_{ii})\epsilon_{\lfloor \frac{n}{2} \rfloor}(A, u, v)^2, \tag{1.1}$$

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where

$$\epsilon_s(A,u,v) \quad := \min_{\emptyset \neq V \subset <n>, |V| \leq s} \frac{\sum_{i \in V, j \in <n> \setminus V} a_{ij} v_i u_j + a_{ji} v_j u_i}{\sum_{i \in V} 2(\rho(A) - a_{ii}) v_i u_i}.$$

The right hand side of (1.1) can also be used to get an upper bound for the spectral radius of principal submatrices of A. Let U be a nonempty subset of $< n >= \{1, \ldots, n\}$ and define

$$\rho_s(A) = \max_{U \subset <n>, |U|=s} \rho(A(U)),$$

where $A(U)$ is the submatrix of A whose rows and columns are in U. It is proved in [FN1] that

$$\rho_s(A) \quad \leq \quad \rho(A) - \frac{1}{2}(\rho(A) - \max_i a_{ii}) \epsilon_s(A,u,v)^2. \tag{1.2}$$

The values $\rho_s(A)$ are used to define a partition of the class of Z–matrices by Fiedler and Markham in [FM] (see also [N] for more details). Moreover, it is established in [FN1] that for any real eigenvalue λ of A different from $\rho(A)$ it holds

$$\lambda \leq \rho_{\lfloor \frac{n}{2} \rfloor}(A).$$

Another bound for $Re(\lambda_{n-1})$ is given by Berman and Zhang. They proved in [BZ]

$$Re(\lambda_{n-1}) \leq \sqrt{\rho(A)^2 - \frac{h(A,u,v)^2}{max_i(u_i v_i)^2}}, \tag{1.3}$$

where

$$h(A,u,v) = \min_{\emptyset \neq U \subset <n>, |U| \leq \lfloor \frac{n}{2} \rfloor} \frac{\sum_{i \in U, j \in <n> \setminus U} a_{ij} v_i u_j + a_{ji} v_j u_i}{2|U|}. \tag{1.4}$$

In this note we compare the above so-called Cheeger-type bounds (1.1) and (1.3). We also establish new bounds for $Re(\lambda_{n-1})$ which improve both bounds (1.1) and (1.3). Moreover we improve (1.2).

We will use $\bar{v} = max_i v_i$ for $v = [v_i] \in \mathrm{I\!R}^n$.

2 Improved bounds

Our results are mainly based on results of [FN2] and [BZ]. There lower bounds for the second smallest eigenvalue of the Laplacian matrix of a connected, undirected weighted graph

$G_w = (V, E(G_w)$ are given. Note that G_w may have loops. Here, $w_{i,j}$ denotes the positive weight $w_{i,j}$ of $(i,j) = (j,i) \in E(G_w)$. The Laplacian matrix $L_w = [l_{i,j}]$ is defined as

$$\begin{aligned} l_{i,j} &= -w_{i,j} \quad \text{for} \quad i \neq j \\ l_{i,i} &= \delta_i - w_{i,i}, \end{aligned}$$

where

$$\delta_i = \sum_{(i,j) \in E(G_w)} w_{i,j},$$

are the weighted degrees. The established bounds use different Cheeger constants $h(G_w)$ and $i(G_w)$:

$$h(G_w) = \min_{\emptyset \neq U \subset V} \frac{|E(U, \bar{U})|}{\min(|U|, |\bar{U}|)} = \min_{\substack{\emptyset \neq U \subset V \\ 0 \neq |U| \leq \lfloor \frac{n}{2} \rfloor}} \frac{|E(U, \bar{U})|}{|U|},$$

$$i(G_w) = \min_{\emptyset \neq U \subset V} \frac{|E(U, \bar{U})|}{\min(|E(U)|, |E(\bar{U})|)} = \min_{\substack{\emptyset \neq U \subset V \\ |E(U)| \leq \frac{1}{2}|E(V)|}} \frac{|E(U, \bar{U})|}{|E(U)|},$$

where

$$|E(U)| = \sum_{i \in U} \delta_i,$$

$\bar{U} =< n > \backslash U$ and $E(U, \bar{U})$ are the edges connecting vertices of U with vertices in $\bar{U}$.

Now let $D = diag(d_1, \ldots, d_n)$ be a positive diagonal matrix. Then Theorem 2.1 of [FN2] gives

$$\lambda_2(D^{-1}L_w) \geq \min_i \frac{\delta_i}{d_i}(1 - \sqrt{1 - i(G_w)^2}). \tag{2.5}$$

while Theorem 2.2 of [BZ] says

$$\lambda_2(D^{-1}L_w) \geq \frac{1}{\max_i d_i} \left(\bar{\delta} - \sqrt{\bar{\delta}^2 - h(G_w)^2} \right). \tag{2.6}$$

Now it is worth to mention that weighted loops do not influence the Laplacian matrix. Thus, we can think of L_w as a Laplacian matrix of a weighted graph with loops or as the Laplacian matrix of a weighted graph without loops. The inequalities (2.5) and (2.6) hold for both cases. (This is not mentioned in [BZ] for (2.6)). But of course one gets different bounds since one has different weighted degrees and a different constant $i(G_w)$. It is shown in [FN2] that for (2.5) loops can increase or decrease the bound. However, since $h(G_w)$ is independent of loops we obtain

Proposition 2.1 *Let G_w be a weighted, undirected, connected graph and $V =< n >$. Let $D = diag(d_1, \dots, d_n)$ be a positive diagonal matrix. Then*

$$\begin{aligned} \lambda_2(D^{-1}L_w) &\geq \frac{1}{\max_i d_i}\left(\bar{\gamma} - \sqrt{\bar{\gamma}^2 - h(G_w)^2}\right) && (2.7)\\ &\geq \frac{1}{\max_i d_i}\left(\bar{\delta} - \sqrt{\bar{\delta}^2 - h(G_w)^2}\right), && (2.8)\end{aligned}$$

where

$$\gamma = [\gamma_i]_{i=1}^n \quad \text{with} \quad \gamma_i = \delta_i - w_{i,i}.$$

Proof. The proof of Theorem 2.2 in [BZ] works for both cases, loops or no loops. Thus we only have to show the second inequality. But the function

$$f(t) = \frac{1}{\max_i d_i}\left(t - \sqrt{t^2 - h(G_w)^2}\right),$$

is monotonicly decreasing for $t \geq \sqrt{t^2 - h(G_w)^2}$. □

We now consider nonnegative matrices $A = [a_{i,j}] \in \mathbb{R}^{n,n}$. Let $U \subset< n >$ and define for convenience

$$\begin{aligned} ||U||_{nl} &:= \sum_{\substack{i\in U, j\in <n>\\ i\neq j}} a_{ij}u_iv_j + a_{ji}u_jv_i = \sum_{i\in U} 2(\rho(A) - a_{ii})v_iu_i\\ ||U||_l &:= \sum_{i\in U, j\in <n>} a_{ij}u_iv_j + a_{ji}u_jv_i = \sum_{i\in U} 2\rho(A)v_iu_i.\end{aligned}$$

Moreover let

$$\begin{aligned} i(A,u,v) &:= \min_{\emptyset\neq U\subset <n>} \frac{\sum_{i\in U, j\in <n>\setminus\ U} a_{ij}v_iu_j + a_{ji}v_ju_i}{\min(||U||_{nl}, ||\bar{U}||_{nl})}\\ &= \min_{\substack{\emptyset\neq U\subset <n>\\ ||U||_{nl}\leq \frac{1}{2}||<n>||_{nl}}} \frac{\sum_{i\in U, j\in <n>\setminus U} a_{ij}v_iu_j + a_{ji}v_ju_i}{\sum_{i\in U} 2(\rho(A) - a_{ii})v_iu_i}\\ l(A,u,v) &:= \min_{\emptyset\neq U\subset <n>} \frac{\sum_{i\in U, j\in <n>\setminus\ U} a_{ij}v_iu_j + a_{ji}v_ju_i}{\min(||U||_l, ||\bar{U}||_l)}\\ &= \min_{\substack{\emptyset\neq U\subset <n>\\ ||U||_l\leq \frac{1}{2}||<n>||_l}} \frac{\sum_{i\in U, j\in <n>\setminus U} a_{ij}v_iu_j + a_{ji}v_ju_i}{\sum_{i\in U} 2\rho(A)v_iu_i}\\ \epsilon(A,U,u,v) &:= \min_{\emptyset\neq W\subset U} \frac{\sum_{i\in W, j\in <n>\setminus W} a_{ij}v_iu_j + a_{ji}v_ju_i}{\sum_{i\in W} 2(\rho(A) - a_{ii})v_iu_i},\end{aligned}$$

We easily obtain

$$\begin{aligned} \epsilon_{\lfloor \frac{n}{2} \rfloor}(A,u,v) &\leq i(A,u,v) && (2.9)\\ h(A,u,v) &\leq max_i u_i v_i \ \rho(A) l(A,u,v). && (2.10) \end{aligned}$$

Then we get

Theorem 2.2 *Let $A = [a_{i,j}] \in \mathrm{I\!R}^{\mathrm{n,n}}$ be a nonnegative irreducible matrix with positive left and right eigenvectors u and v. Then*

$$\begin{aligned} Re(\lambda_{n-1}(A)) &\leq \rho(A) - (\rho(A) - \max_i a_{ii})(1 - \sqrt{1 - i(A,u,v)^2} && (2.11)\\ &\leq \rho(A) - (\rho(A) - \max_i a_{ii})(1 - \sqrt{1 - \epsilon_{\lfloor \frac{n}{2} \rfloor}(A,u,v)^2} && (2.12)\\ &\leq \rho(A) - \frac{1}{2}(\rho(A) - \max_i a_{ii})\epsilon_{\lfloor \frac{n}{2} \rfloor}(A,u,v)^2, && (2.13) \end{aligned}$$

and

$$\begin{aligned} Re(\lambda_{n-1}(A)) &\leq \rho(A)\sqrt{1 - l(A,u,v)^2} && (2.14)\\ &\leq \sqrt{\rho(A)^2 - \frac{h(A,u,v)^2}{max_i(u_i v_i)^2}}. && (2.15) \end{aligned}$$

$$\begin{aligned} Re(\lambda_{n-1}(A)) &\leq min_i a_{ii} + \sqrt{(\rho(A) - min_i a_{ii})^2 - \frac{h(A,u,v)^2}{max_i(u_i v_i)^2}}, && (2.16)\\ &\leq \sqrt{\rho(A)^2 - \frac{h(A,u,v)^2}{max_i(u_i v_i)^2}}. && (2.17) \end{aligned}$$

Moreover, for $s = 1, \ldots, n-1$

$$\begin{aligned} \rho_s(A) &\leq \rho(A) - (\rho(A) - \max_i a_{ii})(1 - \sqrt{1 - \epsilon_s(A,u,v)^2} && (2.18)\\ &\leq \rho(A) - \frac{1}{2}(\rho(A) - \max_i a_{ii})\epsilon_s(A,u,v)^2. && (2.19) \end{aligned}$$

Proof. First assume that $A = A^T$ and $u = v$. We consider the weighted graph G_w, without loops, whose weighted Laplacian matrix is

$$L_w = Q(\rho(A)I - A)Q,$$

where $Q = diag(u_1, \ldots, u_n)$. The weighted degrees δ_i are given by

$$\delta_i = \sum_{1 \leq j \leq n, j \neq i} a_{ij} u_i u_j = (\rho(A) - a_{ii})u_i^2.$$

We get for the second smallest eigenvalue of $Q^{-2}L_w$

$$\begin{aligned}\lambda_2(Q^{-2}L_w) &= \lambda_2(Q^{-1}(\rho(A)-A)Q) = \lambda_2(\rho(A)-A)\\ &= \rho(A)-\lambda_{n-1}(A).\end{aligned}$$

Now let U be a subset of $<n>$ with $|U| = s$. Similarly we obtain for the smallest eigenvalue of $Q^{-2}L_w(U)$

$$\lambda_1(Q^{-2}L_w(U)) = \rho(A) - \rho(A(U)).$$

Now we apply (2.5) and Lemma 2.3 of [FN2] with $D = Q^2$ and get

$$\begin{aligned}\rho(A)-\lambda_{n-1}(A) &\geq \min_i \frac{\delta_i}{d_i}\left(1-\sqrt{1-i(A,u,v)^2}\right)\\ \rho(A)-\rho(A(U)) &\geq \min_i \frac{\delta_i}{d_i}\left(1-\sqrt{1-\epsilon_s(A,u,v)^2}\right).\end{aligned}$$

Thus (2.11) and (2.18) hold for symmetric matrices.

Now assume that A is not symmetric. Let F be the unique positive diagonal matrix such that $Fu = F^{-1}v = x$ and consider $\tilde{A} = FAF^{-1}$. Note that the spectrum of A and $\tilde{A}$ are the same. Moreover $i(A,u,v) = i(\tilde{A},x,x)$ and $\epsilon(A,U,u,v) = \epsilon(\tilde{A},U,x,x)$ for any $U \subset <n>$.

Next consider $B = (\tilde{A}+\tilde{A}^T)/2$. It is well-known that

$$\lambda_{n-1}(B) \geq Re(\lambda_{n-1}(\tilde{A})).$$

We also have $\rho(B) = \rho(\tilde{A})$. Moreover, $i(B,x,x) = i(\tilde{A},x,x)$ and $\epsilon(B,U,x,x) = \epsilon(\tilde{A},U,x,x)$ for all $U \subset <n>$. The maximal characterization of $\rho(B(U))$ implies the inequality $\rho(B(U)) \geq \rho(\tilde{A}(U))$. In particular, $\rho_s(B) \geq \rho_s(A)$. Thus (2.11) and (2.18) hold also for nonsymmetric matrices. With (2.9) we get the inequality (2.12). Since

$$1-\sqrt{1-\epsilon_s(A,u,v)^2} \geq \frac{1}{2}\epsilon_s(A,u,v)^2$$

we obtain (2.13) and (2.19). To prove (2.14) and (2.16) we just consider the weighted graph G_w with loops and use similar arguments as above. The inequality (2.15) follows from (2.10). Moreover (2.17) follows from Proposition 2.1.

□

Theorem 2.2 is also true for reducible matrices. But then we have to consider eigenvalues different from $\rho(A)$.

Next we consider doubly stochastic matrices, i.e. nonnegative matrices A for which $Ae = e$ and $e^TA = e^T$, where $e = (1,\ldots,1)^T$.

Note that we have $||<n>||_{nl} = 2\,(n - \sum_i^n a_{ii})$ and $||<n>||_l = 2n$. We obtain

Corollary 2.3 *Let A be a doubly stochastic matrix. Then for any eigenvalue λ of A different from 1 we obtain*

$$Re(\lambda) \leq \max_i a_{ii} + (1 - \max_i a_{ii})\sqrt{1 - i(A,e,e)^2},$$

and

$$\begin{aligned} Re(\lambda) &\leq \sqrt{1 - l(A,e,e)^2} \\ &\leq \sqrt{1 - h(A,e,e)^2} \end{aligned}$$

Moreover,

$$\begin{aligned} Re(\lambda) &\leq \min_i a_{ii} + \sqrt{(1 - min_i a_{ii})^2 - h(A,e,e)^2} \\ &\leq \sqrt{1 - h(A,e,e)^2} \end{aligned}$$

Proof. For doubly stochastic matrices we have

$$\sum_{i \in U, j \in \bar{U}} a_{ij} = \sum_{i \in U, j \in \bar{U}} a_{ji}.$$

Thus, all inequalities follow from Theorem 2.2. □

Note that one can use all the above inequalities to get lower bounds for the so-called spectral gap, i.e. $|\rho(A) - \lambda|$ where λ is an eigenvalue of A different from $\rho(A)$. One only need to use

$$|\rho(A) - \lambda| \geq |\rho(A) - Re(\lambda)| = \rho(A) - Re(\lambda).$$

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