Geometric Methods in the Representation Theory of Cocommutative Hopf Algebras

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0. Introduction

These notes are an expanded version of a series of lectures given at the University of Bielefeld in January and February of 2000. The primary objective of these talks was two-fold: to provide a first introduction to the theories of Hopf algebras and algebraic groups on the one hand, and to delineate the application of certain geometric techniques to their representation theory on the other. Accordingly, the exposition emphasizes the discussion of illustrative examples and plausibilty considerations highlighting the salient points of the theory. Proofs have been included only when they serve this purpose, and they often only sketch the main ideas rather than embarking on detailed discussions.

These notes are exclusively concerned with the "classical" theory of cocommutative Hopf algebras, leaving the very active field of quantum groups out of the account. Our Hopf algebras will also appear in a geometric guise, as affine algebraic group schemes. Since our point of view veils the historical sources of the subject matter, let me briefly mention that the axioms of our algebras first occurred in the work of the topologist Heinz Hopf [45], defining what are now called graded Hopf algebras.

The first four sections provide a quick tour of Hopf algebras, their associated group schemes, and restricted Lie algebras. Since the selection of the topics is primarily based on what is relevant for the later sections, many interesting aspects have been left out of the account. The reader who is interested in a more in depth introduction may consult [50, 55] for the abstract theory of Hopf algebras, and [48, 81] for group schemes. Aside from establishing the necessary terminology, we point out the division of the modular representation theory of cocommutative Hopf algebras into the classical part concerning finite groups, and the theory of infinitesimal group schemes. The main tools of the latter are geometric methods related to the notions of cohomological support varieties, rank varieties, and schemes of tori.

In §5 we introduce the important notion of the complexity of a module. This number measures the rate of growth of a minimal projective resolution. It can also be interpreted as the dimension of a certain affine variety, the cohomological support variety of a module. These varieties have been available for modules over group algebras of finite groups for quite some time. The extension to arbitrary cocommutative Hopf algebras is based on recent work by Friedlander-Suslin [40], who showed that the cohomology ring of a finite group scheme is finitely generated.

Elementary abelian groups are an important tool in the representation theory of finite groups. In §7 we present recent results by Suslin-Friedlander-Bendel [72, 73] to illustrate that the Frobenius kernels of the additive group play an analogous rôle for infinitesimal group schemes. In particular, they can be used to define rank varieties for modules. In default of a suitable analogue of the Mackey decomposition theorem, our study of infinitesimal groups of finite and tame representation types rests on a detailed study of these varieties.

The techniques involved in the representation theory of infinitesimal groups are often inductive, that is, one reduces problems to the consideration of "small" groups whose structure is well enough understood to be amenable to the methods from the abstract representation theory of Artin algebras. The reduction process involves subgroups as well as quotients, and §8 provides a quick introduction to the latter. For almost all of our purposes it suffices to know that the Hopf algebras associated to quotients are in fact quotient algebras.

Accordingly, the notion of faithfully flat ring extensions, that is necessary for an in depth understanding of quotients, has been omitted.

When studying representations of infinitesimal group schemes one has to deal with two interrelated aspects concerning the structure of the underlying groups, and the Morita equivalence classes of their Hopf algebras, respectively. Solvable groups, that are studied in §9, will frequently play a prominent rôle in our analysis. In this context, one classical result, the Lie-Kolchin theorem, epitomizes a recurring theme of the theory: Frobenius kernels of smooth groups usually behave a lot better than arbitrary infinitesimal groups.

In §10 we employ rank varieties in conjunction with structural features of supersolvable groups to characterize infinitesimal groups of finite representation type. In contrast to finite groups, the Hopf algebras of these groups are Nakayama algebras. Moreover, finite representation type can be detected on the second Frobenius kernel, and the structure of the underlying groups is completely understood.

For Frobenius kernels of reductive groups, the combination of rank varieties with basic results on nilpotent orbits yields the determination of the tame blocks. In fact, they are all Morita equivalent to tame blocks of the first Frobenius kernel of SL(2). In particular, these blocks are special biserial and of domestic representation type.

In sections 12 and 13 we introduce another geometric tool by considering varieties of multiplicative subgroups. These are introduced to understand the structural impact of conditions on rank varieties of restricted Lie algebras. For Lie algebras of smooth groups much information is derived from the so-called root space decomposition associated to a maximal torus. Since any two such tori are mapped onto each other by an automorphism, any maximal torus will do. By contrast, for arbitrary restricted Lie algebras the information encoded in the root space decomposition is highly sensitive to the choice of the torus. Schemes of tori (multiplicative groups of height ≤ 1) are introduced to study all tori simultaneously and to consider algebraic families of Lie algebras.

In section 14 we turn to arbitrary infinitesimal groups of tame representation type. Their study entails the determination of restricted Lie algebras with two-dimensional rank varieties. By combining rank varieties with schemes of tori we classify the semisimple infinitesimal groups of tame representation type. Once again, all tame blocks are special biserial and of domestic representation type.

In the concluding section we survey the current knowledge on the Auslander-Reiten components of infinitesimal group schemes. These are not nearly as well understood as their counterparts for finite groups. Since rank varieties are invariants of AR-components, they can be employed to provide an analogue of Webb's theorem for finite groups: The tree classes of the AR-components are either finite or infinite Dynkin diagrams or Euclidean diagrams. Moreover, components with rank varieties of dimension ≥ 3 are of type $\mathbb{Z}[A_\infty]$.

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1. Definition and Basic Properties

Throughout these notes k will denote a field. The standard example for a Hopf algebra is the group algebra $k[G]$ of some abstract group G. Recall that the G-modules correspond to the $k[G]$-modules. Accordingly, group algebras enjoy special features that ordinary algebras do not have. If M and N are G-modules, then the spaces $M \otimes_k N$ and $\mathrm{Hom}_k(M, N)$ carry the structure of a G-module by setting

$$g(m \otimes n) := g\,m \otimes g\,n \quad \text{ and } \quad (g\,\varphi)(m) := g\varphi(g^{-1}m)$$

for every $g \in G$, $m \in M$, $n \in N$, $\varphi \in \mathrm{Hom}_k(M, N)$, respectively.

Since these structures extend to $k[G]$, we should try to understand them without reference to G. Let's first look at tensor products: the spaces M and N are $k[G]$-modules, so $M \otimes_k N$ naturally carries the structure of a $k[G] \otimes_k k[G]$-module. Note that the map

$$G \times G \longrightarrow k[G] \otimes_k k[G] \ \ ; \ \ (g, h) \mapsto g \otimes h$$

induces an isomorphism $k[G \times G] \cong k[G] \otimes_k k[G]$. Hence the diagonal map $g \mapsto (g, g)$ gives rise to an algebra homomorphism $\Delta : k[G] \longrightarrow k[G] \otimes_k k[G]$. By definition, the $k[G]$-module structure on $M \otimes_k N$ is the pull-back of the $k[G] \otimes_k k[G]$-structure along Δ.

In order to understand the $k[G]$-structure of $\mathrm{Hom}_k(M, N)$, we observe that the map $g \mapsto g^{-1}$ induces an isomorphism $\eta : k[G] \longrightarrow k[G]^{\mathrm{op}}$ from $k[G]$ to its *opposite algebra* $k[G]^{\mathrm{op}}$. The space $\mathrm{Hom}_k(M, N)$ obtains the structure of a $k[G] \otimes_k k[G]^{\mathrm{op}}$-module via

$$((a \otimes b)\varphi)(m) := a\varphi(bm).$$

Our G-structure corresponds to the pull-back of this structure along the algebra homomorphism $(id_{k[G]} \otimes \eta) \circ \Delta$.

The maps Δ and η have various properties that will be listed in the definition below. One obvious relation is $(\Delta \otimes id_{k[G]}) \circ \Delta = (id_{k[G]} \otimes \Delta) \circ \Delta$ ensuring that the natural identification $(X \otimes_k Y) \otimes_k Z \cong X \otimes_k (Y \otimes_k Z)$ is an isomorphism of $k[G]$-modules.

Unless mentioned otherwise a *k-algebra* Λ is meant to be associative with an identity element that acts on all (left) modules via the identity operator. We will occasionally write $m : \Lambda \otimes_k \Lambda \longrightarrow \Lambda$ for the multiplication map. Given a k-algebra Λ, a Λ-module M, and k-linear maps $\varphi : V \longrightarrow \Lambda$, $\psi : W \longrightarrow M$ originating in some k-spaces V, W, we denote by $\varphi\hat{\otimes}\psi : V \otimes_k W \longrightarrow M$ the linear map given by $(\varphi\hat{\otimes}\psi)(v \otimes w) := \varphi(v)\psi(w)$.

Definition. Let H be a k-algebra, $\Delta : H \longrightarrow H \otimes_k H$, $\varepsilon : H \longrightarrow k$, and $\eta : H \longrightarrow H$ k-linear maps. We say that (H, Δ, ε) is a *bialgebra* if

(1) Δ and ε are homomorphisms of k-algebras,

(2) $(\Delta \otimes id_H) \circ \Delta = (id_H \otimes \Delta) \circ \Delta$ (co-associativity),

(3) $(id_H\hat{\otimes}\varepsilon) \circ \Delta = id_H = (\varepsilon\hat{\otimes}id_H) \circ \Delta$ (counit).

If, in addition, we have

(4) $(\eta\hat{\otimes}id_H) \circ \Delta = \varepsilon\,1 = (id_H\hat{\otimes}\eta) \circ \Delta$,

then $(H, \Delta, \varepsilon, \eta)$ is referred to as a *Hopf algebra*.

Remarks. (i). If $H = k[G]$ is the group algebra of a group G, then ε is the unique homomorphism such that $\varepsilon(g) = 1 \ \ \forall\, g \in G$.

(ii). If one writes down the axioms for a k-algebra H as commutative diagrams involving m and the canonical map $k \longrightarrow H$, then (2) and (3) follow by dualizing these diagrams.

(iii). When dealing with the *comultiplication* Δ it is convenient to use the so-called Heyneman-Sweedler notation

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}.$$

For instance, (3) and (4) read as

$$\sum_{(h)} h_{(1)}\varepsilon(h_{(2)}) = h = \sum_{(h)} \varepsilon(h_{(1)})h_{(2)}$$

and

$$\sum_{(h)} \eta(h_{(1)})h_{(2)} = \varepsilon(h) = \sum_{(h)} h_{(1)}\eta(h_{(2)}),$$

respectively.

(iv). The maps ε and η are called the *counit* and the *antipode* of the Hopf algebra H, respectively. The antipode is a homomorphism $H \longrightarrow H^{\mathrm{op}}$. It is bijective whenever H is finite-dimensional.

(v). The ideal $H^\dagger := \ker \varepsilon$ is customarily referred to as the *augmentation ideal* of the Hopf algebra H.

An algebra homomorphism $f : H \longrightarrow H'$ between two bialgebras is a *bialgebra homomorphism* if $(f \otimes f) \circ \Delta = \Delta' \circ f$ and $\varepsilon' \circ f = \varepsilon$. If H and H' are Hopf algebras with antipodes η and η', respectively, then a *Hopf algebra homomorphism* additionally satisfies $\eta' \circ f = f \circ \eta$.

An element $g \neq 0$ of a Hopf algebra H is called *group-like* if $\Delta(g) = g \otimes g$. The set $G(H)$ of group-like elements of H is a subgroup of the group of units of H, the inverse of $g \in G(H)$ being given by $\eta(g)$. Moreover, $G(H)$ is linearly independent, so that $G(H)$ is finite whenever H has finite dimension.

We say that $x \in H$ is *primitive* provided $\Delta(x) = x \otimes 1 + 1 \otimes x$. The set $\mathrm{Lie}(H)$ of primitive elements is a subspace, that is closed under the Lie bracket $[x, y] := xy - yx$.

Examples. (1). Let T be an indeterminate over k. We can endow $k[T]$ and its localization $k[T]_T$ with the following Hopf algebra structures:

(a) $\Delta(T) = T \otimes 1 + 1 \otimes T$, $\varepsilon(T) = 0$, $\eta(T) = -T$.

(b) $\Delta(T) = T \otimes_k T$, $\varepsilon(T) = 1$, $\eta(T) = T^{-1}$.

Note that the Hopf algebra $k[T]_T$ is the group algebra $k[\mathbb{Z}]$ of the group of integers.

(2). An ideal I of a bialgebra H is called a *bi-ideal* if $\Delta(I) \subset H \otimes_k I + I \otimes_k H$ and $\varepsilon(I) = (0)$. In that case H/I canonically obtains the structure of a bialgebra. If, in addition, H is a Hopf algebra and $\eta(I) = I$, then I is called a *Hopf ideal* and H/I inherits the Hopf algebra structure from H.

(3). Let $n \in \mathbb{N}$, and consider the bialgebra $\mathcal{O}(\mathrm{Mat}_n) := k[X_{ij}, 1 \leq i, j \leq n]$ whose comultiplication and counit are given by

$$\Delta(X_{ij}) := \sum_{\ell=1}^{n} X_{i\ell} \otimes X_{\ell j} \quad ; \quad \varepsilon(X_{ij}) = \delta_{ij},$$

respectively. Note that $\det((X_{ij})) \in G(\mathcal{O}(\mathrm{Mat}_n))$.

(4). Given $q \in k \setminus \{0\}$, we consider the k-algebra $\mathcal{O}_q(k^2) := k\langle x, y\rangle/(yx - qxy)$. Then $\mathcal{O}_q(k^2)$ has a bialgebra structure given by

$$\Delta(x) = x \otimes 1 + y \otimes x, \quad \Delta(y) = y \otimes y, \quad \varepsilon(x) = 0, \quad \varepsilon(y) = 1.$$

This bialgebra is often referred to as the *quantum plane*.

Let H be a bialgebra, Λ any k-algebra. Given linear maps $\varphi, \psi : H \longrightarrow \Lambda$, we define the *convolution* $\varphi * \psi : H \longrightarrow \Lambda$ of φ and ψ via

$$(\varphi * \psi)(h) := \sum_{(h)} \varphi(h_{(1)})\psi(h_{(2)}) \qquad \forall\ h \in H.$$

Lemma 1.1 *Let H be a Hopf algebra. Then the following statements hold:*

(1) *H^* is a k-algebra with multiplication given by convolution.*

(2) *If H is finite-dimensional, then H^* is a Hopf algebra with operations $\Delta^*(f) = \sum_{(f)} f_{(1)} \otimes f_{(2)} \Leftrightarrow f(ab) = \sum_{(f)} f_{(1)}(a)f_{(2)}(b) \quad \forall\ a, b \in H$, $\varepsilon^*(f) = f(1)$, and $\eta^*(f) = f \circ \eta$.*

Proof. We illustrate the existence of an identity element: Note that

$$(\varepsilon * f)(h) = \sum_{(h)} \varepsilon(h_{(1)})f(h_{(2)}) = \sum_{(h)} f(\varepsilon(h_{(1)})h_{(2)}) = f(h) \quad \forall\ h \in H.$$

Consequently, ε is the identity element of H^*. $\square$

Suppose that $\dim_k H < \infty$. The group-like elements of H^* are the *characters* of H, i.e., the group $\mathrm{Alg}_k(H, k)$ of k-algebra homomorphisms $H \longrightarrow k$. The space $\mathrm{Lie}(H^*)$ consists of the *derivations* $H \longrightarrow k$, that is, of all linear maps ψ such that $\psi(ab) = \psi(a)\varepsilon(b) + \varepsilon(a)\psi(b)$.

Example. Consider the Hopf algebra $k[T]$ with $\Delta(T) = T \otimes 1 + 1 \otimes T$, $\varepsilon(T) = 0$, $\eta(T) = -T$. Suppose that $\mathrm{char}(k) = p > 0$. Given $n \in \mathbb{N}$ the binomial formula yields $\Delta(T^{p^n}) = T^{p^n} \otimes 1 + 1 \otimes T^{p^n}$. Accordingly, (T^{p^n}) is a Hopf ideal, and $\mathcal{O}(\alpha_{p^n}) := k[T]/(T^{p^n})$ has the structure of a Hopf algebra. We put $t := T + (T^{p^n})$, so that $\mathcal{O}(\alpha_{p^n})$ has basis $\{t^i\ ;\ 0 \le i \le p^n - 1\}$. Let $\{\delta_i\ ;\ 0 \le i \le p^n - 1\}$ be the dual basis within $H(\alpha_{p^n}) = \mathcal{O}(\alpha_{p^n})^*$. Then we have

$$(\delta_i * \delta_j)(t^m) = \begin{cases} \binom{i+j}{i} & i + j = m \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $\delta_i * \delta_j = \binom{i+j}{i}\delta_{i+j}$, where the last term is understood to be zero whenever $i + j \notin \{0, \ldots, p^n - 1\}$. This readily yields $\delta_i^p = 0$ for $i \ge 1$, and it follows that the map $X_i \mapsto \delta_{p^i}$ induces an isomorphism

$$k[X_0, \ldots, X_{n-1}]/(X_0^p, \ldots, X_{n-1}^p) \longrightarrow H(\alpha_{p^n})$$

of k-algebras. Thus, as an algebra, $H(\alpha_{p^n})$ is isomorphic to the group algebra of the elementary abelian p-group $(\mathbb{Z}/(p))^n$. However, the Hopf algebras $H(\alpha_{p^n})$ and $k[(\mathbb{Z}/(p))^n]$ are not isomorphic: the dual of $H(\alpha_{p^n})$ is local (its group scheme is *infinitesimal*), while the dual of $k[(\mathbb{Z}/(p))^n]$ is separable (its group scheme is *étale*).

Let M be a k-vector space. The linear map

$$\tau_M : M \otimes_k M \longrightarrow M \otimes_k M \ \ ; \quad m \otimes n \mapsto n \otimes m$$

is called the *flip*.

Definition. A bialgebra H is called *cocommutative* if and only if $\tau_H \circ \Delta = \Delta$.

Note that a bialgebra is cocommutative exactly when its dual algebra is commutative.

Examples. (1). The Hopf algebras $k[G]$, $k[T]$, $k[T]_T$, and $H(\alpha_{p^n})$ are cocommutative.
(2). The bialgebras $\mathcal{O}_q(k^2)$ and $\mathcal{O}(\mathrm{Mat}_n)$ are not cocommutative.

We continue by recording an important property of a Hopf algebra. Recall that a finite-dimensional k-algebra Λ is referred to as a *Frobenius algebra* if it admits a nondegenerate bilinear form $(\ ,\ ) : \Lambda \times \Lambda \longrightarrow k$ such that

$$(ab, c) = (a, bc) \quad \forall\ a, b, c \in \Lambda.$$

Forms with the latter property are referred to as *associative*. The form $(\ ,\ )$ is usually not symmetric. Its departure from symmetry is measured by the *Nakayama automorphism* $\mu : \Lambda \longrightarrow \Lambda$ that is given by

$$(b, \mu(a)) = (a, b) \quad \forall\ a, b \in \Lambda.$$

Our next results show that cocommutative Hopf algebras are Frobenius algebras with Nakayama automorphisms of finite order.

Given a Hopf algebra H, we put $\int_H^r := \{x \in H \ ;\ xh = \varepsilon(h)x \ \ \forall\ h \in H\}$. The non-zero elements of $\int_H^r$ are called *right integrals* of H.

Theorem 1.2 ([75, 54]) *Let H be a finite-dimensional Hopf algebra.*
(1) *We have* $\dim_k \int_H^r = 1$.
(2) *If $\lambda \in \int_{H^*}^r \setminus \{0\}$, then $H \times H \longrightarrow k$; $(a, b) \mapsto \lambda(ab)$ is a nondegenerate, associative form.* $\square$

Owing to (1) there exists an algebra homomorphism $\zeta : H \longrightarrow k$ such that $hx = \zeta(h)x$ for every $h \in H$ and $x \in \int_H^r$. The function ζ is called the *modular function* of H.

Given an algebra homomorphism (a character) $\zeta : H \longrightarrow k$, the map $\zeta * id_H$ is readily seen to be an automorphism of the algebra H. If H is finite-dimensional, then ζ and $\zeta * id_H$ have finite order.

Proposition 1.3 ([57, 36]) *Let H be a finite-dimensional Hopf algebra with modular function ζ. Then $(\zeta * id_H) \circ \eta^{-2}$ is a Nakayama automorphism of H.* $\square$

If H is cocommutative, then $\eta^2 = id_H$, so that $\zeta * id_H$ is a Nakayama automorphism. Integrals and modular functions are usually hard to compute. In case $H = k[G]$ is the group algebra of a finite group G, the element $\sum_{g \in G} g$ is an integral, and ε is the modular function. In particular, group algebras are *symmetric* algebras (i.e., with a symmetric, nondegenerate associative form). By contrast, the Hopf algebras associated to infinitesimal group schemes are usually not symmetric.

In our prefatory remarks we emphasized the fact that tensor products of modules over a Hopf algebra H are also H-modules. Let me indicate the utility of this concept by giving a result that illustrates the way in which tensor products are exploited.

Lemma 1.4 *Let H be a finite-dimensional Hopf algebra, P a projective H-module, M an arbitrary H-module. Then $P \otimes_k M$ is projective and injective.*

Proof. Recall that the functors $\mathrm{Hom}_k(P \otimes_k M, \cdot)$ and $\mathrm{Hom}_k(P, \mathrm{Hom}_k(M, \cdot))$ are naturally equivalent. Direct computation shows that this equivalence induces an equivalence

$$\mathrm{Hom}_H(P \otimes_k M, \cdot) \cong \mathrm{Hom}_H(P, \mathrm{Hom}_k(M, \cdot)).$$

Consequently, the left-hand functor is, as the composite of two exact functors, exact. Thus, $P \otimes_k M$ is projective, and since H is a Frobenius algebra, $P \otimes_k M$ is also injective. □

Using (1.4) one obtains identities for Ext-groups such as

$$\mathrm{Ext}^n_H(M, N) \cong \mathrm{Ext}^n_H(k, \mathrm{Hom}_k(M, N)),$$

where the right-hand groups are the Hochschild cohomology groups of the augmented algebra (H, ε) with coefficients in the H-module $\mathrm{Hom}_k(M, N)$.

We conclude our general observations by quoting a Hopf algebra freeness theorem that was first verified in [57] for cocommutative Hopf algebras.

Theorem 1.5 ([56]) *Let K be a Hopf subalgebra of the finite-dimensional Hopf algebra H. Then H is a free left and right K-module.* □

2. Group Schemes

In this section we will introduce the geometric interpretation of the theory of cocommutative Hopf algebras. For a more thorough discussion we refer to [48] and [81]. Throughout, M_k and Ens will denote the categories of commutative k-algebras and sets, respectively. A functor $\mathcal{X} : M_k \longrightarrow \mathrm{Ens}$ is called a *k-functor*. The k-functors we will primarily be interested in are the so-called *affine schemes*: given a commutative k-algebra A, we consider the k-functor

$$\mathrm{Spec}_k(A) : M_k \longrightarrow \mathrm{Ens} \quad ; \quad R \mapsto \mathrm{Alg}_k(A, R),$$

where $\mathrm{Alg}_k(A, R)$ is the set of k-algebra homomorphisms from A to R. An affine scheme is called *algebraic* if A is a finitely generated k-algebra. Accordingly, a k-functor $\mathcal{X}$ is affine algebraic if and only if there exist polynomials $f_1, \dots, f_m \in k[X_1, \dots X_n]$ such that

$$\mathcal{X}(R) = \{(r_1, \dots, r_n) \in R^n \ ; \ f_i(r_1, \dots, r_n) = 0 \ , \ 1 \leq i \leq m\},$$

for every commutative k-algebra R. The *rational points* $\mathcal{X}(k)$ of the affine algebraic schemes are the affines varieties from classical algebraic geometry.

Group functors take values in the category Gr of groups. In order to see the connection with Hopf algebras, we require the following basic result:

Theorem 2.1 (Yoneda's Lemma) *Let A, B be two commutative k-algebras. The assignment $\Phi \mapsto \Phi_A(id_A)$ is a bijection between the set of natural transformations $\mathrm{Spec}_k(A) \longrightarrow \mathrm{Spec}_k(B)$ and $\mathrm{Alg}_k(B, A)$. In other words, for every natural transformation $\Phi : \mathrm{Spec}_k(A) \longrightarrow \mathrm{Spec}_k(B)$ there exists a unique homomorphism $\varphi : B \longrightarrow A$ of k-algebras such that $\Phi_R(\lambda) = \lambda \circ \varphi$ for every $\lambda \in \mathrm{Alg}_k(A, R)$ and $R \in M_k$.*

Proof. Each natural transformation $\Phi : \mathrm{Spec}_k(A) \longrightarrow \mathrm{Spec}_k(B)$ is determined by $\Phi_A(id_A) \in \mathrm{Spec}_k(B)(A)$: If $\lambda : A \longrightarrow R$ is a homomorphism of commutative k-algebras, then

$$\Phi_R(\lambda) = \Phi_R(\lambda \circ id_A) = \Phi_R(\mathrm{Spec}_k(A)(\lambda)(id_A)) = \mathrm{Spec}_k(B)(\lambda)(\Phi_A(id_A)) = \lambda \circ \Phi_A(id_A). \qquad \square$$

Let $\Phi : \mathrm{Spec}_k(A) \longrightarrow \mathrm{Spec}_k(B)$ be a natural transformation. The corresponding homomorphism $B \longrightarrow A$ is often denoted Φ^* and referred to as the *comorphism* associated to Φ.

Let A and B be commutative k-algebras. Then there is a natural equivalence

$$\mathrm{Spec}_k(A) \times \mathrm{Spec}_k(B) \longrightarrow \mathrm{Spec}_k(A \otimes_k B)$$

sending a pair (x, y) of algebra homomorphisms with values in R to the unique map

$$x \hat{\otimes} y : A \otimes_k B \longrightarrow R \ \ ; \ \ a \otimes b \mapsto x(a)y(b).$$

Definition. Let $\mathcal{G} : M_k \longrightarrow \mathrm{Gr}$ be a k-functor taking values in the category Gr of groups. We say that $\mathcal{G}$ is a *k-group functor* if

(a) the multiplication $(m_R : \mathcal{G}(R) \times \mathcal{G}(R) \longrightarrow \mathcal{G}(R))_{R \in M_k}$ is a natural transformation, and

(b) the inverse map $(\iota_R : \mathcal{G}(R) \longrightarrow \mathcal{G}(R))_{R \in M_k}$ is a natural transformation.

We say that $\mathcal{G}$ is an *affine group scheme* if the k-functor $\mathcal{G}$ is affine. If the representing algebra A is finitely generated, then the affine group scheme is called *algebraic*. In that case $\mathcal{G}$ is often referred to as an (*affine*) *algebraic k-group*.

Examples. (1). Consider the k-group functor $\mathrm{GL}_n : M_k \longrightarrow \mathrm{Gr}$

$$\mathrm{GL}_n(R) := \{(r_{ij}) \in \mathrm{Mat}_n(R) \ ; \ \det((r_{ij})) \text{ is invertible}\}, \qquad R \in M_k.$$

Observe that $\mathrm{GL}_n = \mathrm{Spec}_k(\mathcal{O}(\mathrm{Mat}_n)_{\det(X_{ij})})$ is an affine algebraic group.

(2). Consider $\alpha_k : M_k \longrightarrow \mathrm{Gr}$; $\alpha_k(R) := (R, +)$ for every $R \in M_k$.

(3). Consider $\mu_k : M_k \longrightarrow \mathrm{Gr}$; $\mu_k(R) := (U(R), \cdot)$ for every $R \in M_k$. The k-group functor $\mu_k = \mathrm{GL}_1$ is represented by the Hopf algebra $k[T]_T$.

(4). Suppose that $\mathrm{char}(k) = p > 0$. Given $n \in \mathbb{N}$, we consider the group α_{p^n} that is given by

$$\alpha_{p^n}(R) := \{r \in \alpha_k(R) \ ; \ r^{p^n} = 0\}.$$

Note that α_k and α_{p^n} are represented by the commutative Hopf algebras $k[T]$ and $k[T]/(T^{p^n})$, respectively. For α_k the points of R are identified with the values of $x(T)$, where $x \in \mathrm{Spec}_k(k[T])(R)$. The operation of α_k is induced by the comultiplication of $k[T]$:

$$x(T) + y(T) = (x \hat{\otimes} y)(T \otimes 1 + 1 \otimes T) = ((x \hat{\otimes} y) \circ \Delta)(T).$$

Inverses are given by the antipode:

$$-x(T) = x(-T) = (x \circ \eta)(T).$$

For every $R \in M_k$ the group $\alpha_{p^n}(R) = \mathrm{Spec}_k(k[T]/(T^{p^n}))(R)$ is contained in $\alpha_k(R)$, and the inclusion is induced by the surjective map $k[T] \longrightarrow k[T]/(T^{p^n})$. This is an example of a closed subgroup of an algebraic group.

The group GL_n is an example of a reduced group scheme. Recall the bialgebra $\mathcal{O}(\mathrm{Mat}_n)$. This algebra represents the monoid functor Mat_n that associates to every commutative k-algebra R the multiplicative monoid $\mathrm{Mat}_n(R)$ of $(n \times n)$-matrices with coefficients in R. Since this algebra is reduced (i.e., with zero being the only nilpotent element), its localization $\mathcal{O}(\mathrm{Mat}_n)_{\det(X_{ij})}$ has the same property. The algebraic groups represented by reduced Hopf algebras, the so-called *reduced group schemes* may be analyzed by studying their rational points. By contrast, $\alpha_{p^n}(k) = \{0\}$, so the rational points don't provide any information here.

Suppose that $(A, \Delta, \varepsilon, \eta)$ is a commutative Hopf algebra. Given $R \in M_k$ we define a multiplication on $\mathrm{Spec}_k(A)(R)$ via convolution:

$$(x * y)(a) = \sum_{(a)} x(a_{(1)}) y(a_{(2)}). \tag{1}$$

(Observe that we need the commutativity of R to ensure that $x * y$ is a homomorphism of k-algebras.) Then $(\mathrm{Spec}_k(A)(R), *)$ is a group with identity element ε and inverse $x^{-1} = x \circ \eta$. These operations endow $\mathrm{Spec}_k(A)$ with the structure of an affine group scheme.

The following result shows that all affine group schemes arise in this fashion.

Proposition 2.2 *Let A be a commutative k-algebra such that $\mathrm{Spec}_k(A)$ is a group scheme. Then A has the structure of a Hopf algebra such that the group structure on $\mathrm{Spec}_k(A)$ is given by* (1).

Proof. Let m be the multiplication on $\mathcal{G} := \mathrm{Spec}_k(A)$. Then $m : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$ is a natural transformation. We combine this transformation with the natural equivalence $\gamma : \mathrm{Spec}_k(A \otimes_k A) \longrightarrow \mathcal{G} \times \mathcal{G}$. By the Yoneda Lemma, there exists an algebra homomorphism $\Delta : A \longrightarrow A \otimes_k A$ such that

$$(m \circ \gamma)(x) = x \circ \Delta \qquad \forall\, x \in \mathrm{Spec}_k(A \otimes_k A)(R),\ R \in M_k.$$

Let $g, h \in \mathcal{G}(R)$. Then $g \hat{\otimes} h$ is the pre-image of (g, h) under γ. We thus have

$$(g \cdot h)(a) = ((g \hat{\otimes} h) \circ \Delta)(a) = \sum_{(a)} g(a_{(1)})h(a_{(2)}) = (g * h)(a).$$

The Yoneda Lemma also provides an algebra homomorphism $\eta : A \longrightarrow A$ such that $g^{-1} = g \circ \eta$ for every element $g \in \mathcal{G}(R)$. The Hopf algebra axioms are now readily seen to correspond to the group axioms. $\square$

Remark. Since $(g^{-1})^{-1} = g$ for every $g \in \mathcal{G}(R)$, we conclude that the antipode η of a commutative Hopf algebra satisfies $\eta^2 = id$. In view of (1.1) the antipode of a finite-dimensional cocommutative Hopf algebra satisfies the same identity. By the same token, an algebra homomorphism $\varphi : H \longrightarrow H'$ between two cocommutative Hopf algebras satisfying $\Delta' \circ \varphi = (\varphi \otimes \varphi) \circ \Delta$ is a homomorphism of Hopf algebras.

Let $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ be a natural transformation between two group functors. We say that φ is a *homomorphism of k-group functors* if for every $R \in M_k$ the map $\varphi_R : \mathcal{G}(R) \longrightarrow \mathcal{H}(R)$ is a group homomorphism. The Yoneda Lemma then shows that the homomorphisms between affine group schemes correspond to the Hopf algebra homomorphisms. More precisely, we have:

Proposition 2.3 *The categories of affine group schemes and commutative Hopf algebras are anti-equivalent.*

Proof. We only have to understand how to retrieve A from $\mathrm{Spec}_k(A)$. For any scheme $\mathcal{X}$ we define $\mathcal{O}(\mathcal{X})$ to be the set of natural transformations $\mathcal{X} \longrightarrow \mathrm{Spec}_k(k[T])$. This set naturally has the structure of a k-algebra. In case $\mathcal{X} = \mathrm{Spec}_k(A)$, the Yoneda Lemma provides an identification $\mathcal{O}(\mathcal{X}) \cong \mathrm{Spec}_k(k[T])(A) \cong A$. $\square$

Definition. Let $\mathcal{X}$ be a k-functor. Then the k-algebra $\mathcal{O}(\mathcal{X})$ of all natural transformations $\mathcal{X} \longrightarrow \mathrm{Spec}_k(k[T])$ is called the *function algebra* of $\mathcal{X}$.

Given a homomorphism $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ of affine group schemes, we consider the functor $\ker \varphi$ that is defined by $\ker \varphi\,(R) := \ker \varphi_R$ for every $R \in M_k$. Let $\varphi^* : \mathcal{O}(\mathcal{H}) \longrightarrow \mathcal{O}(\mathcal{G})$ be the Hopf algebra homomorphism corresponding to φ, and let $I := \mathcal{O}(\mathcal{G})\varphi^*(\mathcal{O}(\mathcal{H})^\dagger)$ be the ideal generated by the image of the augmentation ideal $\mathcal{O}(\mathcal{H})^\dagger$ of $\mathcal{O}(\mathcal{H})$. Then $x \in \ker \varphi\,(R)$ if and only if $x(I) = (0)$. In other words, the functor $\ker \varphi$ is represented by the Hopf algebra $\mathcal{O}(\mathcal{G})/I$.

Suppose that k is algebraically closed, and let A be a finitely generated, commutative k-algebra. Every finite-dimensional semisimple subalgebra $S \subset A$ gives rise to a finite subset of mutually orthogonal idempotents of A. Since A is noetherian, it follows that A possesses a unique maximal finite-dimensional semisimple subalgebra $\pi_0(A)$. Moreover, if A is a Hopf algebra, then $\pi_0(A)$ is a Hopf subalgebra of A.

Now let $\mathcal{G} = \mathrm{Spec}_k(A)$ be an affine algebraic k-group. We put $\pi_0(\mathcal{G}) := \mathrm{Spec}_k(\pi_0(A))$ and consider the homomorphism $\pi : \mathcal{G} \longrightarrow \pi_0(\mathcal{G})$ that is given by restriction. The subgroup $\mathcal{G}^0 := \ker \pi$ is called the *connected component* of $\mathcal{G}$.

Definition. An affine group scheme $\mathcal{G} := \mathrm{Spec}_k(A)$ is *connected* if A possesses exactly one idempotent.

Note that $\mathcal{G}^0$ is a connected, affine algebraic group scheme. An arbitrary affine algebraic group $\mathcal{G} = \mathrm{Spec}_k(A)$ is connected if and only if the prime ideal spectrum of A is connected.

3. Distribution Algebras

In this section we will interpret cocommutative Hopf algebras as the algebras of measures on the finite algebraic k-groups. For simplicity we will assume throughout that k is algebraically closed.

Definition. An affine group scheme $\mathcal{G}$ is said to be *finite* if its function algebra $\mathcal{O}(\mathcal{G})$ is finite-dimensional. Given such a scheme $\mathcal{G}$, we call $H(\mathcal{G}) := \mathcal{O}(\mathcal{G})^*$ the *algebra of measures* on $\mathcal{G}$. The number $\mathrm{ord}(\mathcal{G}) := \dim_k \mathcal{O}(\mathcal{G})$ is referred to as the *order* of the finite algebraic group $\mathcal{G}$.

Observe that $H(\mathcal{G})$ is a finite-dimensional cocommutative Hopf algebra. Our previous results now show that the category of finite-dimensional cocommutative Hopf algebras is equivalent to the category of finite group schemes. Indeed, if H is such a Hopf algebra, then $\mathcal{G}_H := \mathrm{Spec}_k(H^*)$ is a finite group scheme such that $H \cong H(\mathcal{G}_H)$.

Algebras of measures can be viewed as "group algebras" of finite group schemes: Given a k-vector space V, we consider the k-functor $V_a : M_k \longrightarrow \mathrm{Ens}$; $V_a(R) := V \otimes_k R$. In particular, we can consider $H(\mathcal{G})_a$, and note that $H(\mathcal{G}) \otimes_k R$ has the structure of a Hopf algebra over R with comultiplication

$$\Delta_R : H \otimes_k R \longrightarrow H \otimes_k H \otimes_k R \cong (H \otimes_k R) \otimes_R (H \otimes_k R) \;\; ; \;\; h \otimes r \mapsto \sum_{(h)} h_{(1)} \otimes h_{(2)} \otimes r.$$

There is an embedding $\iota_{\mathcal{G}} : \mathcal{G} \hookrightarrow H(\mathcal{G})_a$, which interprets an element $g : \mathcal{O}(\mathcal{G}) \longrightarrow R \in \mathcal{G}(R)$ as a homomorphism $\mathcal{O}(\mathcal{G}) \otimes_k R \longrightarrow R$ of R-algebras. Since $\mathrm{Hom}_R(\mathcal{O}(\mathcal{G}) \otimes_k R, R) \cong \mathrm{Hom}_k(\mathcal{O}(\mathcal{G}), R) \cong H(\mathcal{G}) \otimes_k R$ this amounts to identifying $\mathcal{G}(R)$ with $G(H(\mathcal{G}) \otimes_k R)$. Given any morphism $f : \mathcal{G} \longrightarrow V_a$ there exists a unique k-linear map $\hat{f} : H(\mathcal{G}) \longrightarrow V$ such that

$$f_R = (\hat{f} \otimes id_R) \circ (\iota_{\mathcal{G}})_R \qquad \forall \; R \in M_k.$$

The interested reader may consult [78] for more details.

Definition. Let $\mathcal{G}$ be a k-group, V a k-vector space. We say that V is a $\mathcal{G}$-*module* if there exists a natural transformation $\mathcal{G} \times V_a \longrightarrow V_a$ such that for every $R \in M_k$ the map $\mathcal{G}(R) \times (V \otimes_k R) \longrightarrow V \otimes_k R$ is an action of the group $\mathcal{G}(R)$ on $V \otimes_k R$ by R-linear transformations.

Suppose that $\mathcal{G}$ is a finite algebraic group. The universal property of $H(\mathcal{G})$ entails that the notions of $\mathcal{G}$-module and $H(\mathcal{G})$-module coincide: every $\mathcal{G}$-module possesses a unique $H(\mathcal{G})$-module structure and vice versa.

Examples. (1). Let $\mathrm{char}(k) = p > 0$ and consider the group α_{p^n}. We have observed before that $H(\alpha_{p^n}) \cong k[X_0, \dots, X_{n-1}]/(X_0^p, \dots, X_{n-1}^p)$ as an algebra . Recall that the generator X_i corresponds to the functional δ_{p^i}, where $\{\delta_0, \dots, \delta_{p^n-1}\}$ is the basis dual to $\{1, t, \dots, t^{p^n} - 1\}$ of the function algebra $k[T]/(T^{p^n})$. Direct computation shows that

$$\Delta(\delta_i) = \sum_{j=0}^{i} \delta_j \otimes \delta_{i-j} \quad ; \quad \varepsilon(\delta_i) = \delta_{i,0} \quad ; \quad \eta(\delta_i) = (-1)^i \delta_i.$$

Thus, $\mathrm{Lie}(H(\alpha_{p^n})) = k\delta_1 \cong kX_0$, while $G(H(\alpha_{p^n})) = \{\delta_0\}$. In particular, $H(\alpha_{p^n})$ is generated by $\mathrm{Lie}(H(\alpha_{p^n}))$ if and only if $n = 1$.

(2). Given $n > 0$ we consider the finite group scheme $\mu_n : M_k \longrightarrow \mathrm{Gr}$,

$$\mu_n(R) := \{r \in R \; ; \; r^n = 1\}.$$

Note that $\mu_n = \mathrm{Spec}_k(k[T]/(T^n - 1))$. If $t := T + (T^n - 1)$, then we have $\Delta(t) = t \otimes t$, $\varepsilon(t) = 1$, $\eta(t) = t^{n-1}$. Given $g, h \in H(\mu_n)$ we have

$$(g * h)(t^i) = g(t^i)h(t^i),$$

whence $H(\mu_n) \cong k^n$ is semisimple. If $\{\delta_0, \dots, \delta_{n-1}\}$ is the basis dual to $\{1, t, \dots, t^{n-1}\}$, then the δ_i are the primitive idempotents of $H(\mu_n)$. We also have

$$\Delta(\delta_i) = \sum_{j=0}^{n-1} \delta_j \otimes \delta_{i-j} \quad ; \quad \varepsilon(\delta_i) = \delta_{i,0} \quad ; \quad \eta(\delta_i) = \delta_{-i},$$

where the subscripts are considered elements of $\mathbb{Z}/(n)$. Here we have $\mathrm{Lie}(H(\mu_n)) = (0)$ for $p \not| n$, and $\dim_k \mathrm{Lie}(H(\mu_n)) = 1$, otherwise.

So far, the characteristic of the underlying base field k has not played a major rôle. The following fundamental result shows how the classical characteristic zero theory differs from the modular theory. It says that algebraic groups in characteristic zero are reduced and hence are completely determined by their rational points.

Theorem 3.1 (Cartier) *Suppose that* $\mathrm{char}(k) = 0$. *Then every commutative Hopf algebra* A *over* k *is reduced.* $\square$

Corollary 3.2 *If H is a finite-dimensional, cocommutative Hopf algebra such that H^* is reduced, then there exists a finite group G such that $H \cong k[G]$. In particular, all finite-dimensional cocommutative Hopf algebras of characteristic zero are semisimple.*

Proof. By assumption $A := H^*$ is a finite-dimensional reduced algebra and thus a product of copies of k. We let $G := \mathrm{Alg}_k(A, k)$ be the character group of A, endowed with the convolution product. Then $\mathrm{ord}(G) = \dim_k A$, and $G \subset H$ is linearly independent. Consequently, the canonical map $k[G] \longrightarrow H$ is an isomorphism of Hopf algebras.

If $\mathrm{char}(k) = 0$, then Cartier's Theorem implies that $H \cong k[G]$. Owing to Maschke's Theorem the latter algebra is semisimple. □

Since we will be mainly interested in questions related to the representation type of an algebra, (3.2) shows that we will ultimately be studying Hopf algebras that are defined over fields of positive characteristic.

Suppose that $\mathrm{char}(k) = p > 0$, and consider the group μ_n, where $n = p^s\ell$ with p not dividing ℓ. The map $r \mapsto (r^{p^s}, r^\ell)$ is an isomorphism $\mu_n \cong \mu_\ell \times \mu_{p^s}$. Note that the first factor is represented by $k[T]/(T^\ell - 1) \cong k^\ell$, while the second has local function algebra $k[T]/(T^{p^s} - 1)$. We shall now see that finite algebraic groups behave like this in general.

Let H be a finite-dimensional Hopf algebra with counit ε. As an algebra, H decomposes into its blocks $H = \mathcal{B}_0 \oplus \mathcal{B}_1 \oplus \cdots \oplus \mathcal{B}_n$. Here $\mathcal{B}_0$ is the block to which the *trivial H-module k*, with H acting via ε, belongs. In other words, $\mathcal{B}_0$ is determined by the property $\varepsilon(\mathcal{B}_0) \neq (0)$. This block is usually referred to as the *principal block* $\mathcal{B}_0(H)$ of H. It is not true in general that the principal block of a Hopf algebra is a Hopf subalgebra. In fact, if H is commutative, then $\mathcal{B}_0(H)$ is a Hopf subalgebra if and only if $\mathcal{B}_0(H) = H$ ($\mathcal{B}_0(H)$ is local with unique idempotent e_0. Thus, $\Delta(e_0) = e_0 \otimes e_0$, and e_0 is invertible, whence $e_0 = 1$).

Definitions. Let $\mathcal{G}$ be an affine group scheme. A subfunctor $\mathcal{H} \subset \mathcal{G}$ is called a (*closed*) *subgroup* if there exists a Hopf ideal $I \subset \mathcal{O}(\mathcal{G})$ such that $\mathcal{H}(R) = \mathcal{V}(I)(R) := \{g \in \mathcal{G}(R)\ ;\ g(I) = (0)\}$ for every commutative k-algebra R.

A homomorphism $\mathcal{G} \longrightarrow \mathcal{G}'$ of affine group schemes is called a *closed embedding* if the associated map $\mathcal{O}(\mathcal{G}') \longrightarrow \mathcal{O}(\mathcal{G})$ of k-algebras is surjective.

Let $\mathcal{N} \subset \mathcal{G}$ be a subgroup. We say that $\mathcal{N}$ is a *normal* subgroup of $\mathcal{G}$ if $\mathcal{N}(R)$ is normal in $\mathcal{G}(R)$ for every $R \in M_k$.

Note that the comorphism $c^* : \mathcal{O}(\mathcal{G}) \longrightarrow \mathcal{O}(\mathcal{G}) \otimes_k \mathcal{O}(\mathcal{G})$ of the conjugation $c : \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}\ ;\ (g, h) \mapsto ghg^{-1}$ is given by $c^*(a) = \sum_{(a)} a_{(1)}\eta(a_{(3)}) \otimes a_{(2)}$. Hence $\mathcal{N} = \mathcal{V}(I)$ is normal if and only if $c^*(I) \subset \mathcal{O}(\mathcal{G}) \otimes_k I$.

Theorem 3.3 *Let $\mathcal{G} = \mathrm{Spec}_k(A)$ be a finite algebraic group. Then $\mathcal{G}$ is a semidirect product $\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{\mathrm{red}}$, where $\mathcal{G}^0$ is normal and $\mathcal{G}_{\mathrm{red}} \cong \pi_0(\mathcal{G})$. As a k-functor, the connected component $\mathcal{G}^0$ is represented by the principal block $\mathcal{B}_0(A)$.*

Proof. We decompose $A = \bigoplus_{i=0}^{n} \mathcal{B}_i$ into its blocks, and denote the primitive idempotents by $\{e_0, \dots, e_n\}$ with $\mathcal{B}_0 := \mathcal{B}_0(A) = Ae_0$. Since each block $\mathcal{B}_i = Ae_i$ is local and of the form $\mathcal{B}_i = ke_i \oplus \mathrm{rad}(\mathcal{B}_i)$, the subalgebra $S := \sum_{i=0}^{n} ke_i$ is the largest semisimple subalgebra of A, and we have $A = S \oplus \mathrm{rad}(A)$.

Since $\mathrm{rad}(A)$ is the set of nilpotent elements of A, it is a Hopf ideal. Consider the closed subgroup $\mathcal{G}_{\mathrm{red}} := \mathcal{V}(\mathrm{rad}(A))$ of $\mathcal{G}$.

Note that $\{e_i \otimes e_j \ ; \ 0 \leq i, j \leq n\}$ is the set of orthogonal primitive idempotents of $A \otimes_k A$. Since $\Delta(e_i)$ is an idempotent of $A \otimes_k A$, we have $\Delta(e_i) \in S \otimes_k S$. Consequently, S is a Hopf subalgebra of A, and the composition $\lambda \circ \iota$ of the canonical projection $A \xrightarrow{\lambda} A/\mathrm{rad}(A)$ with the inclusion $\iota : S \hookrightarrow A$ is an isomorphism $S \cong A/\mathrm{rad}(A)$ of Hopf algebras. Accordingly, the corresponding isomorphism $\mathrm{Spec}_k(\lambda \circ \iota) : \mathcal{G}_{\mathrm{red}} \longrightarrow \pi_0(\mathcal{G})$ factors as

$$\mathcal{G}_{\mathrm{red}} \xrightarrow{\mathrm{Spec}_k(\lambda)} \mathcal{G} \xrightarrow{\pi} \pi_0(\mathcal{G}),$$

so that $\mathcal{G} = \mathcal{G}^0 \times \mathcal{G}_{red}$. By definition, $\mathcal{G}^0 = \ker \pi$ is normal in $\mathcal{G}$. Since $\mathcal{G}^0 = \mathcal{V}(AS^\dagger)$ and $AS^\dagger = \bigoplus_{i=1}^{n} \mathcal{B}_i$, we see that the representing algebra $A/AS^\dagger$ is isomorphic to $\mathcal{B}_0(A)$. $\square$

The foregoing result can also be interpreted at the level of Hopf algebras. Recall from (3.2) that, given a reduced finite algebraic group $\mathcal{G}$, we have $H(\mathcal{G}) \cong k[G]$, where $G = \mathrm{Alg}_k(\mathcal{O}(\mathcal{G}), k) = \mathcal{G}(k)$ is the finite group of rational points of $\mathcal{G}$. Now let $\mathcal{G} = \mathcal{G}^0 \times \mathcal{G}_{\mathrm{red}}$ be an arbitrary finite algebraic group. Since the connected component $\mathcal{G}^0$ is represented by a local algebra, we have

$$\mathcal{G}(k) = \mathcal{G}^0(k) \times \mathcal{G}_{\mathrm{red}}(k) = \mathcal{G}_{\mathrm{red}}(k).$$

It follows that

$$H(\mathcal{G}) \cong H(\mathcal{G}^0)\#H(\mathcal{G}_{\mathrm{red}}) \cong H(\mathcal{G}^0)\#k[\mathcal{G}(k)] \cong H(\mathcal{G}^0)[\mathcal{G}(k)]$$

is the *smash product* of $H(\mathcal{G}^0)$ with the group algebra of the rational points of $\mathcal{G}$. The right-hand term interprets the smash product as a *skew group algebra*. When studying the representations of a cocommutative Hopf algebra one thus has to understand three disciplines, all of which require different methods and yield different results:

(a) The modular representation theory of finite groups. By now, this field is rather well-understood.

(b) The representation theory of the infinitesimal group $\mathcal{G}^0$, which will be the focal point of these lectures.

(c) The fusion of (a) and (b). Here one has to study the Frobenius extension $H(\mathcal{G}) : H(\mathcal{G}^0)$, and results are only known in special cases (cf. [29, 30, 67]).

It has turned out that the methods figuring prominently in (a), such as the Mackey decomposition theorem, usually break down for infinitesimal group schemes. This has ultimately led to the approach via the geometric methods to be outlined below.

Definition. A finite group scheme $\mathcal{G}$ is *infinitesimal* if its function algebra $\mathcal{O}(\mathcal{G})$ is local. In that case $H(\mathcal{G})$ is also called the *distribution algebra* of $\mathcal{G}$.

Note that an affine algebraic k-group $\mathcal{G}$ is infinitesimal if and only if $\mathcal{G}(k) = \{1\}$. By Cartier's theorem any infinitesimal group of characteristic zero is trivial.

For the remainder of this section we assume that $\mathrm{char}(k) = p > 0$.

Examples. (1). The groups α_{p^n} and μ_{p^n} are infinitesimal for every $n > 0$.

(2). Consider the closed subgroup of GL(2) given by

$$\mathrm{GL}(2)_n(R) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)(R) \ ; \ a^{p^n} = 1 = d^{p^n} \ , \ b^{p^n} = 0 = c^{p^n}\}$$

for every commutative k-algebra R. The group $\mathrm{GL}(2)_n$ is readily seen to be the kernel of the homomorphism

$$F^n : \mathrm{GL}(2) \longrightarrow \mathrm{GL}(2) \ \ ; \ \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^{p^n} & b^{p^n} \\ c^{p^n} & d^{p^n} \end{pmatrix}.$$

Accordingly,

$$\mathcal{O}(\mathrm{GL}(2)_n) \cong \mathcal{O}(\mathrm{GL}(2))/(\{X_{11}^{p^n} - 1, X_{22}^{p^n} - 1, X_{12}^{p^n}, X_{21}^{p^n}\}),$$

so that $\mathrm{GL}(2)_n$ is finite. Since $\mathrm{GL}(2)_n(k) = \{1\}$ the group scheme $\mathrm{GL}(2)_n$ is infinitesimal.

The preceding example shows how one can generate infinitesimal groups from reduced groups. The relevant notion in this context is that of the Frobenius homomorphism. Suppose that $\mathrm{char}(k) = p > 0$. If V is a k-vector space, we denote by $V^{(p^n)}$ the k-vector space with underlying abelian group V and action given by

$$\alpha \cdot v := \alpha^{p^{-n}} v \qquad \forall\, \alpha \in k,\, v \in V.$$

Given an affine k-group $\mathcal{G}$, we let $\mathcal{G}^{(p^n)} := \mathrm{Spec}_k(\mathcal{O}(\mathcal{G})^{(p^n)})$ be the affine group scheme defined by the twisted function algebra of $\mathcal{G}$.

Definition. Let $\mathcal{G}$ be an affine algebraic group scheme over the algebraically closed field k of characteristic $p > 0$. The homomorphism $F : \mathcal{G} \longrightarrow \mathcal{G}^{(p)}$ satisfying

$$F_R : \mathcal{G}(R) \longrightarrow \mathcal{G}^{(p)}(R) \ \ ; \ \ F_R(\lambda)(x) = \lambda(x)^p \qquad \forall\, \lambda \in \mathcal{G}(R),\, x \in \mathcal{O}(\mathcal{G}),\, R \in M_k$$

is called the *Frobenius homomorphism* of $\mathcal{G}$. The kernel $\mathcal{G}_n$ of its iterate $F^n : \mathcal{G} \longrightarrow \mathcal{G}^{(p^n)}$ is referred to as the *n-th Frobenius kernel* of $\mathcal{G}$.

Remarks. (1). Note that $F_R(\lambda)$ is indeed a k-linear map:

$$F_R(\lambda)(\alpha \cdot x) = \lambda(\alpha^{\frac{1}{p}} x)^p = \alpha F_R(\lambda)(x)$$

for every $x \in \mathcal{O}(\mathcal{G})$ and $\alpha \in k$.

(2). We have $F^n = \mathrm{Spec}_k(\varphi_n)$, where

$$\varphi_n : \mathcal{O}(\mathcal{G})^{(p^n)} \longrightarrow \mathcal{O}(\mathcal{G}) \ \ ; \ \ x \mapsto x^{p^n}.$$

It follows that $\mathcal{O}(\mathcal{G}_n) = \mathcal{O}(\mathcal{G})/\mathcal{O}(\mathcal{G})\varphi_n(\mathcal{O}(\mathcal{G})^\dagger) = \mathcal{O}(\mathcal{G})/\mathcal{O}(\mathcal{G})\{x^{p^n} \ ; \ x \in \mathcal{O}(\mathcal{G})^\dagger\}$. Consequently, $\mathcal{G}_n$ is an infinitesimal k-group whenever $\mathcal{G}$ is algebraic.

(3). Certain problems on representations of algebraic groups can already be decided on sufficiently large Frobenius kernels. For instance, two finite-dimensional $\mathcal{G}$-modules are isomorphic if and only if their restrictions to a suitable $\mathcal{G}_n$ enjoy this property.

Examples. (1). The group α_{p^n} is the n-th Frobenius kernel of α_k.

(2). For $n \geq 1$, let

$$\mathcal{A}_{[n]}(R) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)(R) \ ; \ a^{p^n} = 1 = d^{p^n} \ , \ b^{p^2} = 0 = c^p\} \quad , \quad R \in M_k.$$

Then $\mathcal{A}_{[n]} \subset \mathrm{GL}(2)_n$ is an infinitesimal subgroup of $\mathrm{GL}(2)$ containing $\mathrm{GL}(2)_1$ which is not a Frobenius kernel of $\mathrm{GL}(2)$.

Proposition 3.4 *Let $\mathcal{G}$ be a finite algebraic k-group. Then $\mathcal{G}$ is infinitesimal if and only if $\mathcal{G} = \mathcal{G}_n$ for some $n \geq 0$.*

Proof. Suppose that $\mathcal{G}$ is infinitesimal, and put $I := \mathcal{O}(\mathcal{G})^\dagger$. By assumption, the ideal I is nilpotent, so there exists an $n \geq 0$ such that $x^{p^n} = 0$ for every $x \in I$. By our remark above, this implies that $\mathcal{G}_n = \mathcal{V}(\mathcal{O}(\mathcal{G})\{x^{p^n} \ ; \ x \in I\}) = \mathcal{V}((0)) = \mathcal{G}$. $\square$

Definition. Let $\mathcal{G}$ be an infinitesimal k-group. Then $\mathrm{ht}(\mathcal{G}) := \min\{n \in \mathbb{N}_0 \ ; \ \mathcal{G} = \mathcal{G}_n\}$ is called the *height* of $\mathcal{G}$.

The algebras of measures of an infinitesimal group scheme are a special case of a more general construction that applies to arbitrary affine group schemes. We briefly indicate the definition; a thorough account can be found in Jantzen's book [48, I.§7]. Let $\mathcal{G} = \mathrm{Spec}_k(A)$ be an affine group scheme. Then

$$\mathrm{Dist}(\mathcal{G}) := \{h \in A^* \ ; \ h((A^\dagger)^n) = (0) \text{ for some } n \in \mathbb{N}\}$$

is a subalgebra of A^*. If $\mathcal{G}$ is algebraic, then the definition of the comultiplication for finite-dimensional A still works, and $\mathrm{Dist}(\mathcal{G})$ has the structure of a Hopf algebra. This Hopf algebra is called the *distribution algebra* of $\mathcal{G}$. If $\mathcal{G}$ is a finite algebraic group, then $\mathrm{Dist}(\mathcal{G}) \subset H(\mathcal{G})$ with equality holding if and only if $\mathcal{G}$ is infinitesimal.

Definition. Let $\mathcal{G}$ be an affine algebraic k-group. Then $\mathrm{Lie}(\mathcal{G}) := \mathrm{Lie}(\mathrm{Dist}(\mathcal{G}))$ is called the *Lie algebra of $\mathcal{G}$*.

Let $x \in \mathrm{Lie}(\mathcal{G})$. Then we have $x(ab) = x(a)\varepsilon(b) + \varepsilon(a)x(b)$ for $a, b \in \mathcal{O}(\mathcal{G})$, so that $x((\mathcal{O}(\mathcal{G})^\dagger)^2) = (0)$. Consequently, the natural map $\mathrm{Dist}(\mathcal{G}_n) \hookrightarrow \mathrm{Dist}(\mathcal{G})$ induces an isomorphism $\mathrm{Lie}(\mathcal{G}_n) \cong \mathrm{Lie}(\mathcal{G})$.

Let $\mathcal{G}$ be an affine algebraic k-group. Since $e_k = \mathcal{V}(\mathcal{O}(\mathcal{G}^\dagger))$ is a normal subgroup, the comorphism $c^* : \mathcal{O}(\mathcal{G}) \longrightarrow \mathcal{O}(\mathcal{G}) \otimes_k \mathcal{O}(\mathcal{G})$ of the conjugation action satisfies $c^*(\mathcal{O}(\mathcal{G})^\dagger) \subset \mathcal{O}(\mathcal{G}) \otimes_k \mathcal{O}(\mathcal{G})^\dagger$. Hence we have an action

$$\mathcal{O}(\mathcal{G})^* \otimes_k \mathrm{Dist}(\mathcal{G}) \longrightarrow \mathrm{Dist}(\mathcal{G}) \quad ; \quad (\varphi \cdot \psi)(a) := ((\varphi \hat{\otimes} \psi) \circ c^*)(a) = \sum_{(a)} \varphi(a_{(1)}\eta(a_{(3)}))\psi(a_{(2)})$$

for every $a \in \mathcal{O}(\mathcal{G})$. Direct computation shows that $\varphi \cdot \psi = \sum_{(\varphi)} \varphi_{(1)} * \psi * \eta^*(\varphi_{(2)})$ for $\varphi, \psi \in \mathrm{Dist}(\mathcal{G})$. Thus, our action specializes to the (left) adjoint representation of the Hopf algebra $\mathrm{Dist}(\mathcal{G})$. One verifies that

$$\varphi \cdot \psi \in \mathrm{Lie}(\mathcal{G}) \quad \forall\, \varphi \in \mathcal{O}(\mathcal{G})^*,\ \psi \in \mathrm{Lie}(\mathcal{G}).$$

In particular, the group $\mathcal{G}(k)$ acts on Lie(G) via the *adjoint representation*:

$$g \cdot \psi := g * \psi * g^{-1} \quad \forall\, g \in \mathcal{G}(k),\ \psi \in \mathrm{Lie}(\mathcal{G}).$$

Finally, we have $\varphi \cdot \psi = \varphi * \psi - \psi * \varphi \quad \forall\, \varphi, \psi \in \mathrm{Lie}(\mathcal{G})$ for all $n \geq 1$.

4. Restricted Lie Algebras

Given a Hopf algebra H, we have defined the associated Lie algebra $\mathrm{Lie}(H)$ via

$$\mathrm{Lie}(H) := \{x \in H\ ;\ \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

In general, $\mathrm{Lie}(H)$ is closed under the commutator product $[x, y] := xy - yx$ of H, that is, $\mathrm{Lie}(H)$ is a Lie subalgebra of the Lie algebra $(H, [,])$. If $\mathrm{char}(k) = p > 0$ we also have $x^p \in \mathrm{Lie}(H)$ for every $x \in \mathrm{Lie}(H)$. Lie algebras with the latter property are called restricted Lie algebras.

Throughout this section we assume that k is an algebraically closed field with $\mathrm{char}(k) = p > 0$. The abstract notion of a restricted Lie algebra arose first in work by N. Jacobson concerning a Galois theory for purely inseparable field extensions of exponent 1.

Given a Lie algebra L, the left multiplication effected by the element $x \in L$ is customarily denoted $\mathrm{ad}\, x : L \longrightarrow L\ ;\ y \mapsto [x, y]$. If $(L, [,])$ is a Lie algebra over k, and R is a commutative k-algebra, then $L \otimes_k R$ obtains the structure of a Lie algebra over R via $[x \otimes r, y \otimes s] := [x, y] \otimes rs$ for all $x, y \in L,\ r, s \in R$.

Definition. A *restricted Lie algebra* $(L, [p])$ is a pair consisting of a Lie algebra L and a map $[p] : L \longrightarrow L$ such that
(1) $\mathrm{ad}\, x^{[p]} = (\mathrm{ad}\, x)^p \quad \forall\, x \in L$,
(2) $(\alpha\, x)^{[p]} = \alpha^p x^{[p]} \quad \forall\, \alpha \in k,\ x \in L$,
(3) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$, where the s_i are given by the identity $(\mathrm{ad}\,(x \otimes T + y \otimes 1))^{p-1}(x \otimes 1) = \sum_{i=1}^{p-1} i\, s_i(x, y) \otimes T^{i-1}$
in $L \otimes_k k[T]$.
A map $[p] : L \longrightarrow L$ satisfying (1)-(3) is called a *p-map*.

Given an associative k-algebra Λ, its *commutator algebra* $(\Lambda^-, [,])$ with product $[x, y] = xy - yx$ is restricted with respect to the ordinary p-power operator $x \mapsto x^p$.

If $(L, [p])$ is a restricted Lie algebra, a subalgebra $K \subset L$ (an ideal $I \subset L$) is called a *p-subalgebra* (*a p-ideal*) if $x^{[p]} \in K\ \ \forall\, x \in K$ ($x^{[p]} \in I\ \ \forall\, x \in I$). The notions of homomorphisms and factor algebras of restricted Lie algebras are defined in the canonical fashion.

Suppose that L is an *abelian* Lie algebra, i.e., $[x, y] = 0$ for $x, y \in L$. Then the p-maps on L are just the p-semilinear maps. These are determined by their values on a basis. The following result shows that this holds for arbitrary p-maps.

Theorem 4.1 *Let L be a Lie algebra with basis $(e_i)_{1\leq i\leq n}$. Suppose there exist $x_i \in L$ $(1 \leq i \leq n)$ such that*

$$(\operatorname{ad} e_i)^p = \operatorname{ad} x_i \qquad 1 \leq i \leq n.$$

Then there exists a unique p-map $[p] : L \longrightarrow L$ such that $e_i^{[p]} = x_i$ $\;1 \leq i \leq n$. $\square$

The foregoing result enables us to construct simple examples.

Examples. (1). We consider the (2n+1)-dimensional *Heisenberg algebra H_n* with basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$. The Lie product is given by

$$[x_i, y_j] = \delta_{ij} z \quad ; \quad [x_i, x_j] = 0 = [y_i, y_j] \quad ; \quad [z, H_n] = (0).$$

We endow H_n with the following p-maps:

(a) $x_i^{[p]} = 0 = y_i^{[p]}$; $z^{[p]} = 0$.

(b) $x_i^{[p]} = 0 = y_i^{[p]}$; $z^{[p]} = z$.

(2). Let $L = kt \oplus kx$, $[t, x] = x$. Then L possesses exactly one p-map namely the one satisfying

$$t^{[p]} = t \quad ; \quad x^{[p]} = 0.$$

Let L be a Lie algebra with *universal enveloping algebra* $(U(L), \iota)$. By definition $\iota : L \longrightarrow U(L)^-$ is a homomorphism of Lie algebras satisfying the following universal property: for any associative k-algebra Λ and any homomorphism $f : L \longrightarrow \Lambda^-$ of Lie algebras there exists a unique homomorphism $\varphi : U(L) \longrightarrow \Lambda$ of associative algebras such that $\varphi \circ \iota = f$.

It will be convenient to employ multi-index notation. Let Λ be an k-algebra, $a := (a_1, \ldots, a_\ell) \in \Lambda^\ell$, and $i = (i_1, \ldots, i_\ell) \in \mathbb{N}_0^n$. Then we put

$$a^i := \prod_{j=1}^{\ell} a_j^{i_j}.$$

Given ℓ-tuples $r = (r_1, \ldots, r_\ell)$; $s = (s_1, \ldots, s_\ell) \in \mathbb{N}_0^\ell$ we define

$$r \leq s :\Leftrightarrow r_i \leq s_i \qquad 1 \leq i \leq \ell.$$

We also put $\tau := (p-1, \ldots, p-1)$.

The following result is usually referred to as the PBW-Theorem:

Theorem 4.2 (Poincaré-Birkhoff-Witt) *Let L be a Lie algebra with basis $\{x_1, \ldots, x_\ell\}$. Then $\{\iota(x)^n \;;\; n \in \mathbb{N}_0^\ell\}$ is a basis of $U(L)$ over k.* $\square$

In particular, $\iota : L \longrightarrow U(L)$ is an embedding, and we will henceforth consider L a subalgebra of $U(L)^-$. Since $x \mapsto x \otimes 1 + 1 \otimes x$ is a homomorphism $L \longrightarrow (U(L) \otimes_k U(L))^-$ of Lie algebras, there is a unique extension $\Delta : U(L) \longrightarrow U(L) \otimes_k U(L)$ of associative k-algebras. By the same token, there exist unique homomorphisms $\eta : U(L) \longrightarrow U(L)^{\mathrm{op}}$ and $\varepsilon : U(L) \longrightarrow k$

such that $\eta(x) = -x$ and $\varepsilon(x) = 0$ for every $x \in L$. Consequently, $U(L)$ is a Hopf algebra that is generated by L.

Many features from the theory of finite-dimensional Hopf algebras lose their validity for $U(L)$. For instance, the global dimension of $U(L)$ coincides with $\dim_k L$. By contrast, Frobenius algebras never have finite non-zero global dimension. Moreover, $U(L)$ is free of zero divisors. In view of (1.2) this shows that $k \cdot 1$ is the only finite-dimensional Hopf subalgebra of $U(L)$. Thus, while $\mathrm{Dist}(\mathcal{G}) \cong U(\mathrm{Lie}(\mathcal{G}))$ in case $\mathrm{char}(k) = 0$ (cf. [14, II.§6]), it will follow from (4.4) below that these algebras are not isomorphic over fields of positive characteristic.

If $(L, [p])$ is a restricted Lie algebra with basis $\{x_1, \dots, x_\ell\}$, we define $z_i := x_i^p - x_i^{[p]} \in U(L)$. Then one can modify the PBW-Theorem to show that $\{x^i z^j \ ; \ 0 \leq i \leq \tau, \ j \in \mathbb{N}_0^\ell\}$ is a basis of $U(L)$ over k.

Definition. Let $(L, [p])$ be a restricted Lie algebra with universal enveloping algebra $U(L)$. Let $I \subset U(L)$ be the two-sided ideal generated by $\{x^p - x^{[p]} \ ; \ x \in L\}$. Then

$$u(L) := U(L)/I$$

is called the *restricted enveloping algebra* of L.

Now suppose that $\{x_1, \dots, x_\ell\} \subset L$ is a basis, and consider the natural map $\iota : L \longrightarrow u(L)$. The modified PBW-Theorem readily yields

Corollary 4.3 *The set* $\{\iota(x)^r \ ; \ 0 \leq r \leq \tau\}$ *is a basis of* $u(L)$ *over* k. *In particular,* ι *is injective, and* $\dim_k u(L) = p^{\dim_k L}$. $\square$

Accordingly, we will henceforth consider L a subalgebra of $u(L)^-$. Since I is generated by primitive elements, I is a Hopf ideal, and $u(L)$ inherits the Hopf algebra structure from $U(L)$. As $u(L)$ is generated by primitive elements, it is a cocommutative Hopf algebra. Note that $u(L)$ has the following universal property: for any k-algebra Λ and any homomorphism $f : L \longrightarrow \Lambda^-$ of restricted Lie algebras, there exists a unique homomorphism $\varphi : u(L) \longrightarrow \Lambda$ of associative k-algebras such that $\varphi|_L = f$.

Example. For restricted enveloping algebras integrals, modular functions, and Nakayama automorphisms can be written down explicitly. Let $(L, [p])$ be a restricted Lie algebra with basis $\{e_1, \dots, e_n\}$, and corresponding basis $\{e^r \ ; \ 0 \leq r \leq \tau\}$ of $u(L)$. We consider the linear form

$$\zeta : u(L) \longrightarrow k \ \ ; \ \ \sum_{0 \leq r \leq \tau} \alpha_r e^r \mapsto \alpha_\tau.$$

Since each e_i is primitive, we have $\Delta(e^r) = \sum_{0 \leq s \leq r} \binom{r}{s} e^s \otimes e^{r-s}$, where $\binom{r}{s} := \prod_{i=1}^n \binom{r_i}{s_i}$. For $\varphi \in u(L)^*$ we thus have

$$(\zeta * \varphi)(e^r) = \sum_{0 \leq s \leq r} \binom{r}{s} \zeta(e^s) \varphi(e^{r-s}) = \delta_{r,\tau} \varphi(1) = \varepsilon(\varphi) \zeta(e^r).$$

Hence ζ is a right integral. It was shown in [28] that $\zeta(xe^r) = \zeta(e^r(x - \mathrm{tr}(\mathrm{ad}\, x)1)$ for every $x \in L$. Consequently, the unique automorphism $\mu : u(L) \longrightarrow u(L)$ satisfying $\mu(x) = x - \mathrm{tr}(\mathrm{ad}\, x)1$

for every $x \in L$ is a Nakayama automorphism of $u(L)$. In view of (1.2) $x \mapsto -\mathrm{tr}(\mathrm{ad}\, x)1$ gives rise to the modular function of $u(L)$.

In contrast to group algebras of finite groups, restricted enveloping algebras are usually not symmetric. In fact, they are symmetric precisely when $\mathrm{tr}(\mathrm{ad}\, x) = 0$ for every $x \in L$, a fact that was first observed by Schue [69].

Let $\mathcal{G}$ be a reduced affine algebraic k-group. Then $\mathcal{G}$ acts on $H(\mathcal{G}_r)$ via the adjoint representation Ad, and a combination of [48, (I.9.7)] and [48, p.135] shows that the character $g \mapsto \det(\mathrm{Ad}(g))^{-1}$ defines a modular function of $H(\mathcal{G}_r)$.

We will return to the structure of restricted Lie algebras in §13 when we study schemes of tori. Presently, we are interested in the interpretation of Lie algebras as infinitesimal groups of height ≤ 1.

Theorem 4.4 *Let $\mathcal{G}$ be an infinitesimal k-group.*

(1) *There is an embedding $u(\mathrm{Lie}(\mathcal{G})) \hookrightarrow H(\mathcal{G})$ of Hopf algebras.*

(2) *The group $\mathcal{G}$ has height ≤ 1 if and only if $u(\mathrm{Lie}(\mathcal{G})) \cong H(\mathcal{G})$.*

Proof. Let $L := \mathrm{Lie}(\mathcal{G})$. The universal property of $u(L)$ guarantees the existence of an algebra homomorphism $\zeta : u(L) \longrightarrow H(\mathcal{G})$ such that $\zeta|_L = id_L$. Thus, $(((\zeta \otimes \zeta) \circ \Delta)|_L = (\Delta \circ \zeta)|_L$, $\varepsilon_{\mathcal{G}} \circ \zeta = \varepsilon_L$, and $(\eta_{\mathcal{G}} \circ \zeta)|_L = (\zeta \circ \eta_L)|_L$, so that ζ is in fact a homomorphism of Hopf algebras.

Let $\{x_1, \ldots, x_n\}$ be a basis of L over k. By (4.3) $\{x^r \ ; \ 0 \leq r \leq \tau\}$ is a basis of $u(L)$ over k. Let $\{\delta_r \ ; \ 0 \leq r \leq \tau\}$ be the dual basis of the commutative Hopf algebra $\mathcal{O}(L) := u(L)^*$. Direct computation shows that $\delta_r * \delta_s = \binom{r+s}{r}\delta_{r+s}$. If ϵ_i denotes the n-tuple with i-th entry 1 and all other entries zero, then the map $X_i \mapsto \delta_{\epsilon_i}$ induces an isomorphism $k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p) \cong \mathcal{O}(L)$ of k-algebras. Moreover $\mathcal{O}(L)^\dagger = (X_1, \ldots, X_n)$.

Recall that L is the space of derivations $\mathcal{O}(\mathcal{G}) \longrightarrow k$. This space is isomorphic to $\mathcal{O}(\mathcal{G})^\dagger/(\mathcal{O}(\mathcal{G})^\dagger)^2$. Application of this argument to $\mathcal{O}(L)$ shows that $\dim_k \mathrm{Lie}(u(L)) = n$, so that $L = \mathrm{Lie}(u(L))$.

(1). Consider the transpose map $\zeta^t : \mathcal{O}(\mathcal{G}) \longrightarrow \mathcal{O}(L)$. Since $\zeta|_L = id_L$, we see that ζ^t induces an isomorphism $\mathcal{O}(\mathcal{G})^\dagger/(\mathcal{O}(\mathcal{G})^\dagger)^2 \cong \mathcal{O}(L)^\dagger/(\mathcal{O}(L)^\dagger)^2$. Consequently,

$$\mathcal{O}(L) = \zeta^t(\mathcal{O}(\mathcal{G})) + (\mathcal{O}(L)^\dagger)^2 = \zeta^t(\mathcal{O}(\mathcal{G})) + \mathrm{rad}(\mathcal{O}(L))^2,$$

so that ζ^t is surjective. Accordingly, ζ is injective.

(2). Suppose that $\mathcal{G}$ is infinitesimal of height ≤ 1. Then $x^p = 0$ for every $x \in \mathcal{O}(\mathcal{G})^\dagger$. Since $\dim_k L = n$, the local algebra $\mathcal{O}(\mathcal{G})$ is generated by n elements of $\mathcal{O}(\mathcal{G})^\dagger$. Thus, the resulting surjective homomorphism $k[X_1, \ldots, X_n] \longrightarrow \mathcal{O}(\mathcal{G})$ factors through the truncated polynomial ring $k[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$. Hence $\dim_k \mathcal{O}(\mathcal{G}) \leq p^n$, and the injection ζ is surjective. $\square$

Remark. Thanks to (4.4) the functors $\mathcal{G} \mapsto \mathrm{Lie}(\mathcal{G})$ and $L \mapsto \mathrm{Spec}_k(u(L)^*)$ induce equivalences between the categories of infinitesimal groups of height ≤ 1 and restricted Lie algebras, respectively.

Proposition 4.5 *The distribution algebra of an infinitesimal k-group has dimension a power of p.*

Proof. Let A be a local, commutative Hopf algebra. We show that A has dimension a power of p. We proceed inductively, and consider the Hopf subalgebra $B := \{a^p \; ; \; a \in A\}$. If $B = k1$, then $\mathcal{G} := \operatorname{Spec}_k(A)$ is an infinitesimal group of height ≤ 1 with function algebra A. Hence our assertion follows from a consecutive application of (4.3) and (4.4).

Alternatively, (1.5) implies that A is free over B. Moreover, $\operatorname{rk}_B(A) = \dim_k A/AB^\dagger$. Since $A/AB^\dagger$ is also a local Hopf algebra, the inductive hypothesis ensures that $\dim_k B$ and $\operatorname{rk}_B(A)$ are p-powers. □

5. Complexity and Representation Type

Throughout this section we will be working with a finite-dimensional self-injective algebra Λ, defined over an arbitrary field k. The notion of the complexity of a module, first introduced by Alperin for group algebras of finite groups [2], has proven to be an effective tool in representation theory. Its main feature is that it makes methods from homological algebra amenable to applications.

Let $(a_i)_{i\geq 0}$ be a sequence of natural numbers. We call

$$\gamma((a_i)_{i\geq 0}) := \min\{s \in \mathbb{N} \cup \{\infty\} \; ; \; \exists \lambda > 0 \text{ such that } a_n \leq \lambda n^{s-1} \quad \forall\, n \geq 1\}$$

the *rate of growth* of the sequence $(a_i)_{i\geq 0}$. If $\mathcal{V} := (V_i)_{i\geq 0}$ is a sequence of finite-dimensional k-vector spaces, then we write $\gamma(\mathcal{V}) := \gamma((\dim_k V_i)_{i\geq 0})$.

Definition. Let M a finite-dimensional Λ-module, $\mathcal{P} := (P_i, \partial_i)_{i\geq 0}$ a minimal projective resolution of M. Then $c_\Lambda(M) := \gamma(\mathcal{P})$ is called the *complexity* of M.

Remarks. (1). Since any two minimal projective resolutions are isomorphic, the complexity of a module is well-defined.

(2). Projective modules obviously have complexity zero. Conversely, if $c_\Lambda(M) = 0$, and $\mathcal{P} := (P_i, \partial_i)_{i\geq 0}$ is a minimal projective resolution of M, then there is $n \geq 1$ such that $P_n = (0)$. Thus, M is, as a module of finite projective dimension over a self-injective algebra, projective.

By our last observation, semisimple algebras are characterized by the property that all their modules have complexity zero. In this section we want to provide similar characterizations for algebras of finite- and tame representation type. We begin with an interpretation of the complexity in terms of extension groups. We let $\mathcal{S}$ denote a complete set of representatives for the isomorphism classes of the simple Λ-modules. The projective cover of the simple Λ-module S will be denoted $P(S)$.

The following result, due to Alperin-Evens [3], relates the complexity of a module to the growth of certain Ext-groups.

Proposition 5.1 *Let M be a finite-dimensional Λ-module. Then*

$$c_\Lambda(M) = \max_{S\in\mathcal{S}} \gamma((\operatorname{Ext}^n_\Lambda(M,S))_{n\geq 0}).$$

Proof. Given a minimal projective resolution $(P_n)_{n\geq 0}$ of M, we decompose each P_n into its indecomposable constituents and write $P_n = \bigoplus_{T\in\mathcal{S}} \ell_{n,T} P(T)$. Basic properties of Ext yield

$$\dim_k \operatorname{Ext}^n_\Lambda(M,S) = \sum_{T\in\mathcal{S}} \ell_{n,T} \dim_k \operatorname{Hom}_\Lambda(P(T),S) = \ell_{n,S} \dim_k \operatorname{Hom}_\Lambda(S,S).$$

Consequently,

$$c_\Lambda(M) = \max_{S\in\mathcal{S}} \gamma((\ell_{n,S})_{n\geq 0}) = \max_{S\in\mathcal{S}} \gamma((\operatorname{Ext}^n_\Lambda(M,S))_{n\geq 0}),$$

as desired. □

In the sequel we let $\operatorname{mod}(\Lambda)$ denote the category of finite-dimensional Λ-modules. Let $M \in \operatorname{mod}(\Lambda)$. Given a minimal projective resolution $(P_i, \partial_i)_{i\geq 0}$ of M, the syzygies $\Omega^n_\Lambda(M) := \ker \partial_{n-1}$ $(n \geq 1)$ are uniquely determined up to isomorphism. Hence $\Omega_\Lambda := \Omega^1_\Lambda$ is a well-defined operator on the isoclasses of Λ-modules. This operator is customarily referred to as the *Heller operator* or *loop space operator*. Note that Ω_Λ induces a functor on the stable module category $\underline{\operatorname{mod}}(\Lambda)$. By definition, we have $\Omega^m_\Lambda \circ \Omega^n_\Lambda = \Omega^{m+n}_\Lambda$ for $m, n \geq 0$.

Dually, we can construct for $n \geq 1$ operators Ω^{-n}_Λ by setting $\Omega^{-n}_\Lambda(M) := \operatorname{coker} \partial^{n-1}$, where $(E_i, \partial^i)_{i\geq 0}$ is a minimal injective resolution of M.

Given $M \in \operatorname{mod}(\Lambda)$, the theorem of Krull-Remak-Schmidt implies a decomposition

$$M = M_P \oplus (\text{proj}),$$

in which the first summand is the sum of all non-projective indecomposable constituents of M. In the following we shall consider the category $\operatorname{mod}(\Lambda)_P$ consisting of those Λ-modules M for which $M = M_P$. Since Λ is self-injective, each element of $\operatorname{mod}(\Lambda)_P$ is characterized by the property that it admits no non-zero projective submodules. Note that $\Omega^n_\Lambda(M) \in \operatorname{mod}(\Lambda)_P$ for all $n \in \mathbb{Z} \setminus \{0\}$, $M \in \operatorname{mod}(\Lambda)$.

Lemma 5.2 *Let M, $N \in \operatorname{mod}(\Lambda)_P$. Then the following statements hold :*

(1) $\Omega_\Lambda(M \oplus N) \cong \Omega_\Lambda(M) \oplus \Omega_\Lambda(N)$.
(2) $\Omega^{-1}_\Lambda(M \oplus N) \cong \Omega^{-1}_\Lambda(M) \oplus \Omega^{-1}_\Lambda(N)$.
(3) *M is indecomposable if and only if $\Omega_\Lambda(M)$ is indecomposable.*
(4) *M is indecomposable if and only if $\Omega^{-1}_\Lambda(M)$ is indecomposable.*

Proof. (3). Suppose M to be indecomposable, and write $\Omega_\Lambda(M) = X \oplus Y$. Since $\Omega^{-1}_\Lambda(\Omega_\Lambda(M)) \cong M$, (2) yields

$$M \cong \Omega^{-1}_\Lambda(X) \oplus \Omega^{-1}_\Lambda(Y),$$

so that we may assume without loss of generality that $\Omega^{-1}_\Lambda(X) = (0)$. As a result, $X = X_P \cong \Omega_\Lambda(\Omega^{-1}_\Lambda(X)) = (0)$, proving that $\Omega_\Lambda(M)$ is indecomposable. □

Definition. A finite-dimensional Λ-module M is said to be *periodic* if there exists $n > 0$ such that $\Omega^n_\Lambda(M) \oplus (\text{proj}) \cong M$.

Example. Let $\Lambda = H(\alpha_p) = k[X]/(X^p)$. Since

$$(0) \longrightarrow kx^{p-1} \longrightarrow k[X]/(X^p) \xrightarrow{x\cdot} k[X]/(X^p) \xrightarrow{\varepsilon} k \longrightarrow (0)$$

are the initial terms of a minimal projective resolution, we have $\Omega^2_{H(\alpha_p)}(k) \cong k$.

Note that every periodic module M has complexity $c_\Lambda(M) \leq 1$. Indeed, periodicity implies the existence of a minimal projective resolution $(P_i, \partial_i)_{i\geq 0}$ satisfying $P_{i+r} \cong P_i$ for some $r > 0$. Hence the dimensions of the P_i are bounded and $c_\Lambda(M) \leq 1$.

Definition. The algebra Λ is *representation-finite* or of *finite representation type* if it admits only finitely many isoclasses of finite-dimensional indecomposable modules.

Theorem 5.3 ([42]) *If Λ has finite representation type, then every finite-dimensional Λ-module is periodic.*

Proof. Let $\mathcal{X}$ be the set of isomorphism types of the non-projective indecomposable A-modules. Owing to (5.2) $\Omega|_{\mathcal{X}}$ is bijective. Since $\mathcal{X}$ is finite, there exists $n \geq 1$ such that $\Omega^n_{\mathcal{X}} = id_{\mathcal{X}}$. Our result is now a direct consequence of (5.2) and the theorem of Krull-Remak-Schmidt. $\square$

The converse of Heller's Theorem does not hold in general. For instance, the group algebra of the quaternion group over a field of characteristic 2 is known to possess only periodic modules. However, since every local algebra of finite representation type is a truncated polynomial ring $k[X]/(X^n)$, this algebra does not have finite representation type (in fact, it is tame). In §10 we shall see that distribution algebras of infinitesimal groups satisfy the converse of Heller's Theorem.

By Heller's theorem the complexity of any module of a representation-finite algebra is bounded by 1. Our next result, which is due to Rickard, provides a similar criterion for tame algebras. Its proof is considerably harder as it employs deep results by Crawley-Boevey [13] concerning the structure of the Auslander-Reiten quiver of tame algebras.

Definition. A k-algebra Λ is said to be *tame* if it is not representation-finite, and if for each $d > 0$ there exist $(\Lambda, k[X])$-bimodules $M_1, \ldots, M_{n(d)}$ that are finitely generated and free over $k[X]$, so that all but finitely many d-dimensional indecomposable Λ-modules are isomorphic to $M_i \otimes_{k[X]} k_\lambda$ for some $i \in \{1, \ldots, n(d)\}$ and $\lambda \in \mathrm{Alg}_k(k[X], k)$.

Theorem 5.4 ([68]) *Suppose that Λ is tame. Then $c_\Lambda(M) \leq 2$ for every $M \in \mathrm{mod}(\Lambda)$.* $\square$

We continue by collecting a few basic properties of the complexity of modules.

Proposition 5.5 *Let Λ be self-injective, $M \in \mathrm{mod}(\Lambda)$.*

(1) *If $S_1, \ldots, S_n$ are the composition factors of M, then $c_\Lambda(M) \leq \max_{1\leq i\leq n} c_\Lambda(S_i)$.*

(2) *If $\Gamma \subset \Lambda$ is a self-injective subalgebra such that Λ is a projective left Γ-module, then $c_\Gamma(M) \leq c_\Lambda(M)$.*

(3) *If Λ is a Hopf algebra, then $c_\Lambda(M) \leq c_\Lambda(k)$.*

Proof. (2). Let $\mathcal{P} := (P_i)_{i\geq 0}$ be a minimal projective resolution of the Λ-module M. By assumption, the functor $\mathrm{Hom}_\Gamma(P_i,\cdot) \cong \mathrm{Hom}_\Lambda(P_i, \mathrm{Hom}_\Gamma(\Lambda,\cdot))$ is exact. Hence each P_i is a projective Γ-module, and $c_\Gamma(M) \leq \gamma(\mathcal{P}) = c_\Lambda(M)$.

(3). Let $\mathcal{P} := (P_i)_{i\geq 0}$ be a minimal projective resolution of the trivial Λ-module k. Thanks to (1.4) the complex $\mathcal{P}\otimes_k M := (P_i \otimes_k M)_{i\geq 0}$ is a projective resolution of M. Hence $c_\Lambda(M) \leq \gamma(\mathcal{P}\otimes_k M) = c_\Lambda(k)$. $\square$

Example. Consider the Hopf algebra $H(\alpha_{p^2}) \cong k[X,Y]/(X^p,Y^p)$. Let $\mathcal{P} = (P_i)_{i\geq 0}$ be a minimal projective resolution of the trivial $k[X]/(X^p)$-module k, i.e., $P_i = k[X]/(X^p)$ for every $i \geq 0$. Setting $Q_i := \sum_{j=0}^{i} P_j \otimes_k P_{i-j}$ we obtain that the complex $\mathcal{Q} := (Q_i)_{i\geq 0}$ is a minimal projective resolution of the $H(\alpha_{p^2})$-module $k\otimes_k k \cong k$. Since $\dim_k Q_i = (i+1)p^2$, we have $c_{H(\alpha_{p^2})}(k) = 2$. One can iterate this process to see that $c_{H(\alpha_{p^r})}(k) = r$.

By (5.5(3)) we thus have $c_{H(\alpha_{p^2})}(M) \leq 2$ for every $M \in \mathrm{mod}(H(\alpha_{p^2}))$. However, $H(\alpha_{p^2}) \cong k[X,Y]/(X^p,Y^p)$ is wild unless $p = 2$. Consequently, the converse of Rickard's Theorem does not obtain.

Remark. By combining (1.5) with (5.4) and (5.5), we see that Hopf subalgebras of tame or representation-finite Hopf algebras come close to being tame or representation-finite. In fact, by special properties of extensions of group algebras, the representation type is inherited by subgroups of finite groups (cf. [9, Prop.2]). It is considerably harder to establish similar results for infinitesimal group schemes.

6. Support Varieties

In this section we will provide a geometric interpretation for the complexity of modules of a finite algebraic k-group $\mathcal{G}$. As before we will be working over an algebraically closed field k of characteristic $p > 0$. Given a module M of a commutative ring R, we recall that the *support* $\mathrm{Supp}(M) := \{P \in \mathrm{Max}(R)\ ;\ M_P \neq (0)\}$ of M is the set of those maximal ideals $P \subset R$ for which the localization of M at P is not trivial. If R is noetherian, and M is finitely generated then $\mathrm{Supp}(M) = Z(\mathrm{Ann}_R(M)) := \{P \in \mathrm{Max}(R)\ ;\ \mathrm{Ann}_R(M) \subset P\}$ is the zero locus of the annihilator of M. Thus, if R is an affine k-algebra, then $\mathrm{Supp}(M)$ is an affine variety.

Let $\mathcal{G}$ be a finite algebraic group. If M is a $H(\mathcal{G})$-module, we denote by

$$H^n(\mathcal{G}, M) := \mathrm{Ext}^n_{H(\mathcal{G})}(k, M) \quad (n \geq 0)$$

the n-th cohomology group of $\mathcal{G}$ with coefficients in M. Note that these are just the Hochschild cohomology groups of the augmented algebra $(H(\mathcal{G}), \varepsilon)$.

These cohomology groups were first studied by Hochschild [44] in the context of restricted Lie algebras. He related them to the Chevalley-Eilenberg cohomology and provided interpretations for H^1 and H^2. Further early results can be found in [60].

Given three $H(\mathcal{G})$-modules X, Y, Z we recall the *Yoneda product*

$$\mathrm{Ext}^m_{H(\mathcal{G})}(Y,Z) \times \mathrm{Ext}^n_{H(\mathcal{G})}(X,Y) \longrightarrow \mathrm{Ext}^{m+n}_{H(\mathcal{G})}(X,Z).$$

This product endows $\operatorname{Ext}^\bullet_{H(\mathcal{G})}(X,X) := \bigoplus_{n\geq 0}\operatorname{Ext}^n_{H(\mathcal{G})}(X,X)$ with the structure of a $\mathbb{Z}$-graded k-algebra. Moreover, the spaces $\operatorname{Ext}^\bullet_{H(\mathcal{G})}(Y,X)$ and $\operatorname{Ext}^\bullet_{H(\mathcal{G})}(X,Y)$ are graded left and right $\operatorname{Ext}^\bullet_{H(\mathcal{G})}(X,X)$-modules, respectively. In particular, $H^\bullet(\mathcal{G},M)$ is a graded right module over the *cohomology ring* $H^\bullet(\mathcal{G},k)$. This ring is known to be *graded commutative*, i.e., we have

$$yx = (-1)^{\deg(x)\deg(y)}xy$$

for any two homogeneous elements $x,y\in H^\bullet(\mathcal{G},k)$. Consequently, the subring

$$H^{\mathrm{ev}}(\mathcal{G},k) = \bigoplus_{i\geq 0} H^{2i}(\mathcal{G},k)$$

is a commutative, $\mathbb{Z}$-graded k-algebra (see [10, §6] for details).

The following result by Friedlander and Suslin, which generalizes earlier work by Venkov [77] and Evens [20] for finite groups, and Friedlander-Parshall [39] for infinitesimal groups of height ≤ 1, is fundamental for everything that follows.

Theorem 6.1 ([40]) *Let $\mathcal{G}$ be a finite algebraic k-group, M a finite-dimensional $H(\mathcal{G})$-module. Then the following statements hold:*

(1) *$H^{\mathrm{ev}}(\mathcal{G},k)$ is a finitely generated k-algebra.*

(2) *$H^\bullet(\mathcal{G},M)$ is a finitely generated $H^{\mathrm{ev}}(\mathcal{G},k)$-module.* $\square$

In some cases, the even cohomology ring can be computed explicitly. If $\mathcal{G}$ is smooth, semisimple and simply connected and p exceeds the Coxeter number of $\mathcal{G}$, then $H^{\mathrm{ev}}(\mathcal{G}_1,k)\cong\mathcal{O}(\mathcal{N})$, where $\mathcal{N} := \{x\in \operatorname{Lie}(\mathcal{G})\ ;\ x^{[p]^n}=0 \text{ for some } n\in\mathbb{N}\}$ is the *nullcone* of $\operatorname{Lie}(\mathcal{G})$ (see [38]).

Let M be a finite-dimensional $H(\mathcal{G})$-module, $(P_i,\partial_i)_{i\geq 0}$ a projective resolution of the trivial module. Since $(P_i\otimes_k M, \partial_i\otimes id_M)_{i\geq 0}$ is a projective resolution of M, we obtain a homomorphism

$$\Phi_M : H^{\mathrm{ev}}(\mathcal{G},k) \longrightarrow \operatorname{Ext}^\bullet_{H(\mathcal{G})}(M,M) \quad ; \quad [f]\mapsto [f\hat{\otimes} id_M]$$

of graded k-algebras. According to (6.1) this map endows the Yoneda algebra with the structure of a finitely generated $H^{\mathrm{ev}}(\mathcal{G},k)$-module. We define the *cohomological support variety* of M via

$$\mathcal{V}_{\mathcal{G}}(M) := Z(\ker\Phi_M)\subset \operatorname{Maxspec}(H^{\mathrm{ev}}(\mathcal{G},k)).$$

Since $\ker\Phi_M$ is a homogeneous ideal, the affine variety $\mathcal{V}_{\mathcal{G}}(M)$ is conical.

Lemma 6.2 *Let $\mathcal{G}'\subset\mathcal{G}$ be a subgroup, $M\in\operatorname{mod}(H(\mathcal{G}))$.*

(1) *$\dim\mathcal{V}_{\mathcal{G}}(M) = c_{H(\mathcal{G})}(M)$.*

(2) *$\dim\mathcal{V}_{\mathcal{G}'}(M)\leq\dim\mathcal{V}_{\mathcal{G}}(M)$.*

Proof. (1). Thanks to (6.1) the Yoneda algebra $\operatorname{Ext}^\bullet_{H(\mathcal{G})}(M,M)$ is a finitely generated $H^{\mathrm{ev}}(\mathcal{G},k)$-module. Consequently, we have

$$\dim\mathcal{V}_{\mathcal{G}}(M) = \dim H^{\mathrm{ev}}(\mathcal{G},k)/\ker\Phi_M = \gamma(H^{\mathrm{ev}}(\mathcal{G},k)/\ker\Phi_M) = \gamma(\operatorname{Ext}^\bullet_{H(\mathcal{G})}(M,M)).$$

In view of (5.1) the latter number is bounded by $c_{H(\mathcal{G})}(M)$. To verify the reverse inequality, we let S be a simple $H(\mathcal{G})$-module. Thanks to (6.1), the space $\mathrm{Ext}^\bullet_{H(\mathcal{G})}(M,S) \cong H^\bullet(\mathcal{G}, M^* \otimes_k S)$ is a finitely generated $H^{\mathrm{ev}}(\mathcal{G},k)$-module. Consequently, it is also a finitely generated right $\mathrm{Ext}^\bullet_{H(\mathcal{G})}(M,M)$-module. Accordingly, we have

$$\gamma(\mathrm{Ext}^\bullet_{H(\mathcal{G})}(M,S)) \leq \gamma(\mathrm{Ext}^\bullet_{H(\mathcal{G})}(M,M)),$$

so that another application of (5.1) yields $c_{H(\mathcal{G})}(M) = \gamma(\mathrm{Ext}^\bullet_{H(\mathcal{G})}(M,M))$, as desired.

(2). This follows directly from (1), (5.5) and (1.5). □

As an immediate application, we record the following basic criteria for blocks of finite and tame representation types.

Theorem 6.3 *Let $\mathcal{B} \subset H(\mathcal{G})$ be a block, $M \in \mathrm{mod}(\mathcal{B})$.*

(1) *If $\mathcal{B}$ is representation-finite, then $\dim \mathcal{V}_\mathcal{G}(M) \leq 1$.*

(2) *If $\mathcal{B}$ is tame, then $\dim \mathcal{V}_\mathcal{G}(M) \leq 2$.* □

Example. Suppose that $p \geq 3$. From the Künneth formula one obtains an isomorphism $H^\bullet(\alpha_{p^n}, k) \cong k[X_1, \dots X_n] \otimes_k \Lambda(Y_1, \dots Y_n)$, where the X_i and Y_i have degrees 2 and 1, respectively (cf. [10, (7.6)]). Consequently, $k[X_1, \dots, X_n]$ is a Noether normalization of $H^{\mathrm{ev}}(\alpha_{p^n}, k)$ and $\dim \mathcal{V}_{\alpha_{p^n}}(k) = n$. Note that this agrees with our earlier observations.

7. Rank Varieties

Early work by Quillen [64, 65] showed that the support variety of the trivial module of a finite group G may be described as the union of the corresponding supports for the elementary abelian subgroups of G. This result was later extended to arbitrary modules by Avrunin and Scott [5].

For elementary abelian groups a second notion is available, that of the so-called *rank varieties*. These were introduced by Jon Carlson [11, 12]. They can roughly be described as follows: Given an elementary abelian group E, one considers a subspace $V \subset k[E]$ with $\dim_k V = \mathrm{rk}(E)$ whose nonzero elements have the following property:

$$x^p = 0, \text{ and } k[x] \text{ is a local algebra of dimension } p \text{ such that } k[E]|_{k[x]} \text{ is free.}$$

We may then define

$$\hat{\mathcal{V}}_E(M) := \{x \in V \ ; \ M|_{k[x]} \text{ is not free}\} \cup \{0\}.$$

Avrunin and Scott [5] showed that $\hat{\mathcal{V}}_E(M) \cong \mathcal{V}_E(M)$, and they thus obtained an intrinsic characterization of the cohomological support variety.

In the mid 80's Friedlander-Parshall [39] introduced support varieties for infinitesimal groups of height ≤ 1. For the trivial module Jantzen [47] gave an intrinsic characterization in terms of a subvariety of the associated Lie algebra. These rank varieties also occurred in

Voigt's work [80]. The description of support varieties of arbitrary modules in terms of rank varieties was given by Friedlander and Parshall in [37]. Recently, Suslin, Friedlander and Bendel [72, 73] have employed "higher nullcones" (cf. [41]) to generalize these concepts and results to infinitesimal groups of arbitrary height.

Recall our identification of the distribution algebra of $\alpha_{p^n} = \mathrm{Spec}_k(k[T]/(T^{p^n}))$. Setting $t := T + (T^{p^n})$, we consider the basis $\{\delta_0, \ldots, \delta_{p^n-1}\} \subset H(\alpha_{p^n})$ that is dual to $\{t^0, \ldots, t^{p^n-1}\}$. The assignment $X_i \mapsto \delta_{p^i}$ then defines an isomorphism $k[X_0, \ldots, X_{n-1}]/(X_0^p, \ldots, X_{n-1}^p)$. Hence, as an algebra, $H(\alpha_{p^n})$ looks like the group algebra of the elementary abelian group $(\mathbb{Z}/(p))^n$. For the definition of the rank variety we have to consider elementary abelian subgroups of a given infinitesimal group $\mathcal{G}$. We do this by considering homomorphisms $\alpha_{p^n} \longrightarrow \mathcal{G}$. (For finite groups this would amount to considering all rank varieties of elementary abelian subgroups of rank $\leq n$). Let $0 \leq \ell \leq n-1$. The embedding $\alpha_{p^\ell} \hookrightarrow \alpha_{p^n}$ is induced by the projection map $k[t] \longrightarrow k[t]/(t^{p^\ell})$. Accordingly, $H(\alpha_{p^\ell}) = k[\delta_0, \ldots, \delta_{p^\ell-1}]$. We consider the *subalgebra* that does not meet the Hopf ideal $H(\alpha_{p^n})H(\alpha_{p^{n-1}})^\dagger = (\delta_1, \ldots, \delta_{p^{n-1}-1})$ and put

$$A_n := k[\delta_{p^{n-1}}] \cong k[X_{n-1}]/(X_{n-1}^p).$$

Now let $\mathcal{G}$ be an affine algebraic group, M a $\mathcal{G}$-module. Given a homomorphism $\varphi : \alpha_{p^n} \longrightarrow \mathcal{G}$, i.e., a homomorphism $\varphi : H(\alpha_{p^n}) \longrightarrow H(\mathcal{G}_n)$ of Hopf algebras, the module M obtains, via pull-back, the structure of an $H(\alpha_{p^n})$-module. Suppose that $\varphi \neq 0$. From the subgroup structure of α_{p^n} we obtain $\ker \varphi \subset \alpha_{p^{n-1}}$, so that the kernel of the corresponding homomorphism $\varphi : H(\alpha_{p^n}) \longrightarrow H(\mathcal{G}_n)$ is contained in $(\delta_1, \ldots, \delta_{p^{n-1}-1})$. Consequently, (1.5) implies that $H(\mathcal{G}_n)$ is free over A_n. We have thus emulated the set-up of Carlson's rank varieties.

Definition. Let $\mathcal{G}$ be an affine algebraic group, M a finite-dimensional $\mathcal{G}$-module. For $n \geq 1$ we put

$$\hat{\mathcal{V}}_{\mathcal{G}_n}(M) := \{\varphi \in \mathrm{Hom}(\alpha_{p^n}, \mathcal{G}) \ ; \ M|_{A_n} \text{ is not free}\}.$$

Examples. (1). For $n = 1$, we have $A_1 = H(\alpha_p)$. Recall that $H(\alpha_p) = k[x] \cong k[X]/(X^p)$, where $kx = \mathrm{Lie}(\alpha_p)$. If $\mathcal{G}$ is any algebraic group, then the homomorphisms $\alpha_p \longrightarrow \mathcal{G}$ correspond to the points of $x \in \mathrm{Lie}(\mathcal{G})$ satisfying $x^p = 0$. It follows that

$$\hat{\mathcal{V}}_{\mathcal{G}_1}(M) := \{x \in \mathrm{Lie}(\mathcal{G}) \ ; \ x^p = 0 \text{ and } M|_{k[x]} \text{ is not free}\} \cup \{0\}.$$

(2). Consider the group α_{p^2}. We have seen before that $\mathrm{Lie}(\alpha_{p^2}) = kx$, where $x^p = 0$. Consequently, $\hat{\mathcal{V}}_{(\alpha_{p^2})_1}(k) = \mathrm{Lie}(\alpha_{p^2})$ has dimension 1.

For $n \geq 2$ we consider $\hat{\mathcal{V}}_{(\alpha_{p^2})_n}(k) = \mathrm{Hom}(\alpha_{p^n}, \alpha_{p^2})$. By the Yoneda Lemma, the latter space corresponds to $\mathrm{Hom}_{\mathrm{Hopf}}(k[T]/(T^{p^2}), k[Z]/(Z^{p^n}))$. A homomorphism $\varphi : k[T]/(T^{p^2}) \longrightarrow k[Z]/(Z^{p^n})$ is determined by the image $\varphi(t)$ of the primitive element $t := T + (T^{p^2})$. We put $z := Z + (Z^{p^n})$ and note that $\bigoplus_{i=0}^{n-1} kz^{p^i}$ is the space of primitive elements of $k[Z]/(Z^{p^n})$. Since $\varphi(t)^{p^2} = 0$, we have $\varphi(t) = \lambda_{n-2}z^{p^{n-2}} + \lambda_{n-1}z^{p^{n-1}}$. Consequently, $\hat{\mathcal{V}}_{(\alpha_{p^2})_n}(k) \cong k \times k$ has dimension 2 for $n \geq 2$.

From the last example, we see that the rank variety $\hat{\mathcal{V}}_{(\alpha_{p^2})_n}(k)$ has dimension $\dim \mathcal{V}_{\alpha_{p^2}}(k)$ as soon as n equals the height of the infinitesimal group α_{p^2}. The following important result shows that this is not accidental:

Theorem 7.1 ([73]) *Suppose that $\mathcal{G}$ is an infinitesimal group of height n. Then*

$$\dim \hat{\mathcal{V}}_{\mathcal{G}_n}(M) = \dim \mathcal{V}_{\mathcal{G}}(M)$$

for every $M \in \mathrm{mod}(H(\mathcal{G}))$. $\square$

In general rank varieties are hard to compute. In special cases, however, we have more information:

Examples. (1). Let $\mathcal{G} = \mathrm{GL}(n)$. We have

$$\hat{\mathcal{V}}_{\mathrm{GL}(n)_r}(k) = \{(x_1, \ldots, x_r) \in \mathrm{Mat}_n(k)^r \ ; \ x_i^p = 0 = [x_j, x_\ell] \ \ 1 \leq i, j, \ell \leq r\},$$

where $[x, y] = xy - yx$ denotes the commutator product.

(2). Let $\mathcal{G} = \mathrm{SL}(2)$. Then $\hat{\mathcal{V}}_{\mathcal{G}_1}(k)$ is the variety of nilpotent (2×2)-matrices, so that $\dim \hat{\mathcal{V}}_{\mathcal{G}_1}(k) = 2$.

The variety $\hat{\mathcal{V}}_{\mathcal{G}_2}(k)$ corresponds to the set of pairs of commuting nilpotent matrices, and $\dim \hat{\mathcal{V}}_{\mathcal{G}_2}(k) = 3$.

Consequently, the representation theory quickly gets complicated for higher Frobenius kernels. By (7.1) and (6.3) we see that $\mathrm{SL}(2)_2$ is wild. It turns out that the group $\mathrm{SL}(2)_1$ is tame (see [35]).

(3). Consider the Borel subgroup $\mathrm{B} \subset \mathrm{SL}(2)$ of upper triangular matrices. The first Frobenius kernel B_1 affords a character

$$\lambda : \mathrm{B}_1 \longrightarrow \mu_k \ \ ; \ \ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto a^{p-1}.$$

Recall that λ corresponds to a group-like element of $\mathcal{O}(\mathrm{B}_1) = H(\mathrm{B}_1)^*$. Hence we can view λ as a character $H(\mathrm{B}_1) \longrightarrow k$. The corresponding one-dimensional $H(\mathrm{B}_1)$-module with action given by $h \cdot \alpha := \lambda(h)\alpha$ will be denoted k_λ. One can show that the induced module $\mathrm{St} := H(\mathcal{G}_1) \otimes_{H(\mathrm{B}_1)} k_\lambda$ is projective. This module is called the *Steinberg module*. However, the canonical map

$$\varphi : \alpha_p \longrightarrow \mathrm{B}_1 \ \ ; \ \ b \mapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

defines a non-trivial element of $\hat{\mathcal{V}}_{\mathrm{B}_1}(k_\lambda)$. Consequently, k_λ is not projective and thus not a direct summand of $(H(\mathcal{G}_1) \otimes_{H(\mathrm{B}_1)} k_\lambda)|_{H(\mathrm{B}_1)}$. The Mackey decomposition theorem ensures that this phenomenon does not occur in the respresentation theory of finite groups.

Theorem 7.2 *Let $\mathcal{G}$ be an infinitesimal group.*

(1) *If $(0) \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow (0)$ is an exact sequence of finite-dimensional $H(\mathcal{G})$-modules, then $\hat{\mathcal{V}}_{\mathcal{G}_n}(X) \subset \hat{\mathcal{V}}_{\mathcal{G}_n}(M) \cup \hat{\mathcal{V}}_{\mathcal{G}_n}(N)$.*

(2) *If the sequence splits, then $\hat{\mathcal{V}}_{\mathcal{G}_n}(X) = \hat{\mathcal{V}}_{\mathcal{G}_n}(M) \cup \hat{\mathcal{V}}_{\mathcal{G}_n}(N)$.*

(3) *$\hat{\mathcal{V}}_{\mathcal{G}_n}(M \otimes_k N) = \hat{\mathcal{V}}_{\mathcal{G}_n}(M) \cap \hat{\mathcal{V}}_{\mathcal{G}_n}(N)$ for M, $N \in \mathrm{mod}(H(\mathcal{G}))$.*

Proof. We only verify (1) and (2). The proof of (3) is more involved. Let φ be an element of $\hat{\mathcal{V}}_{\mathcal{G}_n}(X)$. If $\varphi \notin \hat{\mathcal{V}}_{\mathcal{G}_n}(M)$, then M is a projective A_n-module, and the given sequence splits over A_n. Consequently, N is not a projective A_n-module, proving $\varphi \in \hat{\mathcal{V}}_{\mathcal{G}_n}(N)$. Since direct summands of projective modules are projective, we also have (2). $\square$

8. Quotients

Let $\mathcal{G}$ be an affine algebraic k-group, $\mathcal{N} \subset \mathcal{G}$ a normal subgroup. We would like to define a quotient $\mathcal{G}/\mathcal{N}$. The naive approach, setting $\mathcal{G}/\mathcal{N}(R) := \mathcal{G}(R)/\mathcal{N}(R)$ does not work, as this k-functor is not necessarily representable (see the example below). What we need is a categorical quotient, i.e., a map $\pi : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$ with kernel $\mathcal{N}$ that satisfies the following universal property:

If $\varphi : \mathcal{G} \longrightarrow \mathcal{G}'$ is any group homomorphism of affine group schemes such that $\mathcal{N} \subset \ker \varphi$, then there exists a unique homomorphism $\psi : \mathcal{G}/\mathcal{N} \longrightarrow \mathcal{G}'$ such that $\psi \circ \pi = \varphi$.

Let us see what this condition amounts to: Let $\pi^* : \mathcal{O}(\mathcal{G}/\mathcal{N}) \longrightarrow \mathcal{O}(\mathcal{G})$ be the associated Hopf algebra homomorphism. Setting $I := \ker \pi^*$ we consider the subgroup $\mathcal{H} := \mathcal{V}(I) \subset \mathcal{G}/\mathcal{N}$ as well as the canonical embedding $\iota : \mathcal{H} \longrightarrow \mathcal{G}/\mathcal{N}$. By construction, there is a homomorphism $\gamma^* : \mathcal{O}(\mathcal{H}) \longrightarrow \mathcal{O}(\mathcal{G})$ of Hopf algebras such that $\gamma^* \circ \iota^* = \pi^*$. By the universal property, the corresponding homomorphism $\gamma : \mathcal{G} \longrightarrow \mathcal{H}$ induces a unique map $\omega : \mathcal{G}/\mathcal{N} \longrightarrow \mathcal{H}$ with $\omega \circ \pi = \gamma$. Accordingly, we have $\pi = \iota \circ \gamma = (\iota \circ \omega) \circ \pi$, and unicity implies $\iota \circ \omega = id_{\mathcal{G}/\mathcal{N}}$. Thus $\omega^* \circ \iota^* = id_{\mathcal{O}(\mathcal{G}/\mathcal{N})}$, so that $I = \ker \iota^* = (0)$. We therefore make the following definition:

Definition. A homomorphism $\pi : \mathcal{G} \longrightarrow \mathcal{H}$ between to affine k-groups is a *quotient map* if the corresponding Hopf algebra map $\mathcal{O}(\mathcal{H}) \longrightarrow \mathcal{O}(\mathcal{G})$ is injective.

It is by no means clear that for any normal subgroup $\mathcal{N} \subset \mathcal{G}$ a quotient map with kernel $\mathcal{N}$ exists. It turns out that the algebra $\mathcal{O}(\mathcal{G})^{\mathcal{N}}$ of invariants gives rise to the quotient group. Thus, if $\mathcal{G}$ is algebraic, then the question as to whether $\mathcal{G}/\mathcal{N}$ inherits this property is related to Hilbert's fourteenth problem. Of course, for finite algebraic groups we don't need to worry about such issues.

Theorem 8.1 *Let $\mathcal{G}$ be an affine k-group, $\mathcal{N} \subset \mathcal{G}$ a normal subgroup. Then there exists a quotient map $\pi : \mathcal{G} \longrightarrow \mathcal{H}$ with kernel $\mathcal{N}$. If $\mathcal{G}$ is algebraic, so is $\mathcal{H}$.* $\square$

Remarks. (1). By the universal property, the pair $(\mathcal{H}, \pi)$ is unique up to isomorphism. One thus writes $\mathcal{G}/\mathcal{N} := \mathcal{H}$ and calls $\mathcal{G}/\mathcal{N}$ the *factor group of $\mathcal{G}$ by $\mathcal{N}$*.

(2). Note that the quotient map $\pi : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$ is usually not surjective at each point. Suppose it is, then there exists $\lambda \in \mathcal{G}(\mathcal{O}(\mathcal{G}/\mathcal{N}))$ such that $\pi(\lambda) = id_{\mathcal{O}(\mathcal{G}/\mathcal{N})}$. Accordingly, the comorphism $\pi^* : \mathcal{O}(\mathcal{G}/\mathcal{N}) \longrightarrow \mathcal{O}(\mathcal{G})$ is split injective.

Example. Consider the group α_{p^2} with function algebra $\mathcal{O}(\alpha_{p^2}) = k[T]/(T^{p^2})$. We write $t := T + (T^{p^2})$ and observe that the primitive element t^p generates a Hopf ideal I. Thus, the corresponding normal subgroup $\mathcal{N} := \mathcal{V}(I)$ is isomorphic to α_p. Consider the group scheme

$$\mathcal{X} : M_k \longrightarrow \mathrm{Gr} \quad ; \quad R \mapsto \mathcal{G}(R)/\mathcal{N}(R).$$

Then $\pi := (\pi_R : \mathcal{G}(R) \longrightarrow \mathcal{X}(R))_{R \in M_k}$ is a homomorphism of k-group functors. If $\mathcal{X} = \mathrm{Spec}_k(\mathcal{O}(\mathcal{X}))$ is representable, then $\pi^* : \mathcal{O}(\mathcal{X}) \longrightarrow \mathcal{O}(\mathcal{G})$ is split injective, and there exists an ideal $J \subset \mathcal{O}(\mathcal{G})$ such that

$$\mathcal{O}(\mathcal{G}) = \mathcal{O}(\mathcal{X}) \oplus J.$$

Since $\mathcal{O}(\mathcal{G})$ is free over $\mathcal{O}(\mathcal{X})$, we have $\dim_k \mathcal{O}(\mathcal{X}) = p$. Consequently, $x^p = 0$ for every $x \in \mathcal{O}(\mathcal{X})^\dagger$, so that $\mathcal{O}(\mathcal{X})^\dagger \subset \bigoplus_{i=p}^{p^2-1} kt^i$. The condition $\Delta(\mathcal{O}(\mathcal{X})) \subset \mathcal{O}(\mathcal{X}) \otimes_k \mathcal{O}(\mathcal{X})$ then implies $\mathcal{O}(\mathcal{X}) = k[t^p]$. Thus, $x^p \in \mathcal{O}(\mathcal{X})$ for every $x \in \mathcal{O}(\mathcal{G})$, and the p-power map is trivial on J. Since $\mathcal{O}(\mathcal{G})^\dagger = \mathcal{O}(\mathcal{X})^\dagger \oplus J$, we obtain $x^p = 0 \;\; \forall\, x \in \mathcal{O}(\mathcal{G})^\dagger$, a contradiction.

We will employ quotients mainly in inductive arguments. To that end we have to identify the algebra of measures of a quotient. In this context, the following notion is convenient:

Let $\mathcal{G}$ be a finite algebraic k-group. If η denotes the antipode of $H(\mathcal{G})$, then the action given by

$$h \cdot x := \sum_{(h)} h_{(1)} x \, \eta(h_{(2)}) \qquad \forall\, h, x \in H(\mathcal{G})$$

is called the *(left) adjoint representation of* $H(\mathcal{G})$.

Proposition 8.2 *Let $\mathcal{G}$ be a finite algebraic k-group, $\mathcal{N} \subset \mathcal{G}$ a normal subgroup. Then $H(\mathcal{G})H(\mathcal{N})^\dagger$ is a Hopf ideal, and $H(\mathcal{G}/\mathcal{N}) \cong H(\mathcal{G})/H(\mathcal{G})H(\mathcal{N})^\dagger$.*

Proof. Consider the natural map $\mathcal{G} \times \mathcal{N} \longrightarrow \mathcal{N}\ ;\ (g, n) \mapsto gng^{-1}$. By the universal property there exists a unique linear map $H(\mathcal{G} \times \mathcal{N}) \longrightarrow H(\mathcal{N})$ extending the above action. Direct computation, using $H(\mathcal{G} \times \mathcal{N}) \cong H(\mathcal{G}) \otimes_k H(\mathcal{N})$, shows that this map is given by the adjoint representation. Accordingly,

$$h \cdot x \in H(\mathcal{N}) \qquad \forall\, h \in H(\mathcal{G}),\ x \in H(\mathcal{N}).$$

Given $x \in H(\mathcal{N})^\dagger$ and $h \in H(\mathcal{G})$, we therefore have

$$xh = \sum_{(h)} \varepsilon(h_{(1)}) x h_{(2)} = \sum_{(h)} h_{(1)} \eta(h_{(2)}) x h_{(3)} = \sum_{(h)} h_{(1)} (\eta(h_{(2)}) \cdot x) \in H(\mathcal{G})H(\mathcal{N})^\dagger.$$

Thus, $H(\mathcal{G})H(\mathcal{N})^\dagger$ is a two-sided ideal, and it readily follows that it is also a Hopf ideal.

Let $\pi : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$ and $\iota : \mathcal{N} \hookrightarrow \mathcal{G}$ be the quotient map and the canonical embedding, respectively. Since $\pi^* : \mathcal{O}(\mathcal{G}/\mathcal{N}) \longrightarrow \mathcal{O}(\mathcal{G})$ is injective, the induced map $\hat{\pi} : H(\mathcal{G}) \longrightarrow H(\mathcal{G}/\mathcal{N})$ is surjective. By the same token, $\hat{\iota} : H(\mathcal{N}) \longrightarrow H(\mathcal{G})$ is injective. Since $\pi \circ \iota = 1$, we have $\hat{\pi} \circ \hat{\iota} = \varepsilon$. Consequently, $H(\mathcal{G})H(\mathcal{N})^\dagger \subset \ker \hat{\pi}$.

Consider the cocommutative Hopf algebra $H := H(\mathcal{G})/H(\mathcal{G})H(\mathcal{N})^\dagger$ as well as $\mathcal{H} := \mathrm{Spec}_k(H^*)$. There results a factorization $\hat{\pi} = \hat{\zeta} \circ \hat{\gamma}$, with a surjective homomorphism $\hat{\gamma} : H(\mathcal{G}) \longrightarrow H$ of Hopf algebras. Note that $\hat{\gamma}$ corresponds to a homomorphism $\gamma : \mathcal{G} \longrightarrow \mathcal{H}$. Since $\hat{\gamma} \circ \hat{\iota} = \varepsilon$, we have $\gamma \circ \iota = 1$, whence $\mathcal{N} \subset \ker \gamma$. The universal property now provides a homomorphism $\omega : \mathcal{G}/\mathcal{N} \longrightarrow \mathcal{H}$ such that $\omega \circ \pi = \gamma$. Consequently,

$$(\hat{\omega} \circ \hat{\zeta}) \circ \hat{\gamma} = \hat{\omega} \circ \hat{\pi} = \hat{\gamma},$$

so that the surjectivity of $\hat{\gamma}$ yields $\hat{\omega} \circ \hat{\zeta} = id_H$. As a result, $\hat{\zeta}$ is injective, so that $\ker \hat{\pi} = \ker(\hat{\zeta} \circ \hat{\gamma}) = \ker \hat{\gamma} = H(\mathcal{G})H(\mathcal{N})^\dagger$. $\square$

9. Hopf Algebras of Solvable Groups

In this section we are going to collect a few properties of distribution algebras of infinitesimal solvable group schemes and their representations. We retain our general conventions and assume that k is an algebraically closed field of characteristic $p > 0$.

Definitions. An affine group scheme $\mathcal{G}$ is said to be *abelian* if $\mathcal{G}(R)$ is abelian for every $R \in M_k$. We say that $\mathcal{G}$ is *solvable* if there exists a sequence $e_k = \mathcal{G}_{[0]} \subset \mathcal{G}_{[1]} \subset \mathcal{G}_{[2]} \subset \cdots \subset \mathcal{G}_{[n]} = \mathcal{G}$ of closed subgroups such that $\mathcal{G}_{[i-1]}$ is normal in $\mathcal{G}_{[i]}$ and $\mathcal{G}_{[i]}/\mathcal{G}_{[i-1]}$ is abelian $1 \leq i \leq n$.

Lemma 9.1 *An infinitesimal group scheme $\mathcal{G}$ is solvable if and only if its first Frobenius kernel $\mathcal{G}_1$ is solvable.*

Proof. We use induction on the height $\mathrm{ht}(\mathcal{G})$ of $\mathcal{G}$. If $r := \mathrm{ht}(\mathcal{G}) > 1$, then $\mathcal{G}_{r-1}$ is solvable. Consider the quotient map $\pi : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}_{r-1}$. By the universal property, there exists an embedding $\gamma : \mathcal{G}/\mathcal{G}_{r-1} \hookrightarrow \mathcal{G}^{(p^{r-1})}$ such that $\gamma \circ \pi = F^{r-1}$. Let $F_1 : \mathcal{G}/\mathcal{G}_{r-1} \longrightarrow (\mathcal{G}/\mathcal{G}_{r-1})^{(p)}$ be the Frobenius homomorphism, $\gamma^{(p)} : (\mathcal{G}/\mathcal{G}_{r-1})^{(p)} \hookrightarrow \mathcal{G}^{(p^r)}$ the map induced by γ. Then we have

$$\gamma^{(p)} \circ F_1 \circ \pi = F \circ \gamma \circ \pi = F \circ F^{r-1} = 0.$$

Consequently, $F_1 \circ \pi = 0$, and since π is a quotient map this entails $F_1 = 0$. Accordingly, $\mathcal{G}/\mathcal{G}_{r-1}$ is a closed subgroup of $\mathcal{G}^{(p^{r-1})}$ of height ≤ 1. As $(\mathcal{G}^{(p)})_1$ is solvable, we see that $\mathcal{G}/\mathcal{G}_{r-1}$ is solvable. It follows that $\mathcal{G}$ is solvable. $\square$

Remark. According to (4.4) the distribution algebra $H(\mathcal{G})$ of an infinitesimal group of height ≤ 1 is isomorphic to the restricted enveloping algebra $u(\mathrm{Lie}(\mathcal{G}))$ of its Lie algebra. Since $\mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G}_1)$ we can interpret (9.1) by saying that an infinitesimal group is solvable if and only if its Lie algebra enjoys this property.

To illustrate some subtle points of the theory, we continue by quoting a classical result, the so-called *Lie-Kolchin Theorem*. Given $n \geq 1$, we consider the closed subgroup $\mathrm{Upp}(n) \subset \mathrm{GL}(n)$ of upper triangular matrices. Recall that an affine algebraic group $\mathcal{G}$ is *connected* provided the identity element is the only idempotent of its function algebra $\mathcal{O}(\mathcal{G})$.

Theorem 9.2 (Lie-Kolchin) *Let $\mathcal{G}$ be a connected, reduced (=smooth), solvable, affine algebraic k-group. Then there exists a closed embedding $\mathcal{G} \hookrightarrow \mathrm{Upp}(n)$ for some $n \geq 1$.* $\square$

Accordingly, the Frobenius kernels of the smooth solvable algebraic groups can be put into triangular form. This is equivalent to saying that all simple modules of the distribution algebra are one-dimensional, that is, this algebra is basic. Unfortunately, the representation theory of arbitrary solvable infinitesimal groups is considerably more complicated.

We say that a finite algebraic group $\mathcal{G}$ is *trigonalizable* if its algebra of measures $H(\mathcal{G})$ is basic.

Example. Let X be an indeterminate, and consider the following operators of the k-vector space $V := k[X]/(X^p)$:

$$\partial : x^i \mapsto ix^{i-1} \ ; \ \mu : x^i \mapsto x^{i+1} \ ; \ id_V,$$

where $x := X + (X^p)$. Since $[\partial, \mu] = id_V$ the vector space $L := k\partial \oplus k\mu \oplus k\, id_V$ is closed under the Lie bracket operation. It is in fact the three-dimensional Heisenberg algebra with p-map given by the ordinary p-power of linear transformations, i.e., $\partial^p = 0 = \mu^p$, $id_V^p = id_V$. One can show that V is a simple module for L. Moreover, the Lie algebra L is solvable (even nilpotent). Accordingly, the group $\mathcal{G} := \operatorname{Spec}_k(u(L)^*)$ is a solvable group, that is not trigonalizable.

Definition. An affine algebraic group $\mathcal{M}$ is said to be *diagonalizable* or *multiplicative* if there exists an abelian group C such that $\mathcal{O}(\mathcal{M}) = k[C]$ is the group algebra of C.

Remarks. (1). The name "diagonalizable" derives from the fact, that such groups may be embedded into some group $\operatorname{diag}(n)$ of diagonal $(n \times n)$-matrices. Since group algebras are cocommutative, every multiplicative group $\mathcal{M}$ is abelian.

(2). Let $\mathcal{M}$ be multiplicative and infinitesimal. Then $k[C]$ is local, so that C is an abelian p-group. Consequently, $C \cong \prod_{i=1}^{\ell} \mathbb{Z}/(p^{n_i})$, and $\mathcal{M} \cong \prod_{i=1}^{\ell} \mu_{p^{n_i}}$.

(3). Suppose that $\mathcal{M}$ is a finite multiplicative group. Then C is the set of group-like elements of $H(\mathcal{M})^*$, i.e., $C = \operatorname{Hom}_{\mathrm{Alg}}(H(\mathcal{M}), k)$ is the character group of $\mathcal{M}$. Let $h \in H(\mathcal{M})$ be a nilpotent element. Then $\lambda(h) = 0$ for every $\lambda \in C$. Thus $\lambda(h) = 0$ for every $\lambda \in \mathcal{O}(\mathcal{M})$, and $h = 0$. Consequently, the commutative distribution algebra $H(\mathcal{M})$ is semisimple. We will see later, that the multiplicative infinitesimal groups are the only infinitesimal groups with semisimple distribution algebra. Thus, in contrast to finite groups, the structure of linearly reductive infinitesimal groups is completely understood.

(4). Multiplicative group schemes are *rigid*. If a connected group acts on such a group via automorphisms, then the action is trivial (see [81, (7.7)] for details). This implies that every infinitesimal group $\mathcal{G}$ possesses a unique maximal normal multiplicative subgroup $\mathcal{M}(\mathcal{G})$. This group is called the *multiplicative center* of $\mathcal{G}$.

Definition. Let $\mathcal{G}$ be an infinitesimal group.

(1) The group $\mathcal{G}$ is *supersolvable* if there exists a chain $e_k = \mathcal{G}_{[0]} \subset \mathcal{G}_{[1]} \subset \cdots \subset \mathcal{G}_{[n]} = \mathcal{G}$ of normal subgroups of $\mathcal{G}$ such that $\mathcal{G}_{[i]}/\mathcal{G}_{[i-1]} \cong \alpha_p$ or μ_p for every $i \in \{1, \ldots, n\}$.

(2) The group $\mathcal{G}$ is *unipotent* if $H(\mathcal{G})$ is local.

(3) The group $\mathcal{G}$ is *nilpotent* if the factor group $\mathcal{G}/\mathcal{M}(\mathcal{G})$ is unipotent.

The unipotent groups are precisely those that can be embedded into a group of strictly upper triangular matrices (i.e, upper triangular matrices with 1's on the main diagonal). Supersolvable and nilpotent infinitesimal groups may be characterized in terms of the principal blocks of their distribution algebras:

Theorem 9.3 ([78]) *Let $\mathcal{G}$ be an infinitesimal group.*

(1) *$\mathcal{G}$ is nilpotent if and only if $\mathcal{B}_0(\mathcal{G})$ is local.*

(2) *$\mathcal{G}$ is supersolvable if and only if $\mathcal{B}_0(\mathcal{G})$ is basic.* $\square$

Theorem 9.3 implies that distribution algebras with basic principal block satisfy a useful linkage relation. Given simple modules S, T of an arbitrary block $\mathcal{B} \subset H(\mathcal{G})$, Schur's Lemma ensures that $\mathcal{M}(\mathcal{G})$ operates on both modules via the same character. Consequently, $\mathcal{M}(\mathcal{G})$ operates trivially on $\mathrm{Hom}_k(S, T)$, so that the latter becomes a module for the trigonalizable group $\mathcal{G}/\mathcal{M}(\mathcal{G})$. There thus exists a character $\lambda : H(\mathcal{G}) \longrightarrow k$ and a one-dimensional submodule $k_\lambda \subset \mathrm{Hom}_k(S, T)$. Any generator φ of this submodule is an isomorphism $S \cong T \otimes_k k_{\lambda\circ\eta}$. In particular, all simple $\mathcal{B}$-modules have the same dimension.

Proposition 9.4 *Let $\mathcal{G}$ be a supersolvable, infinitesimal k-group.*

(1) *If $\mathcal{G}$ is trigonalizable, then $H(\mathcal{G})$ possesses* $\dim_k H(\mathcal{M}(\mathcal{G}))$ *blocks, each of which is isomorphic to $H(\mathcal{G}/\mathcal{M}(\mathcal{G}))$.*

(2) *The canonical projection induces an isomorphism $\mathcal{B}_0(\mathcal{G}) \cong H(\mathcal{G}/\mathcal{M}(\mathcal{G}))$.*

Proof. (1). Since $H(\mathcal{G})$ is trigonalizable, the group $\mathcal{G}$ decomposes into a semidirect product

$$\mathcal{G} = \mathcal{U} \rtimes \mathcal{M},$$

where $\mathcal{M} \supset \mathcal{M}(\mathcal{G})$ is multiplicative, and $\mathcal{U}$ is a unipotent normal subgroup of $\mathcal{G}$ (cf. [14, (IV.§2(3.5))]). The function algebra $\mathcal{O}(\mathcal{M})$ of the multiplicative group $\mathcal{M}$ is isomorphic to the group algebra $k[G(\mathcal{O}(\mathcal{M}))]$. In the sequel we will occasionally identify $G(\mathcal{O}(\mathcal{M}))$ with the image of $G(\mathcal{O}(\mathcal{G}))$ under the canonical restriction map.

In view of the above decomposition, every simple $H(\mathcal{G})$-module is isomorphic to k_γ for a suitably chosen $\gamma \in G(\mathcal{O}(\mathcal{M})) =: \Gamma$.

We let $H(\mathcal{M})$ operate on $H(\mathcal{U})$ via the adjoint representation, and decompose

$$H(\mathcal{U}) = \bigoplus_{\psi \in R} H(\mathcal{U})_\psi$$

into its weight spaces. Thus, $R \subset \Gamma$, and we let $\Psi \subset \Gamma$ be the subgroup generated by R.

Given $\gamma \in \Gamma$, the module $P(\gamma) := H(\mathcal{G}) \otimes_{H(\mathcal{M})} k_\gamma$ is projective. The isomorphism

$$H(\mathcal{G}) \cong H(\mathcal{U}) \# H(\mathcal{M})$$

induces an isomorphism $H(\mathcal{G}) \cong H(\mathcal{U}) \otimes_k H(\mathcal{M})$ of $(H(\mathcal{U}), H(\mathcal{M}))$-bimodules. In particular, $P(\gamma)|_{H(\mathcal{U})} \cong H(\mathcal{U})$, so that $P(\gamma)$ is indecomposable. By the same token, we have

$$P(\gamma)|_{H(\mathcal{M})} \cong \bigoplus_{\psi \in \Psi} H(\mathcal{U})_\psi \otimes_k k_\gamma.$$

Let $\mathcal{B}_\gamma \subset H(\mathcal{G})$ be the block containing k_γ. By the above isomorphism, k_ζ belongs to $\mathcal{B}_\gamma$ if and only if $\zeta \in \Psi * \gamma$.

Consider $\mathcal{M}' := \mathrm{Spec}_k(k[\Gamma/\Psi]) \subset \mathcal{M}$. Since $\mathcal{M}(\mathcal{G})$ lies in the center of $\mathcal{G}$, we have $\mathcal{M}(\mathcal{G}) \subset \mathcal{M}'$. On the other hand, $\mathcal{M}'$ centralizes $H(\mathcal{U})$, and thus belongs to the center of $\mathcal{G}$. This implies $\mathcal{M}' \subset \mathcal{M}(\mathcal{G})$. It follows that $\dim_k H(\mathcal{M}(\mathcal{G})) = [\Gamma : \Psi]$. Consequently, $H(\mathcal{G})$ possesses $\dim_k H(\mathcal{M}(\mathcal{G}))$ blocks, and

$$\dim_k \mathcal{B}_\gamma = \bigoplus_{\psi \in \Psi} \dim_k P(\psi * \gamma) = \dim_k H(\mathcal{U}) \mathrm{ord}(\Psi) = \dim_k H(\mathcal{U}) \frac{\mathrm{ord}(\Gamma)}{[\Gamma : \Psi]} = \dim_k H(\mathcal{G}/\mathcal{M}(\mathcal{G})).$$

It remains to show that $\mathcal{B}_\gamma \cong H(\mathcal{G}/\mathcal{M}(\mathcal{G}))$. We first verify (2).

(2). Since $\mathcal{G}$ is supersolvable, the factor group $\mathcal{G}' := \mathcal{G}/\mathcal{M}(\mathcal{G})$ is trigonalizable with $\mathcal{M}(\mathcal{G}') = e_k$ (cf. [29]). According to what was shown in (1) the algebra $H(\mathcal{G}')$ is connected. It follows that the restriction $\pi : \mathcal{B}_0(\mathcal{G}) \longrightarrow H(\mathcal{G}')$ of the canonical projection maps the primitive central idempotent of $\mathcal{B}_0(\mathcal{G})$ onto the identity. Consequently, π is surjective. Since the ideal $H(\mathcal{G})H(\mathcal{M}(\mathcal{G}))^\dagger$ is generated by central idempotents not belonging to $\mathcal{B}_0(\mathcal{G})$, the map π is also injective, and our assertion follows.

Returning to (1) we observe that the identity $\varepsilon \circ (\gamma * id_{H(\mathcal{G})}) = \gamma$ implies $(\gamma * id_{H(\mathcal{G})})(\mathcal{B}_\gamma) = \mathcal{B}_0(\mathcal{G})$. Accordingly, (2) yields $\mathcal{B}_\gamma \cong \mathcal{B}_0(\mathcal{G}) \cong H(\mathcal{G}/\mathcal{M}(\mathcal{G}))$. $\square$

Theorem 9.5 ([30, 79]) *Suppose that $p \geq 3$. Let $\mathcal{G}$ be a solvable, infinitesimal group, S a simple $H(\mathcal{G})$-module.*

(1) *There exists a subgroup $\mathcal{K} \subset \mathcal{G}$ containing $\mathcal{M}(\mathcal{G})$, and a character $\lambda : H(\mathcal{K}) \longrightarrow k$ such that $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$.*

(2) *If $\mathcal{K}$ is supersolvable, and $\mathcal{M} \subset \mathcal{K}$ is a maximal multiplicative subgroup, then we have $P(S)|_{H(\mathcal{K})} \cong H(\mathcal{K}) \otimes_{H(\mathcal{M})} k_{\lambda\circ\nu}$ for a suitable automorphism $\nu \in \mathrm{Aut}_k(H(\mathcal{K}))$.* $\square$

Let $\mathcal{G}$ be supersolvable. In view of (4.5) and (1.5), the foregoing result shows that the dimensions of the simple modules and their principal indecomposables are powers of p.

We illustrate the above result by providing two consequences:

Corollary 9.6 *Let $\mathcal{G}$ be a nilpotent infinitesimal group of characteristic $p \geq 3$.*

(1) *Every block $\mathcal{B} \subset H(\mathcal{G})$ is primary.*

(2) *$H(\mathcal{G})$ does not admit any tame blocks.*

Proof. (1). Let S be a simple $\mathcal{B}$-module. By (9.5) there exists a subgroup $\mathcal{K} \subset \mathcal{G}$ containing $\mathcal{M}(\mathcal{G})$ such that $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$. Since $\mathcal{K}$ is nilpotent, $\mathcal{M}(\mathcal{G})$ is the unique maximal multiplicative subgroup of $\mathcal{K}$, and (2) of (9.5) now yields $P(S)|_{H(\mathcal{K})} \cong H(\mathcal{K}) \otimes_{H(\mathcal{M}(\mathcal{G}))} k_\lambda$. By the proof of (9.4) this module is indecomposable with k_λ being the only composition factor. Accordingly, each composition factor T of $P(S)$ contains a copy of k_λ, and there thus exists a surjection $S \longrightarrow T$. This implies $S \cong T$, as desired.

(2). By (1) each block of $H(\mathcal{G})$ is Morita equivalent to a local algebra. One can now combine information on the Ext-groups of certain modules (cf. [23]) with Ringel's classification of tame, local algebras (cf. [16]) to obtain (2). $\square$

Corollary 9.7 *Let $\mathcal{G}$ be an infinitesimal group. If $p \geq 3$, then $H(\mathcal{G})$ does not admit any tame local blocks.*

Proof. Let $\mathcal{B} \subset H(\mathcal{G})$ be a local block. Then there exists an algebra homomorphism $\lambda : H(\mathcal{G}) \longrightarrow k$ that sends the primitive central idempotent of $\mathcal{B}$ to 1. As in the proof of (9.4) we have $(\lambda * id_{H(\mathcal{G})})(\mathcal{B}) = \mathcal{B}_0(\mathcal{G})$. Hence $\mathcal{B}_0(\mathcal{G})$ is local, and (9.3) implies that $\mathcal{G}$ is nilpotent. In view of (9.6), the block $\mathcal{B}$ is not tame. $\square$

Remark. All of the above fails at even characteristic. For instance, the restricted enveloping algebra $u(H)$ of the three-dimensional Heisenberg algebra $H := kx \oplus ky \oplus kz$, $x^{[2]} = y^{[2]} = z^{[2]} = 0$ is isomorphic to the group algebra $k[D_4]$ of the dihedral group of order 8. The latter is known to be tame.

10. Infinitesimal Groups of Finite Module Type

In this section we shall investigate two basic, interrelated problems concerning an infinitesimal group $\mathcal{G}$ whose distribution algebra $H(\mathcal{G})$ has finite representation type:

(a) What can be said about the structure of $\mathcal{G}$?

(b) What structure do the blocks of $H(\mathcal{G})$ have?

Note that both problems are understood for finite groups. According to Higman's theorem [43] a finite group G has a representation-finite group algebra if and only if its p-Sylow subgroups are cyclic. Thus, there is not much information on the structure of G. One has detailed information on the structure of the representation-finite blocks of group algebras (cf. [6]).

We begin with the analogue of Maschke's Theorem. The main reason for presenting its proof is to illustrate the impact of certain conditions imposed on rank varieties on the structure of the underlying groups.

Theorem 10.1 *Let $\mathcal{G}$ be an infinitesimal group. The following statements are equivalent:*

(1) *The principal block $\mathcal{B}_0(\mathcal{G})$ is semisimple.*

(2) $\hat{\mathcal{V}}_{\mathcal{G}_1}(k) = \{0\}$.

(3) *$\mathcal{G}$ is diagonalizable.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathcal{B}_0(\mathcal{G})$ is semisimple. Then k is a projective $H(\mathcal{G})$-module, and the conical variety $\hat{\mathcal{V}}_{\mathcal{G}_1}(k)$ has dimension zero. Hence $\{x \in \operatorname{Lie}(\mathcal{G}) \ ; \ x^p = 0\} = \{0\}$, as desired.

(2) $\Rightarrow$ (3). We proceed by induction on the order of $\mathcal{G}$. The assumption implies that for every $x \in \operatorname{Lie}(\mathcal{G})$, the left multiplication

$$\operatorname{ad} x : \operatorname{Lie}(\mathcal{G}) \longrightarrow \operatorname{Lie}(\mathcal{G}) \quad ; \quad y \mapsto [x,y]$$

is diagonalizable. Let $\alpha \neq 0$ be an eigenvalue for $\operatorname{ad} x$. Then there exists $y \in \operatorname{Lie}(\mathcal{G})\setminus\{0\}$ such that $[x,y] = \alpha y$. Accordingly, the subspace $V := kx \oplus ky$ is invariant under $\operatorname{ad} y$, and $\operatorname{ad} y|_V$ is non-trivial and nilpotent, a contradiction. We conclude that 0 is the only eigenvalue of $\operatorname{ad} x$, whence $\operatorname{ad} x = 0$. Consequently, the Lie algebra $\operatorname{Lie}(\mathcal{G})$ is abelian. In particular, the p-map $[p] : \operatorname{Lie}(\mathcal{G}) \longrightarrow \operatorname{Lie}(\mathcal{G})$ is semilinear, and bijective. Let $\{x_1, \ldots, x_n\}$ be a basis of $\operatorname{Lie}(\mathcal{G})$. Then $\{x_1^{[p]^n}, \ldots, x_n^{[p]^n}\}$ is also a basis of $\operatorname{Lie}(\mathcal{G})$, and a two-fold application of (4.3) ensures that $u(\operatorname{Lie}(\mathcal{G}))$ has no nilpotent elements. Thanks to (4.4) the algebra $H(\mathcal{G}_1) \cong u(\operatorname{Lie}(\mathcal{G}))$ is reduced, and (3.2) implies that $\mathcal{G}_1$ is diagonalizable.

The embedding $\mathcal{G}/\mathcal{G}_1 \hookrightarrow \mathcal{G}^{(p)}$ induces an embedding $\operatorname{Lie}(\mathcal{G}/\mathcal{G}_1) \hookrightarrow \operatorname{Lie}(\mathcal{G}^{(p)})$. It follows that the rank variety of $\mathcal{G}/\mathcal{G}_1$ is also trivial. By inductive hypothesis the factor group $\mathcal{G}/\mathcal{G}_1$ is diagonalizable.

Let S be a simple $H(\mathcal{G})$-module. According to Schur's Lemma the central subalgebra $H(\mathcal{G}_1)$ of $H(\mathcal{G})$ operates on S via a character. Consequently, $H(\mathcal{G}_1)$ acts trivially on $\operatorname{End}_k(S)$, and the latter space has the structure of an $H(\mathcal{G}/\mathcal{G}_1)$-module. Since $\mathcal{G}/\mathcal{G}_1$ is multiplicative, $\operatorname{End}_k(S)$ is a semisimple $H(\mathcal{G})$-module, and there results a decomposition

$$\operatorname{End}_k(S) = \bigoplus_{\lambda \in C} \operatorname{End}_k(S)_\lambda,$$

where $C \subset \mathrm{Alg}_k(H(\mathcal{G}), k)$, and $\mathrm{End}_k(S)_\lambda = \{\varphi \in \mathrm{End}_k(S) \ ; \ h \cdot \varphi = \lambda(h)\varphi \ \ \forall \ h \in H(\mathcal{G})\}$. One readily verifies that $\varphi \circ \psi \in \mathrm{End}_k(S)_{\lambda * \gamma} \quad \forall \ \varphi \in \mathrm{End}_k(S)_\lambda, \ \psi \in \mathrm{End}_k(S)_\gamma$.

Let φ be an element of $\mathrm{End}_k(S)_\lambda$. Since $\varphi(hs) = ((\lambda \circ \eta) * id_{H(\mathcal{G})})(h)\varphi(s)$ for $h \in H(\mathcal{G})$ and $s \in S$, we see that $\ker \varphi$ is a submodule of S. Accordingly, every non-zero element of $\mathrm{End}_k(S)_\lambda$ is invertible.

Direct computation shows that $\mathrm{tr}(h \cdot \varphi) = \varepsilon(h)\mathrm{tr}(\varphi)$ for $\varphi \in \mathrm{End}_k(S)$ and $h \in H(\mathcal{G})$. This implies that $\mathrm{tr}(\mathrm{End}_k(S)_\lambda) = (0)$ whenever $\lambda \neq \varepsilon$.

Let $\varphi \in \mathrm{End}_k(S)_\lambda \setminus \{0\}$ for some $\lambda \neq \varepsilon$. Owing to (1.5) and (4.5) the character λ has order p^n for some $n \geq 1$, whence $\varphi^{p^n} \in \mathrm{End}_k(S)_\varepsilon$. By Schur's Lemma the latter space coincides with $k\, id_S$, so that there exists $\alpha \in k$ such that $\varphi^{p^n} = \alpha\, id_S$. Since φ is invertible, we have $\alpha \neq 0$. From the identity

$$\mathrm{tr}(\alpha\, id_S) = \mathrm{tr}(\varphi^{p^n}) = \mathrm{tr}(\varphi)^{p^n} = 0,$$

we conclude that $\mathrm{tr}(\mathrm{End}_k(S)) = \mathrm{tr}(\mathrm{End}_k(S)_\varepsilon) = (0)$, a contradiction. As a result, $\mathrm{End}_k(S) = \mathrm{End}_k(S)_\varepsilon$ is one-dimensional, so that $\dim_k S = 1$.

Since the functor $\mathrm{Hom}_{H(\mathcal{G})}(k, \cdot) \cong \mathrm{Hom}_{H(\mathcal{G}/\mathcal{G}_1)}(k, \cdot) \circ \mathrm{Hom}_{H(\mathcal{G}_1)}(k, \cdot)$ is exact, k is a projective $H(\mathcal{G})$-module. In view of (1.4) this entails that every $H(\mathcal{G})$-module is projective. Consequently, $H(\mathcal{G})$ is semisimple. Now we decompose $\mathcal{O}(\mathcal{G}) = H(\mathcal{G})^*$ into its simple, one-dimensional $H(\mathcal{G})$-constituents and obtain

$$\mathcal{O}(\mathcal{G}) = \bigoplus_{\lambda \in D} \mathcal{O}(\mathcal{G})_\lambda,$$

where $D \subset G(\mathcal{O}(\mathcal{G}))$ and $\mathcal{O}(\mathcal{G})_\lambda = \{x \in \mathcal{O}(\mathcal{G}) \ ; \ h \cdot x = \lambda(h)x \ \ \forall \ h \in H(\mathcal{G})\}$ is the λ-weight space of $\mathcal{O}(\mathcal{G})$. Direct computation shows that $\mathcal{O}(\mathcal{G})_\lambda = k\,(\lambda \circ \eta)$. Hence the Hopf algebra $\mathcal{O}(\mathcal{G})$ is generated by group-like elements, and $\mathcal{G}$ is diagonalizable.

(3) $\Rightarrow$ (1). This was noted earlier. $\square$

According to (10.1), semisimplicity may be detected by studying the support variety of the first Frobenius kernel. The group α_{p^2} shows that the analogous test does not work for representation-finite groups. Nevertheless the structure of groups with rank varieties of dimension ≤ 1 is not arbitrary. More precisely, we have

Proposition 10.2 ([29]) *Let $\mathcal{G}$ be an infinitesimal group. If $\dim \hat{\mathcal{V}}_{\mathcal{G}_1}(k) \leq 1$, then $\mathcal{G}$ is supersolvable.* $\square$

The main problem is that we do not have an anologue of (9.1) for supersolvable groups. Hence one needs a rather detailed analysis for groups of height ≤ 1. This can be accomplished either by homological methods (cf. [21]), or via schemes of tori. An example illustrating the latter approach can be found in §13.

Proposition 10.2 allows us to bring our knowledge of solvable groups to bear: If the principal block $\mathcal{B}_0(\mathcal{G})$ is representation-finite, then a consecutive application of (6.3), (7.1), and (10.2) yields the supersolvability of $\mathcal{G}$. By general theory, the factor group $\mathcal{G}/\mathcal{M}(\mathcal{G})$ of such a group is trigonalizable, and thus decomposes into a semidirect product

$$\mathcal{G} \cong \mathcal{U} \rtimes \mathcal{M},$$

with a normal unipotent subgroup $\mathcal{U}$ and a multiplicative subgroup $\mathcal{M}$. One can show that $\mathcal{B}_0(\mathcal{G}) \cong \mathcal{B}_0(\mathcal{G}/\mathcal{M}(\mathcal{G}))$. Moreover, the latter algebra is representation-finite if and only if $H(\mathcal{U})$ enjoys this property. However, $H(\mathcal{U})$ is local of dimension a p-power. Thus, this algebra is of finite representation type if and only if it is isomorphic to a truncated polynomial ring $k[X]/(X^{p^n})$. In particular, $H(\mathcal{U})$ is a Nakayama algebra, and $\mathcal{U}$ is commutative.

The commutative unipotent groups that give rise to Nakayama algebras are the so-called *V-uniserial* groups. Here the prefix "*V*" refers to the *Verschiebung* $V_{\mathcal{U}} : \mathcal{U}^{(p)} \longrightarrow \mathcal{U}$, a homomorphism that is the dual of the Frobenius homomorphism of the *Cartier dual* $\mathcal{D}(\mathcal{U})$ of $\mathcal{U}$. In the above we have sketched the implication (1) $\Rightarrow$ (2) of the following result.

Theorem 10.3 ([29, 33]) *Let $\mathcal{G}$ be an infinitesimal k-group. Then the following statements are equivalent:*

(1) *$\mathcal{B}_0(\mathcal{G})$ has finite representation type.*
(2) *$\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong \mathcal{U} \rtimes \mu_{p^n}$ is a semidirect product with a V-uniserial normal subgroup $\mathcal{U}$.*
(3) *$H(\mathcal{G})$ is a Nakayama algebra.*
(4) $\dim \hat{\mathcal{V}}_{\mathcal{G}_2}(k) \leq 1$.
(5) *$H(\mathcal{G}_2)$ is a Nakayama algebra.* $\square$

In particular, finite representation type may be detected on the second Frobenius kernel of an infinitesimal group.

Remarks. (i). Special cases of (10.3) can be found in [62, 34, 21].

(ii). The equivalence (1) $\Leftrightarrow$ (4) readily shows that subgroups of representation-finite groups are representation-finite. For finite groups this fact follows directly from Mackey's theorem.

(iii). Let G be a finite group. The example of the quaternion group shows that the analogue of (1) $\Leftrightarrow$ (4) fails in this context. Moreover, representation-finite groups are not necessarily Nakayama algebras. Since the defect group of the principal block of $k[G]$ is a Sylow-p-subgroup, $k[G]$ is representation-finite whenever $\mathcal{B}_0(G)$ has this property.

We now turn to the block structure of $H(\mathcal{G})$. According to (10.3) each block $\mathcal{B}$ of a representation-finite infinitesimal group $\mathcal{G}$ is a self-injective Nakayama algebra. In general, indecomposable Nakayama algebras are determined by their *Kupisch series*, that is, by the number of simple modules, and the lengths of their projective covers (cf. [53]).

Given a finite-dimensional k-algebra Λ, we let $\mathcal{S}_\Lambda$ be the set of isoclasses of the simple Λ-modules. The automorphism group $\mathrm{Aut}_k(\Lambda)$ operates on $\mathcal{S}_\Lambda$ by twisting: For a Λ-module M and an automorphism $\varphi \in \mathrm{Aut}_k(\Lambda)$ we denote by $M^{(\varphi)}$ the Λ-module with underlying k-space M and action given by

$$a \cdot m := \varphi^{-1}(a)m \quad \forall\, a \in \Lambda,\ m \in M.$$

Thus, the algebraic group $\mathrm{Aut}_k(\Lambda)$ operates on $\mathcal{S}_\Lambda$ with its connected component acting trivially.

Let $\mathcal{G}$ be an infinitesimal group. Consider the subgroup $C_\nu \subset \mathrm{Aut}_k(H(\mathcal{G}))$ that is generated by a Nakayama automorphism of ν of the Frobenius algebra $H(\mathcal{G})$. The general theory of Frobenius algebras shows that two Nakayama automorphisms of $H(\mathcal{G})$ differ only by an inner automorphism [59]. Consequently, the operation of C_ν on $\mathcal{S}_{H(\mathcal{G})}$ does not depend on the choice of ν. Moreover, any Nakayama automorphism restricts to an automorphism of each block $\mathcal{B}$ of $H(\mathcal{G})$, so that C_ν also acts on $\mathcal{S}_{\mathcal{B}}$.

For positive integers $a,\, b \in \mathbb{N}$ we consider the algebra $\Lambda(a,b) := k[\tilde{A}_{a-1}]/I_b$ given by the quiver $\tilde{A}_{a-1} := \mathbb{Z}/(a)$ with arrows $x_i := i \to i+1$ and relations defined by the ideal I_b generated by all paths of length b.

Theorem 10.4 ([30]) *Let $\mathcal{G}$ be an infinitesimal group of finite representation type, $\mathcal{B} \subset H(\mathcal{G})$ a block. Then the following statements hold:*

(1) $\mathcal{S}_{\mathcal{B}} = C_\nu \cdot [S] \quad \forall\ [S] \in \mathcal{S}_{\mathcal{B}}$.

(2) *The block $\mathcal{B}$ has dimension a p-power and is either primary or basic.*

(3) *There exist $s, \ell \in \mathbb{N}_0$ such that $\mathcal{B}$ is Morita equivalent to $\Lambda(p^s, p^\ell)$.* $\square$

Remarks. (1). The parameters s, ℓ of (3) can be described as orders of certain subgroups of $\mathcal{G}$ (see [30] for the details).

(2). Part (1) of (10.4) does not hold for finite groups. For $p = 3$ the group algebra of the symmetric group S_3 on 3 letters has finite representation type. Its Sylow-3-subgroup is normal and not central, so that the inertial index of the principal block $\mathcal{B}_0(S_3)$ of the group algebra $k[S_3]$ is 2. According to [6, (6.5.4)], $\mathcal{B}_0(S_3)$ is a symmetric Nakayama algebra with 2 simple modules.

(3). Thanks to [78, (I.2.6)] the blocks of the group algebras of supersolvable infinitesimal groups are full matrix rings over their basic algebras. Parts (2) and (3) therefore imply that a block $\mathcal{B} \subset H(\mathcal{G})$ is either isomorphic to $\mathrm{Mat}_m(k[X]/(X^{p^\ell}))$ or to $\Lambda(p^s, p^\ell)$.

(4). Let $\mathcal{G}$ be an infinitesimal group of height ≤ 1. According to [39] the cohomology ring $H^{\mathrm{ev}}(\mathcal{G}, k)$ is generated in degree 2. This implies that all representation-finite blocks of $H(\mathcal{G})$ are Nakayama algebras (cf. [21]). Since this also holds for distribution algebras of supersolvable infinitesimal groups (cf. [26]), this conceivably is a feature of arbitrary infinitesimal group schemes.

11. Frobenius Kernels of Smooth Group Schemes

Throughout this section, we fix a reduced (=smooth) affine group scheme $\mathcal{G}$. We will illustrate how rank varieties and nilpotent orbits can be employed to classify the representation-finite and tame blocks of the Frobenius kernels of reductive groups. We begin by looking at an important example.

Example. Let $\mathcal{G} := \mathrm{SL}(2)_1$ be the first Frobenius kernel of SL(2). According to (4.4) the algebra $H(\mathcal{G})$ is isomorphic to $u(s\ell(2))$, the restricted enveloping algebra of the Lie algebra of (2×2)-matrices of trace zero.

The block structure of $u(s\ell(2))$ was determined by Pollack [63]. For $p \geq 3$, the algebra $u(s\ell(2))$ possesses exactly $\frac{p+1}{2}$ blocks. There is one simple block, corresponding to the p-dimensional *Steinberg module*, and $\ell := \frac{p-1}{2}$ blocks $\mathcal{B}_0, \ldots, \mathcal{B}_{\ell-1}$ with $\mathcal{B}_i$ having two simple modules of dimensions $i+1$ and $p-i-1$. Pollack also proved that $u(s\ell(2))$ has infinite representation type, which of course also follows from the fact that $\hat{\mathcal{V}}_{\mathcal{G}_1}(k)$ has dimension 2 (see §7).

About 25 years later, Fischer [35] determined the basic algebra of each non-simple block. It is given by the quiver

$$\begin{tikzcd} \bullet \arrow[r, shift left=4.5, "x_1"] \arrow[r, shift left=1.5, "y_1"] & \bullet \arrow[l, shift left=1.5, "x_2"] \arrow[l, shift left=4.5, "y_2"] \end{tikzcd}$$

and relations $x_1y_2 = y_1x_2$, $y_2x_1 = x_2y_1$, $x_1x_2 = 0$, $y_1y_2 = 0$, $x_2x_1 = 0$, $y_2y_1 = 0$. In particular, all non-simple blocks are Morita equivalent. In addition, each of these blocks is special biserial and of domestic representation type.

By work of Pfautsch [61] every tame block of $H(\mathrm{SL}(2)_r)$ is Morita equivalent to a tame block of $H(\mathrm{SL}(2)_1) \cong u(s\ell(2))$.

Definition. An algebraic group $\mathcal{G}$ is said to be *reductive* if it contains no nontrivial unipotent normal subgroups.

The classical groups $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{SO}(n)$, and $\mathrm{Sp}(2n)$ are smooth reductive groups. For the remainder of this section we assume that $p \geq 5$.

Theorem 11.1 ([25]) *Let $\mathcal{G}$ be a smooth, connected, reductive group, $\mathcal{B} \subset H(\mathcal{G}_r)$ a block.*
(1) *If $\mathcal{B}$ is representation-finite, then $\mathcal{B}$ is simple.*
(2) *If $\mathcal{B}$ is tame, then $\mathcal{B}$ is Morita equivalent to a tame block of $H(\mathrm{SL}(2)_1)$.*

Proof. Since $\mathcal{G}_r$ is a normal subgroup of $\mathcal{G}$, the subalgebra $H(\mathcal{G}_r) \subset \mathrm{Dist}(\mathcal{G})$ is stable under the adjoint operation of $\mathrm{Dist}(\mathcal{G})$. Consequently, $\mathcal{G}(k) \subset \mathrm{Dist}(\mathcal{G})$ operates on $H(\mathcal{G}_r)$ via the adjoint representation $\mathrm{Ad} : \mathcal{G}(k) \longrightarrow \mathrm{Aut}_k(H(\mathcal{G}_r))$. Let S be a simple $\mathcal{B}$-module. For $g \in \mathcal{G}(k)$ we put $S^{(g)} := S^{(\mathrm{Ad}(g))}$. Direct computation shows that

$$\hat{\mathcal{V}}_{\mathcal{G}_r}(S^{(g)}) = g \cdot \hat{\mathcal{V}}_{\mathcal{G}_r}(S) \qquad \forall\ g \in \mathcal{G}(k),$$

where $g \cdot \varphi := \mathrm{Ad}(g) \circ \varphi$ for $g \in \mathcal{G}(k)$, and $\varphi \in \hat{\mathcal{V}}_{\mathcal{G}_r}(k)$. Since S is simple and $\mathcal{G}(k)$ is connected, we have $S \cong S^{(g)} \ \ \forall\ g \in \mathcal{G}(k)$, so that $\hat{\mathcal{V}}_{\mathcal{G}_r}(S)$ is invariant under the adjoint action.

We first show that $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(S) \neq 1$. Otherwise, each irreducible component of the conical variety $\hat{\mathcal{V}}_{\mathcal{G}_r}(S)$ is a $\mathcal{G}(k)$-stable line, and $\mathcal{G}_r$ contains a normal subgroup isomorphic to α_{p^s} for some $s \in \{1, \ldots, r\}$. Consequently, $\mathrm{Lie}(\mathcal{G})$ contains a line $kx \neq (0)$ with $x^{[p]} = 0$ that is invariant under the adjoint representation. This contradicts the fact that $\mathcal{G}$ is reductive.

(1). If $\mathcal{B}$ has finite representation type, then $\hat{\mathcal{V}}_{\mathcal{G}_r}(S)$ has dimension ≤ 1 (cf. (6.3), (7.1)). By what we have just seen, this implies $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(S) = 0$. Consequently, S is projective, and $\mathcal{B}$ is simple.

(2). By (6.3), (7.1) and our observation above there exists a simple $\mathcal{B}$-module S such that $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(S) = 2$. Since $\hat{\mathcal{V}}_{\mathcal{G}_r}(S)$ is conical, Borel's Fixed Point Theorem provides a point $[x_0] \in \operatorname{Proj}(\hat{\mathcal{V}}_{\mathcal{G}_r}(S))$ whose stabilizer is a parabolic subgroup $\mathcal{P} \subset \mathcal{G}$. Thus, $\dim \mathcal{G}(k)/\mathcal{P}(k) \leq 1$, and if $\mathcal{G}(k) = \mathcal{P}(k)$ then the above reasoning yields a contradiction. Alternatively, one can show that there exist closed normal subgroups $\mathcal{H}, \mathcal{K} \subset \mathcal{G}$ such that

(a) $\mathcal{G} = \mathcal{H} \cdot \mathcal{K}$, and $\operatorname{Lie}(\mathcal{G}) = \operatorname{Lie}(\mathcal{H}) \oplus \operatorname{Lie}(\mathcal{K})$,

(b) $\mathcal{K} \cong \mathrm{SL}(2)$ or $\mathcal{K} \cong \mathrm{PSL}(2)$, and $\mathcal{H} \subset \mathcal{P}$.

Since $\mathrm{SL}(2)_r \cong \mathrm{PSL}(2)_r$ for every $r \geq 1$, we obtain an isomorphism

$$\mathcal{G}_r \cong \mathrm{SL}(2)_r \times \mathcal{H}_r.$$

Consequently, there are blocks $\mathcal{B}_1 \subset H(\mathrm{SL}(2)_r)$ and $\mathcal{B}_2 \subset H(\mathcal{H}_r)$ such that $\mathcal{B} \cong \mathcal{B}_1 \otimes_k \mathcal{B}_2$. By the same token, $S \cong S_1 \otimes_k S_2$, where S_i is a simple $\mathcal{B}_i$-module. By general properties of varieties, we obtain

$$\hat{\mathcal{V}}_{\mathcal{G}_r}(S) = \hat{\mathcal{V}}_{\mathrm{SL}(2)_r}(S_1) \times \hat{\mathcal{V}}_{\mathcal{H}_r}(S_2).$$

If $\dim \hat{\mathcal{V}}_{\mathrm{SL}(2)_r}(S_1) = 0$, then $\hat{\mathcal{V}}_{\mathcal{G}_r}(S) = \mathcal{V}_{\mathcal{H}_r}(S_2)$ and $\mathcal{G}(k)$ stabilizes the point $[x_0]$, a contradiction. Thus, $\dim \hat{\mathcal{V}}_{\mathrm{SL}(2)_r}(S_1) = 2$, and $\dim \hat{\mathcal{V}}_{\mathcal{H}_r}(S_2) = 0$. Consequently, S_2 is projective and $\mathcal{B}_2$ is a matrix ring over k. There results an isomorphism $\mathcal{B} \cong \operatorname{Mat}_n(\mathcal{B}_1)$. In particular, $\mathcal{B}$ is Morita equivalent to a tame block of $H(\mathrm{SL}(2)_r)$. We may now apply Pfautsch's result [61] to obtain the assertion. $\square$

If $\mathcal{B} = \mathcal{B}_0(\mathcal{G}_r)$ is the principal block, and $S = k$ is the trivial module, then the group $\mathcal{H}_r$ is multiplicative. Since $\dim \hat{V}_{\mathrm{SL}(2)_2}(k) = 3$, it follows that $r = 1$. Consequently, we obtain

Corollary 11.2 *Let $\mathcal{G}$ be a smooth, connected, reductive group. Then $\mathcal{B}_0(\mathcal{G}_r)$ is tame if and only if $r = 1$, and there exists a multiplicative infinitesimal group $\mathcal{M}$ such that $\mathcal{G}_1 \cong \mathrm{SL}(2)_1 \times \mathcal{M}$.* $\square$

12. Algebraic Families of Vector Spaces

In this section we collect a few basic results on algebraic families of modules that we require for the application of schemes of tori. Throughout, we let A be a finitely generated integral domain over k with associated scheme $\mathcal{X} := \operatorname{Spec}_k(A)$. Given a finite-dimensional k-vector space V, we consider the free A-module $V \otimes_k A$. For an A-submodule $P \subset V \otimes_k A$, and $x \in \mathcal{X}(k)$ we denote by $P(x) := (id_V \otimes x)(P) \subset V$ the subspace of V obtained by specialization along x. If P is a direct summand of the A-module $V \otimes_k A$, then $P(x) \cong P \otimes_A k_x$, where k_x denotes the one-dimensional A-module afforded by x. We will be studying the algebraic family $(P(x))_{x \in \mathcal{X}(k)}$ of subspaces of V.

Proposition 12.1 *Let $Y \subset V$ be a conical variety, and suppose that $P \subset V \otimes_k A$ is an A-direct summand. Then the function $\mathcal{X}(k) \longrightarrow \mathbb{N}_0$; $x \mapsto \dim P(x) \cap Y$ is upper semicontinuous.*

Proof. Consider the variety $\mathcal{X}(k) \times \mathrm{Proj}(Y)$ as well as the subset

$$Z := \{(x, [y]) \in \mathcal{X}(k) \times \mathrm{Proj}(Y) \ ; \ \exists \ p \in P \text{ such that } (id_L \otimes x)(p) = y\}.$$

One proceeds by showing that Z is a closed subset of $\mathcal{X}(k) \times \mathrm{Proj}(Y)$.

Taking this for granted, we consider the restriction $\pi : Z \longrightarrow \mathcal{X}(k)$ of the projection onto the first factor. Then we have $\pi^{-1}(x) \cong \mathrm{Proj}(P(x) \cap Y)$ for every $x \in \mathcal{X}(k)$. Thanks to [15, §14] the set

$$\mathcal{X}(k)_d := \{x \in \mathcal{X}(k) \ ; \ \dim \mathrm{Proj}(P(x) \cap Y) \geq d\}$$

is closed for any $d \geq 0$. Since $\dim \mathrm{Proj}(P(x) \cap Y) = \dim P(x) \cap Y - 1$, we see that $x \mapsto \dim P(x) \cap Y$ is upper semicontinuous. $\square$

Corollary 12.2 *Let $Y \subset V$ be conical, $P \subset V \otimes_k A$ an A-direct summand. If $d := \min\{\dim P(x) \cap Y \ ; \ x \in \mathcal{X}(k)\}$, then there exists a dense open subset $U \subset \mathcal{X}(k)$ such that $\dim P(x) \cap Y = d$ for every $x \in U$.* $\square$

Corollary 12.3 *Suppose that $P \subset V \otimes_k A$ is an A-direct summand.*

(1) *The function $\mathcal{X}(k) \longrightarrow \mathbb{N}_0 \ ; \ x \mapsto \dim_k P(x)$ is constant on $\mathcal{X}(k)$.*

(2) *If $P \neq (0)$, then $P(x) \neq (0)$ for every $x \in \mathcal{X}(k)$.*

Proof. (1). By assumption there exists an A-submodule $Q \subset V \otimes_k A$ such that $V \otimes_k A = P \oplus Q$. Consequently, $V = P(x) \oplus Q(x)$ for every $x \in \mathcal{X}(k)$. Let $d_P := \min\{\dim_k P(x) \ ; \ x \in \mathcal{X}(k)\}$ and $d_Q := \min\{\dim_k Q(x) \ ; \ x \in \mathcal{X}(k)\}$. Since $\mathcal{X}(k)$ is irreducible, (12.2) yields $d_P + d_Q = \dim_k V$. Consequently, $d_P \leq \dim_k P(x) = \dim_k V - \dim_k Q(x) \leq \dim_k V - d_Q = d_P$ for an arbitrary $x \in \mathcal{X}(k)$.

(2). Suppose there is $x_0 \in \mathcal{X}(k)$ such that $P(x_0) = (0)$. According to (1) we then have $P(x) = (0)$ for every $x \in \mathcal{X}(k)$. Let $\{v_1, \dots, v_n\}$ be a basis of V. Given $p \in P$ we write $p = \sum_{i=1}^n v_i \otimes a_i$. Since

$$0 = (id_V \otimes x)(p) = \sum_{i=1}^n x(a_i) v_i = 0 \qquad \forall \ x \in \mathcal{X}(k)$$

we see that the zero locus $Z(I)$ of the ideal $I := (\{a_1, \dots, a_n\}) \subset A$ is all of $\mathcal{X}(k)$. Hilbert's Nullstellensatz now yields $I = (0)$, whence $p = 0$. Consequently, $P = (0)$, a contradiction. $\square$

Given an A-direct summand $P \subset V \otimes_k A$, and a subset $U \subset \mathcal{X}(k)$, we put $I_U(P) := \bigcap_{x \in U} P(x)$.

Definition. Let V be a finite dimensional vector space, $\mathcal{F}$ a finite set of irreducible subvarieties of V. A family $(W_i)_{i \in I}$ of subspaces of V is said to be *$\mathcal{F}$-regular* if every subspace W_i contains an element of $\mathcal{F}$.

Lemma 12.4 *Let $\mathcal{F}$ be a finite set of irreducible subvarieties of the finite-dimensional vector space V. If $P \subset V \otimes_k A$ is an A-submodule such that*

(a) *P is an A-direct summand of $V \otimes_k A$, and*

(b) *there exists a dense subset $U \subset \mathcal{X}(k)$ such that $(P(x))_{x \in U}$ is $\mathcal{F}$-regular,*

then the subspace $I_{\mathcal{X}(k)}(P) \subset V$ is $\mathcal{F}$-regular.

Proof. We write $\mathcal{F} = \{Y_1, \dots, Y_n\}$. In view of (a) there exists an A-submodule $Q \subset V \otimes_k A$ such that $V \otimes_k A = P \oplus Q$. Note that this implies $V = P(x) \oplus Q(x)$ for every $x \in \mathcal{X}(k)$. We consider an element $y \in V \otimes_k k \cdot 1$ and write $y = p + q$ with $p \in P$ and $q \in Q$. Then $y \in P(x)$ if and only if $(id_V \otimes x)(q) = 0$. By choosing a basis $\{v_1, \dots, v_m\}$ of V and writing $q = \sum_{i=1}^m v_i \otimes a_i$, we see that the latter condition is equivalent to $x(a_i) = 0$ for $1 \leq i \leq m$. Consequently, the set $Z[y] := \{x \in \mathcal{X}(k) \ ; \ y \in P(x)\}$ is closed, and $Z_j := \{x \in \mathcal{X}(k) \ ; \ Y_j \subset P(x)\} = \bigcap_{y \in Y_j} Z[y]$ has the same property. Condition (b) implies $\mathcal{X}(k) = \bar{U} \subset \bigcup_{j=1}^n Z_j$. Since A is an integral domain, the variety $\mathcal{X}(k)$ is irreducible, and there exists $j_0 \in \{1, \dots, n\}$ such that $Z_{j_0} = \mathcal{X}(k)$. It follows that $Y_{j_0} \subset I_{\mathcal{X}(k)}(P)$. $\square$

Let $\mathcal{G}$ be an affine algebraic k-group, V a $\mathcal{G}$-module. We assume that $\mathcal{G}$ acts on A via algebra homomorphisms. Consider the $\mathcal{G}$-module $V \otimes_k A$ with the diagonal operation

$$g \cdot [(v \otimes_k r) \otimes_R (a \otimes_k s)] := g(v \otimes_k r) \otimes_R g(a \otimes_k s)$$

for $g \in \mathcal{G}(R)$, $r, s \in R$, $v \in V$, $a \in A$.

Lemma 12.5 *Let $P \subset V \otimes_k A$ be a $\mathcal{G}$-stable A-submodule that is also an A-direct summand of $V \otimes_k A$. If $U \subset \mathcal{X}(k)$ is dense, then $I_U(P) \subset V$ is a $\mathcal{G}$-submodule.*

Proof. Let $\mathcal{O}(\mathcal{G})$ be the function algebra of $\mathcal{G}$. Since $V \cong V \otimes_k k \cdot 1$ is $\mathcal{G}$-invariant and k is a field, V is an $\mathcal{O}(\mathcal{G})$-subcomodule of $V \otimes_k A$. By the same token, $V \cap P \subset V \otimes_k A$ is, as a subcomodule of $V \otimes_k A$, also $\mathcal{G}$-invariant. We will conclude the proof by showing that

$$V \cap P = I_U(P).$$

Observing the obvious inclusion $V \cap P \subset I_U(P)$, we proceed by showing that $I_U(P) \subset V \cap P$. By assumption there exists an A-module Q such that $V \otimes_k A = P \oplus Q$. Recall that this implies $V = P(x) \oplus Q(x)$ for every $x \in \mathcal{X}(k)$.

Now let v be an element of $I_U(P)$, and write $v = p + q$ with $p \in P$ and $q \in Q$. It follows that $v = (id_V \otimes x)(p) + (id_V \otimes x)(q) \ \ \forall \ x \in \mathcal{X}(k)$. As v belongs to $I_U(P)$, the element q vanishes after specialization along $x \in U$. Letting $(v_j)_{j \in J}$ be a basis of V, we write $q = \sum_{j \in J} v_j \otimes a_j$. By the above, the ideal $S := \sum_{j \in J} A a_j \subset A$ is annihilated by every $x \in U$. Thus, U is contained in the zero locus $Z(S)$ of S. Since U is a dense subset of $\mathcal{X}(k)$, it follows from Hilbert's Nullstellensatz that $S = (0)$. Consequently, $q = 0$ and v belongs to $V \cap P$. $\square$

13. Schemes of Tori

We have seen how the determination of the representation type of infinitesimal groups leads to conditions on rank varieties. This leaves us with the problem of interpreting the ramifications of these conditions for the structure of the underlying groups. For Frobenius kernels of smooth groups our knowledge of nilpotent orbits suffices to reduce the problem to the study of SL(2), a group whose representation theory is well enough understood to provide us with complete answers. Schemes of tori and their associated algebraic families of Lie algebras help us deal with this problem for arbitrary infinitesimal groups.

Recall that k is an algebraically closed field of characteristic $p > 0$. Throughout this section we will only consider finite-dimensional restricted Lie algebras. Such a Lie algebra $(T, [p])$ is called a *torus* if and only of $\hat{\mathcal{V}}_T := \{t \in T \; ; \; t^{[p]} = 0\} = \{0\}$. Tori are necessarily abelian and linearly reductive: every $u(T)$-module is completely reducible.

If $L = \mathrm{Lie}(\mathcal{G})$ is the Lie algebra of a smooth group, then much of the structure of L can be detected via the so-called *root space decomposition*. One picks a maximal torus $T \subset L$ and decomposes L into its eigenspaces relative to T:

$$L = C_L(T) \oplus \bigoplus_{\alpha \in R} L_\alpha \quad ; \quad R \subset T^* \setminus \{0\}.$$

Here $C_L(T) := \{x \in L \; ; \; [t,x] = 0 \;\; \forall \; t \in T\}$ is the *centralizer of T in L*, and $L_\alpha := \{x \in L \; ; \; [t,x] = \alpha(t)x \;\; \forall \; t \in T\} \neq (0)$ is the *root space* for the *root* $\alpha \in R$. Since $\mathcal{G}$ is smooth, any two maximal tori are conjugate under the adjoint representation, so it doesn't really matter which maximal torus we take. The following example shows that this is no longer true for arbitrary restricted Lie algebras.

Example. For $p \geq 5$ we consider the *Witt algebra* $W(1) := \mathrm{Der}_k(k[X]/(X^p))$ of the derivations of the truncated polynomial ring $k[X]/(X^p)$. Since the p-th power of a derivation is again a derivation, $(W(1), p)$ is a restricted Lie algebra. Let ∂ be the derivation induced by $\frac{d}{dX}$ and set $x := X + (X^p)$ as well as $e_i := x^{i+1}\partial$ for $-1 \leq i \leq p-2$. Then $\{e_{-1}, \dots, e_{p-2}\}$ is a basis of $W(1)$, and we have

$$[e_i, e_j] = (j-i)e_{i+j} \quad ; \quad e_i^p = \delta_{i,0}e_0,$$

where the product is understood to be zero whenever $i+j$ does not lie within $\{-1, \dots, p-2\}$. It follows that $T := ke_0$ is a maximal torus of $W(1)$.

Now we define $f_i := (x+1)^{i+1}\partial$ for $-1 \leq i \leq p-2$. Then $f_0^p = f_0$ and $[f_i, f_j] = (j-i)f_{i+j}$. In particular, $T' := kf_0$ is another maximal torus of $W(1)$. However, now the subscripts have to be interpreted $\mathrm{mod}(p)$, e.g., $[f_1, f_{p-2}] = -3f_{-1}$. Thus, while the root space decomposition relative to T induces a $\mathbb{Z}$-grading, we have a grading with respect to the group $\mathbb{Z}/(p)$ in the latter case. The $\mathbb{Z}$-grading is better to work with because we can for instance read off that $\mathrm{ad}\, e_1$ is a nilpotent transformation.

These observations already indicate that T and T' are really different. In fact, they cannot be mapped onto each other by any automorphism of $W(1)$. Direct computation shows that $W(1)_{(0)} := \sum_{i=0}^{p-2} k\, e_i$ is the unique p-subalgebra of codimension 1 (here we need $p \geq 5$). Hence it is fixed by any automorphism $\varphi \in \mathrm{Aut}_k(W(1))$, and $\varphi(T) \subset W(1)_{(0)}$. In particular, $\varphi(T) \neq T'$.

The foregoing example illustrates our predicament. We have to choose a maximal torus without knowing which choice is good for our purposes. Schemes of tori obviate this difficulty by simultaneously studying all tori of a certain isomorphism type.

Let $(L, [p])$ be a restricted Lie algebra over k, R a commutative k-algebra. Recall that $L \otimes_k R$ obtains the structure of a restricted Lie algebra over R via

$$[x \otimes r, y \otimes s] := [x, y] \otimes rs \quad ; \quad (x \otimes r)^{[p]} = x^{[p]} \otimes r^p \quad \forall\, x, y \in L,\ r, s \in R.$$

Now let (T, L) be a pair of restricted Lie algebras over k. We consider the k-functor $\mathcal{T}_L : M_k \longrightarrow \mathrm{Ens}$ that associates to each commutative k-algebra R the set $\mathcal{T}_L(R)$ of those homomorphisms $\varphi : T \otimes_k R \longrightarrow L \otimes_k R$ of restricted Lie algebras over R that are split injective. Observe that the set $\mathcal{T}_L(k)$ of rational points is just the set of embeddings $T \hookrightarrow L$.

Theorem 13.1 ([31]) *Let T be a torus.*

(1) *$\mathcal{T}_L$ is a smooth, affine, algebraic scheme.*

(2) *If $\mathcal{X} \subset \mathcal{T}_L$ is an irreducible component, then*

$$\dim \mathcal{X} = \dim_k L - \dim_k C_L(\varphi(T)) \quad \forall\, \varphi \in \mathcal{X}(k). \qquad \square$$

One main point of (1) is that the connected components of $\mathcal{T}_L$ coincide with the irreducible components. Thus, if $T \subset L$ is a torus with the embedding $T \hookrightarrow L$ corresponding to a rational point $x_0 \in \mathcal{T}_L(k)$, then there exists exactly one irreducible component $\mathcal{X}_T \subset \mathcal{T}_L$ such that $x_0 \in \mathcal{X}_T(k)$.

If T is not a torus, then $\mathcal{T}_L$ may not even be reduced:

Example. Let $T := kt,\ t^{[p]} = 0$ be the one-dimensional strongly abelian Lie algebra, and consider the abelian p-unipotent Lie algebra $L := ka \oplus ka^{[p]},\ a^{[p]^2} = 0$. Then the natural transformation $\psi : \mathrm{Spec}_k(k[X, Y]/(X^p)) \longrightarrow \mathcal{T}_L$, sending $h \in \mathrm{Spec}_k(k[X, Y]/(X^p))(R)$ to the homomorphism $\psi_R(h) \in \mathcal{T}_L(R)$ that is given by $\psi_R(h)(t) := a \otimes h(\bar{X}) + a^{[p]} \otimes h(\bar{Y})$, is an isomorphism. Accordingly, $\mathcal{T}_L$ is not reduced, and thus not smooth.

We illustrate our result by relating it to the conjugacy of tori. Let $\mathcal{G}$ be an affine algebraic group, $\mathrm{Ad} : \mathcal{G}(k) \longrightarrow \mathrm{Aut}_k(\mathrm{Lie}(\mathcal{G}))$ its adjoint representation. Then $\mathcal{G}(k)$ operates on the affine variety $\mathcal{T}_L(k)$ via

$$g \cdot \varphi := \mathrm{Ad}(g) \circ \varphi \quad \forall\, g \in \mathcal{G}(k),\ \varphi \in \mathcal{T}_L(k).$$

Proposition 13.2 *Let $L = \mathrm{Lie}(\mathcal{G})$ be the Lie algebra of a smooth, connected, affine algebraic group, T a torus. Then the connected components of $\mathcal{T}_L$ are the $\mathcal{G}(k)$-orbits of $\mathcal{T}_L(k)$.*

Proof. Let $\mathcal{X} \subset \mathcal{T}_L$ be a connected component, $\varphi \in \mathcal{X}(k)$ a rational point. Since the orbit $\mathcal{G}(k) \cdot \varphi$ is connected, and $\mathcal{X}(k)$ is a connected component of $\mathcal{T}_L(k)$, we have $g \cdot \varphi \in \mathcal{X}(k)$ for every $g \in \mathcal{G}(k)$. Note that the stabilizer $\mathrm{Stab}_{\mathcal{G}(k)}(\varphi)$ is given by

$$\mathrm{Stab}_{\mathcal{G}(k)}(\varphi) = \{g \in \mathcal{G}(k)\ ;\ \mathrm{Ad}(g)(\varphi(t)) = \varphi(t) \quad \forall\, t \in T\}.$$

Thus, $\mathrm{Lie}(\mathrm{Stab}_{\mathcal{G}(k)}(\varphi)) \subset C_L(\varphi(T))$, and (13.1) yields

$$\dim \overline{\mathcal{G}(k) \cdot \varphi} = \dim \mathcal{G}(k) - \dim \mathrm{Stab}_{\mathcal{G}(k)}(\varphi) \geq \dim_k L - \dim_k C_L(\varphi(T)) = \dim \mathcal{X}(k).$$

Hence every orbit of $\mathcal{X}(k)$ lies dense in $\mathcal{X}(k)$. Consequently, every orbit is closed, and $\mathcal{X}(k) = \mathcal{G}(k) \cdot \varphi$. $\square$

Remarks. (1). Let $T \overset{x_0}{\hookrightarrow} L$ be the canonical embedding. Under the assumptions of (13.2) the morphism $g \mapsto \mathrm{Ad}(g) \circ x_0$ induces a bijective $\mathcal{G}(k)$-equivariant map $\mathcal{G}(k)/\mathrm{Stab}_{\mathcal{G}(k)}(x_0) \longrightarrow \mathcal{X}_T(k)$ of homogeneous spaces. Owing to [31, (1.4)] its differential $T_{\bar{1}}(\mathcal{G}(k)/\mathrm{Stab}_{\mathcal{G}(k)}(x_0)) \longrightarrow T_{x_0}(\mathcal{X}_T)$ is given by the bijection

$$L/C_L(T) \longrightarrow \mathrm{Der}_k(T, L) \ \ ; \ \ [x] \mapsto adx|_T.$$

Consequently, we have an isomorphism $\mathcal{G}(k)/C_{\mathcal{G}(k)}(T) \cong \mathcal{X}_T(k)$. Here $C_{\mathcal{G}(k)}(T) = \{g \in \mathcal{G}(k) \ ; \ \mathrm{Ad}(g)(t) = t \ \ \forall\, t \in T\}$ is the centralizer of T in $\mathcal{G}(k)$.

(2). If T, $T' \subset L$ are maximal tori, then Borel's fixed point theorem implies that T and T' are conjugate under the adjoint action. Consequently, $\mathcal{T}_L \cong \mathcal{G}(k)/C_{\mathcal{G}(k)}(T)$ is connected whenever $T \subset L$ is a maximal torus.

Let $(L, [p])$ be a restricted Lie algebra, $T \subset L$ a maximal torus with embedding $T \hookrightarrow L$ corresponding to a rational point $x_0 \in \mathcal{T}_L(k)$. Thanks to (13.1) the irreducible component $\mathcal{X}_T$ is representable: there exists a finitely generated integral domain A such that $\mathcal{X}_T \cong \mathrm{Spec}_k(A)$. We consider the restricted Lie algebra $\tilde{L} := L \otimes_k A$. Under the above identification $id_A \in \mathrm{Spec}_k(A)(A)$ corresponds to an embedding $j : T \longrightarrow \tilde{L}$ of restricted k-Lie algebras such that the A-submodule $Aj(T) \subset \tilde{L}$ is a direct summand of $\tilde{L}$. Since $\mathcal{X}_T \cong \mathrm{Spec}_k(A)$, any element $\varphi : T \longrightarrow L \otimes_k R$ of $\mathcal{X}_T(R)$ is obtained via specialization: if $\varphi \in \mathcal{X}_T(R)$ corresponds to $x \in \mathrm{Spec}_k(A)(R)$, then we have

$$\varphi = (id_L \otimes x) \circ j.$$

For that reason we call $j : T \hookrightarrow \tilde{L}$ the *universal embedding.*

Note that j endows $\tilde{L}$ with the structure of an infinite-dimensional $u(T)$-module. Since $u(T)$ is commutative and semisimple, there results a weight space decomposition

$$\tilde{L} = \tilde{L}_0 \oplus \bigoplus_{\alpha \in \Phi} \tilde{L}_\alpha$$

of $\tilde{L}$ relative to T. Here $\Phi \subset T^* \setminus \{0\}$ is the set of weights, and for $\alpha \in \Phi \cup \{0\}$ the weight space $\tilde{L}_\alpha = \{v \in \tilde{L} \ ; \ [j(t), v] = \alpha(t)v \ \forall\, t \in T\} \neq (0)$ is an A-direct summand of $\tilde{L}$. Given an arbitrary element $x \in \mathcal{X}_T(k)$, we have

$$L = \tilde{L}_0(x) \oplus \bigoplus_{\alpha \in \Phi} \tilde{L}_\alpha(x),$$

where $\tilde{L}_\alpha(x) = \{v \in L \ ; \ [(id_L \otimes x)(j(t)), v] = \alpha(t)v \ \ \forall\, t \in T\}$ for $\alpha \in \Phi \cup \{0\}$. In other words, if $\varphi := (id_L \otimes x) \circ j$ is the embedding corresponding to $x \in \mathcal{X}_T(k)$, then $\tilde{L}_\alpha(x)$ is the root space with root $\alpha \circ \varphi^{-1}$ relative to the torus $\varphi(T) \subset L$. In particular, specialization along x_0 yields the root space decomposition

$$L = C_L(T) \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

of L relative to T. Thanks to (12.3) this also shows that Φ is a finite set.

A nilpotent, self-normalizing p-subalgebra of L is called a *Cartan subalgebra*. According to general theory, the centralizer $C_L(T)$ of the torus T is a Cartan subalgebra of L if and only if T is a maximal torus (cf. [71, Chapter II]).

Recall that the rank variety of the trivial module of a restricted Lie algebra $(L, [p])$ is given by

$$\hat{\mathcal{V}}_L = \{x \in L \; ; \; x^{[p]} = 0\}.$$

If $\mathcal{G}$ is an infinitesimal group such that $\mathcal{B}_0(\mathcal{G})$ is tame, then $\dim \hat{\mathcal{V}}_{\mathrm{Lie}(\mathcal{G})} = \dim \hat{\mathcal{V}}_{\mathcal{G}_1}(k) \leq 2$. We will occasionally refer to $\dim \hat{\mathcal{V}}_L$ as the *α_p-rank* of the restricted Lie algebra L.

Returning to our general set-up, we let $P \subset \tilde{L}$ be an A-direct summand of $\tilde{L}$. Thanks to (12.2) there exists a non-empty open subset $U_P \subset \mathcal{X}_T(k)$, and a natural number $c_P(L, T) \in \mathbb{N}_0$ such that

$$\dim P(x) \cap \hat{\mathcal{V}}_L = c_P(L, T) \qquad \forall \; x \in U_P.$$

The number $c_P(L, T)$ is the *generic α_p-rank* of the algebraic family $(P(x))_{x \in \mathcal{X}_T(k)}$ of subspaces of L. Given a subset $\Psi \subset T^*$, the A-submodule $\tilde{L}^{\Psi} := \bigoplus_{\alpha \in \Psi} \tilde{L}_\alpha$ is a direct summand of $\tilde{L}$, and we write $c_\Psi(L, T) := c_{\tilde{L}^\Psi}(L, T)$. Let $\mathrm{GF}(p)$ be the Galois field with p elements. For $\alpha \in \Phi \cup \{0\}$ we put $(\alpha) := \mathrm{GF}(p)\alpha$ and define $c_\alpha(L, T) := c_{(\alpha)}(L, T)$. Note that $c_\alpha(L, T)$ is the generic α_p-rank associated to the p-subalgebra $\tilde{L}^{(\alpha)} := \bigoplus_{i=0}^{p-1} \tilde{L}_{i\alpha}$ of $\tilde{L}$.

We illustrate the utility of the generic α_p-rank by giving the following subsidiary result:

Lemma 13.3 *Let $T \subset L$ be a torus.*

(1) *$c_0(L, T) = 0$ if and only if $C_L(T) = T$.*

(2) *If T is a torus of maximal dimension, and $c_0(L, T) = 1$, then $\tilde{L}_0$ is abelian.*

Proof. (1). Suppose that $c_0(L, T) = 0$. Let $x \in U_0$, and put $\varphi := (id_L \otimes x) \circ j$. Then $\hat{\mathcal{V}}_{\tilde{L}_0(x)} = \{0\}$ and $\tilde{L}_0(x) = C_L(\varphi(T))$ is a torus. Since $C_L(\varphi(T))$ is self-normalizing, it is a Cartan subalgebra of L. Thus, $(Aj(T))(x) = \varphi(T)$ is a maximal torus of L, and we conclude that $(Aj(T))(x) = \tilde{L}_0(x)$ for every $x \in U_0$. Since $Aj(T)$ and $\tilde{L}_0$ are A-direct summands of $\tilde{L}$, (12.3) yields

$$\dim_k C_L(T) = \dim_k \tilde{L}_0(x_0) = \dim_k \tilde{L}_0(x) = \dim_k (Aj(T))(x) = \dim_k (Aj(T))(x_0) = \dim_k T,$$

so that $T = C_L(T)$.

Conversely, assume that $T = C_L(T)$. Then we have $\dim_k (Aj(T))(x_0) = \dim_k \tilde{L}_0(x_0)$. As $Aj(T)$ and $\tilde{L}_0$ are A-direct summands of $\tilde{L}$, (12.3) implies $(Aj(T))(x) = \tilde{L}_0(x)$ for every $x \in \mathcal{X}_T(k)$. This shows that $c_0(L, T) = 0$.

(2). By assumption we have $\dim \hat{\mathcal{V}}_{\tilde{L}_0(x)} = 1$ for every $x \in U_0$. Since T has maximal dimension, $(Aj(T))(x)$ is a maximal torus of L, and $\tilde{L}_0(x)$ is nilpotent (cf. [71, Chapter II]). It now follows from the classification of representation-finite restricted Lie algebras (cf. [21]) that $\tilde{L}_0(x)$ is abelian $\forall \; x \in U_0$. Hilbert's Nullstellensatz then yields the assertion. $\square$

Let $(L, [p])$ be a restricted Lie algebra. We denote by $\mathcal{AUT}(L)$ the automorphism scheme of L. For every commutative k-algebra R, $\mathcal{AUT}(L)(R)$ is the set of automorphisms of the restricted R-Lie algebra $L \otimes_k R$. The connected component of $\mathcal{AUT}(L)$ will be denoted $\mathcal{G}_L$. A subspace $I \subset L$ is *$\mathcal{G}_L$-invariant* if $g(I \otimes_k R) = I \otimes_k R$ for every $R \in M_k$ and $g \in \mathcal{G}_L(R)$.

The natural operation of $\mathcal{G}_L$ on L induces an action of $\mathcal{G}_L$ on $\mathcal{T}_L$: For $g \in \mathcal{G}_L(R)$ and $\varphi \in \mathcal{T}_L(R)$ we have $g \cdot \varphi := g \circ \varphi$.

Lemma 13.4 *The following statements hold:*
(1) *Every irreducible component $\mathcal{X} \subset \mathcal{T}_L$ is $\mathcal{G}_L$-invariant.*
(2) *Let $\Psi \subset T^*$. Then $I_{\mathcal{X}_T(k)}(\tilde{L}^{(\Psi)})$ is $\mathcal{G}_L$-invariant.*

Proof. Part (2) hinges on the fact that the induced diagonal action of $\mathcal{G}_L$ on $\tilde{L}$ fixes $j(T)$ pointwise. Consequently, $\tilde{L}^{(\psi)}$ is $\mathcal{G}_L$-invariant, and we may now apply (12.5) to see that $I_{\mathcal{X}_T(k)}(\tilde{L}^{(\Psi)})$ is $\mathcal{G}_L$-invariant. $\square$

In the above situation, $I_{\mathcal{X}_T(k)}(\tilde{L}^{(\Psi)})$ is an ideal of the Lie algebra L. We illustrate the use of these techniques by considering a special case:

Example. Let $(L, [p])$ be a restricted Lie algebra with the following properties:
(a) $\dim \hat{\mathcal{V}}_L = 1$, and
(b) L possesses a self-centralizing torus, and
(c) L admits no nonzero toral ideals.

By (b) there exists a torus $T \subset L$ such that $C_L(T) = T$. Let $\mathcal{X}_T \subset \mathcal{T}_L$ be the irreducible component containing the embedding $T \overset{x_0}{\hookrightarrow} L$, and consider the weight space decomposition

$$\tilde{L} = \tilde{L}_0 \oplus \bigoplus_{\alpha \in \Phi} \tilde{L}_\alpha$$

relative to T.

Since $\dim \hat{\mathcal{V}}_L = 1$ and T is self-centralizing we have $\dim_k L_\alpha = \dim_k L_\alpha(x_0) = 1$ for every $\alpha \in \Phi$. In view of (a) and (b) the set Φ is not empty. Let α be an element of Φ. According to (12.3) we have $\dim_k L_\alpha(x) = 1$ for every $x \in \mathcal{X}_T(k)$. Let $P^\alpha := \tilde{L}^{(\Psi)}$, where $\Psi := \{0, \alpha\}$. By (12.2) $P^\alpha(x)$ is a p-subalgebra of L such that $\dim P^\alpha(x) \cap \hat{\mathcal{V}}_L = 1$ for every $x \in \mathcal{X}_T(k)$. In other words, the algebraic family $(P^\alpha(x))_{x \in \mathcal{X}_T(k)}$ is regular with respect to the irreducible components of $\hat{\mathcal{V}}_L$. Thanks to (12.4) and (13.4) the space $I^\alpha := I_{\mathcal{X}_T(k)}(P^\alpha)$ is a p-ideal such that $\dim \hat{\mathcal{V}}_{I^\alpha} = 1$. We decompose $I^\alpha = I_0 \oplus I_\alpha$ into its weight spaces relative to T, and use (c) to obtain that $1 \leq \dim_k I^\alpha \leq 2$. If $\dim_k I^\alpha = 1$ for every $\alpha \in \Phi$, then $L_\alpha = I^\alpha$ is a p-ideal for each α. Since $\dim \hat{\mathcal{V}}_L = 1$ this implies that $L = T \oplus L_\alpha$ is the two-dimensional, non-abelian Lie algebra. If $\dim_k I^{\alpha_0} = 2$ for some $\alpha_0 \in \Phi$, then $I := I^{\alpha_0}$ is *complete*, that is, centerless with all derivations being inner. There results a decomposition

$$L = I \oplus C_L(I)$$

of L into p-ideals. From $1 = \dim \hat{\mathcal{V}}_L = \dim \hat{\mathcal{V}}_I + \dim \hat{\mathcal{V}}_{C_L(I)}$ we conclude $\dim \hat{\mathcal{V}}_{C_L(I)} = 0$. Hence $C_L(I)$ is a torus, and (c) implies $C_L(I) = \{0\}$. Consequently, $L = I$ is the two-dimensional non-abelian Lie algebra in this case as well.

14. Infinitesimal Groups of Tame Module Type

In this section we are going to apply schemes of tori and rank varieties to the study of infinitesimal groups whose principal blocks have tame representation type. Our results, which are culled from [32], hold for groups that are defined over an algebraically closed field k of characteristic $p \geq 3$.

Theorem 14.1 *Let $\mathcal{G}$ be an infinitesimal, solvable group. Then $\mathcal{B}_0(\mathcal{G})$ is not tame.*

Proof. One considers a counterexample of minimal order $\operatorname{ord}(\mathcal{G}) = \dim_k H(\mathcal{G})$. Since $p \geq 3$ Theorem 9.5 shows that $\mathcal{G}$ is supersolvable with trivial multiplicative center. Hence $\mathcal{G} = \mathcal{U} \times \mathcal{M}$ is the semidirect product of a unipotent normal subgroup $\mathcal{U}$, and a multiplicative group $\mathcal{M}$. It follows that $\mathcal{U}$ contains a normal subgroup $\mathcal{N} \cong \alpha_p$ of $\mathcal{G}$. Since $\mathcal{B}_0(\mathcal{G}/\mathcal{N})$ has finite representation type, (10.3) ensures that the group $\mathcal{U}/\mathcal{N}$ is V-uniserial. This actually implies the commutativity of $\mathcal{U}$. By the same token, we have $V_{\mathcal{U}}(\mathcal{U}^{(p)}) = e_k$. We may now appeal to the classification of groups with this property [14], to obtain an isomorphism $\mathcal{U} \cong \prod_{i=1}^n \alpha_{p^{r_i}}$. Since $c_{H(\mathcal{U})}(k) = 2$ we see that $\sum_{i=1}^n r_i = 2$, whence $\mathcal{U} \cong \alpha_{p^2}$ or $\mathcal{U} \cong \alpha_p \times \alpha_p$. Thanks to (9.4) $H(\mathcal{G}) = \mathcal{B}_0(\mathcal{G})$ in either case.

For the given groups one proceeds by determining the Gabriel quiver and the relations of $H(\mathcal{G})$. In one case $\operatorname{mod}_{H(\mathcal{G})}$ contains the module category of the following bound quiver, in which the relations are marked by dotted lines, as a full subcategory

$$\begin{tikzcd}
\bullet \arrow[d] & & \\
\bullet \arrow[r] & \bullet & \\
\bullet \arrow[u] \arrow[r] \arrow[ur, dotted, no head] & \bullet \arrow[u] \arrow[r] & \bullet \\
& \bullet \arrow[u] \arrow[r] \arrow[ur, dotted, no head] & \bullet \arrow[u]
\end{tikzcd}$$

By results of [76] this module category is wild. Consequently, $H(\mathcal{G})$ is wild, a contradiction. $\square$

The arguments of (9.7) together with (9.3) now yield:

Corollary 14.2 *Let $\mathcal{G}$ be an infinitesimal group.*

(1) *The distribution algebra $H(\mathcal{G})$ does not possess any tame, basic blocks.*
(2) *If $\mathcal{G}$ is trigonalizable, then $H(\mathcal{G})$ has no tame blocks.* $\square$

In §11 we have classified the tame blocks of the Frobenius kernels of the smooth reductive groups. By combining (14.2) with the Lie-Kolchin Theorem (9.2) we readily obtain:

Corollary 14.3 *Let $\mathcal{G}$ be a smooth, solvable k-group, $r \in \mathbb{N}$. Then $H(\mathcal{G}_r)$ does not possess any tame blocks.* $\square$

Given an arbitrary infinitesimal group $\mathcal{G}$, we let $\mathcal{R}(\mathcal{G})$ and $\mathcal{U}(\mathcal{G})$ denote the largest solvable normal subgroup and the largest unipotent normal subgroup of $\mathcal{G}$, respectively.

Theorem 14.4 *Suppose that $\mathcal{B}_0(\mathcal{G})$ is tame.*
(1) *$\mathcal{R}(\mathcal{G})$ is nilpotent.*
(2) *$\mathcal{R}(\mathcal{G})/\mathcal{R}(\mathcal{G})_1$ is multiplicative, and $\mathcal{U}(\mathcal{G})$ has height ≤ 1.* $\square$

Let $\mathcal{G}$ be an infinitesimal group such that $\mathcal{B}_0(\mathcal{G})$ is tame. Since $\mathcal{B}_0(\mathcal{G}/\mathcal{R}(\mathcal{G}))$ is a direct summand of the image of $\mathcal{B}_0(\mathcal{G})$ under the canonical projection $H(\mathcal{G}) \longrightarrow H(\mathcal{G}/\mathcal{R}(\mathcal{G}))$, it has finite or tame representation type. In the former case (10.2) entails the solvability of $\mathcal{G}/\mathcal{R}(\mathcal{G})$. Hence $\mathcal{G}$ is solvable, and (14.1) gives a contradiction. We are thus led to the study of semisimple infinitesimal groups of tame representation type.

The unique largest toral ideal of a restricted Lie algebra $(L, [p])$, the so-called *toral radical* of L, will be denoted $T(L)$. Note that $T(L)$ is contained in the center $C(L)$ of L. If $I \subset L$ is a p-ideal, then $T(I) \subset T(L)$.

Definition. A restricted Lie algebra $(L, [p])$ is called *characteristic semisimple* if it does not possess any non-zero solvable $\mathcal{G}_L$-invariant ideals.

Theorem 14.5 *Let $(L, [p])$ be a characteristic semisimple Lie algebra of α_p-rank 2. Then $L \cong s\ell(2)$.*

Proof. Let $T \subset L$ be a torus of maximal dimension. We consider the component $\mathcal{X}_T \cong \operatorname{Spec}_k(A)$, the corresponding Lie algebra $\tilde{L} := L \otimes_k A$ as well as the weight space decomposition induced by the universal embedding $j : T \hookrightarrow \tilde{L}$:

$$\tilde{L} = \tilde{L}_0 \oplus \bigoplus_{\alpha \in \Phi} \tilde{L}_\alpha.$$

Since L is characteristic semisimple, the Cartan subalgebra $H = C_L(T)$ does not contain any non-zero $\mathcal{G}_L$-invariant p-ideals of L. Thus, (12.4) and (13.4) yield $0 \leq c_0(L, T) \leq 1$. Assume that $c_0(L, T) = 1$.

There exists $\alpha_0 \in \Phi$ such that $\Phi \subset \mathrm{GF}(p)\alpha_0$.
The assumption $c_0(L, T) = 1$, implies the existence of a weight $\alpha_0 \in \Phi$ such that $c_{\alpha_0}(L, T) = 2$. Owing to (12.4) the p-ideal $Q^{(\alpha_0)} = I_{\mathcal{X}_T(k)}(\tilde{L}^{(\alpha_0)})$ has α_p-rank 2. Since $T(Q^{(\alpha_0)}) \subset T(L) = (0)$, we have $\ker \alpha_0|_{T \cap Q^{(\alpha_0)}} = (0)$, so that $\dim_k T \cap Q^{(\alpha_0)} \leq 1$.

If $Q^{(\alpha_0)} \cap T = (0)$, then $Q^{(\alpha_0)}$ is a p-nilpotent $\mathcal{G}_L$-invariant p-ideal of L. This, however, contradicts the characteristic semisimplicity of L.

Thus, $Q^{(\alpha_0)} \cap T$ is one-dimensional, and $C_L(Q^{(\alpha_0)}) \cap Q^{(\alpha_0)} \subset C(Q^{(\alpha_0)}) \subset Q_0^{(\alpha_0)} \subset H$ is a $\mathcal{G}_L$-invariant p-ideal of L. This implies $C_L(Q^{(\alpha_0)}) \cap Q^{(\alpha_0)} = (0)$, so that

$$2 = \dim \hat{\mathcal{V}}_{Q^{(\alpha_0)} \oplus C_L(Q^{(\alpha_0)})} = \dim \hat{\mathcal{V}}_{Q^{(\alpha_0)}} + \dim \hat{\mathcal{V}}_{C_L(Q^{(\alpha_0)})} = 2 + \dim \hat{\mathcal{V}}_{C_L(Q^{(\alpha_0)})}.$$

Hence $C_L(Q^{(\alpha_0)})$ is a toral p-ideal and $C_L(Q^{(\alpha_0)}) = (0)$. Since $L_\beta \subset C_L(Q^{(\alpha_0)})$ for all $\beta \in \Phi \setminus \mathrm{GF}(p)\alpha_0$, we obtain $\Phi \subset \mathrm{GF}(p)\alpha_0$.

Similar arguments then actually show that

$$\Phi = \{\alpha_0, -\alpha_0\} \text{ and } \dim_k L_{\alpha_0} = 1 = \dim_k L_{-\alpha_0}.$$

One then verifies $H = T$, and this contradicts our assumption $c_0(L,T) = 1$ (cf. (13.3)).

Thus, $c_0(L,T) = 0$, and in that case we have a full classification of the possible algebras (cf. [31]). As L is semisimple, we obtain $L \cong s\ell(2)$. $\square$

Now let $\mathcal{G}$ be a semisimple infinitesimal group with Lie algebra $L = \mathrm{Lie}(\mathcal{G})$. The group $\mathcal{G}$ operates on L via the adjoint representation such that L does not possess any solvable $\mathcal{G}$-invariant ideals. It follows that L is characteristic semisimple. Thus, if $\mathcal{B}_0(\mathcal{G})$ is tame, then (14.5) implies that $L \cong s\ell(2)$. Consequently, we have a homomorphism $\mathrm{Ad} : \mathcal{G} \longrightarrow \mathcal{AUT}(s\ell(2))$. This ultimately yields an embedding $\mathcal{G} \hookrightarrow \mathrm{SL}(2)$ so that $\mathcal{G}_1 \cong \mathrm{SL}(2)_1$. We thus have to study closed subgroups of SL(2) whose first Frobenius kernel coincides with $\mathrm{SL}(2)_1$.

Our groups have one more property: we know that $\mathcal{B}_0(\mathcal{G}/\mathcal{G}_1)$ is tame or representation-finite. The Frobenius homomorphism induces an embedding $\mathcal{G}/\mathcal{G}_1 \hookrightarrow \mathrm{SL}(2)$. If the principal block of the factor group is tame, then its Lie algebra is a subalgebra of $s\ell(2)$ of α_p-rank 2. Consequently, it coincides with $s\ell(2)$, and $\mathrm{SL}(2)_2 \subset \mathcal{G}$. However, we have seen in §7 that $3 = c_{H(\mathrm{SL}(2)_2)}(k) \leq c_{H(\mathcal{G})}(k)$, a contradiction. Hence $\mathcal{B}_0(\mathcal{G}/\mathcal{G}_1)$ has finite representation type, and (10.3) determines the structure of $\mathcal{G}/\mathcal{G}_1$. The subgroups $\mathcal{G}$ of SL(2) with

(a) $\mathcal{G}_1 = \mathrm{SL}(2)_1$, and

(b) $\mathcal{B}_0(\mathcal{G}/\mathcal{G}_1)$ is representation-finite

belong to the following list: For a natural number $n \geq 1$ we let $\mathcal{A}_{[n]}$ and $\mathcal{Q}_{[n]}$ be the closed subgroups of SL(2) that are given by

$$\mathcal{A}_{[n]}(R) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)(R) \ ; \ a^{p^n} = 1 = d^{p^n}, \ b^{p^2} = 0 = c^p\}$$

and

$$\mathcal{Q}_{[n]}(R) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2)(R) \ ; \ a^{p^n} = 1 = d^{p^n}, \ b^p = 0 = c^p\}$$

for every commutative k-algebra R, respectively. Note that $\mathcal{Q}_{[n]}$ is the n-th Frobenius kernel of the product $\mathrm{SL}(2)_1\mathrm{T}$ of the first Frobenius kernel of SL(2) with the standard torus $\mathrm{T} \subset \mathrm{SL}(2)$ of diagonal matrices.

To see which groups are tame we are again forced to write down the quiver and some of the relations of $H(\mathcal{G})$ for $\mathcal{G} = \mathcal{A}_{[n]}, \ \mathcal{Q}_{[n]}$. The main reason for the feasability of this project is the following: Let M be a simple or principal indecomposable $H(\mathrm{SL}(2)_1)$-module. By work of Curtis (cf. [48]) and Jeyakumar [49] these modules have an SL(2)-structure. Since the restriction $S|_{H(\mathcal{G}_1)}$ of a simple $H(\mathcal{G})$-module S is simple, this enables us to determine all simple $H(\mathcal{G})$-modules and their projective covers. Thanks to Humphreys' work [46] the Loewy series of the principal indecomposables can also be determined. This provides enough information to see that $H(\mathcal{A}_{[n]})$ is actually wild. We thus arrive at the following result:

Theorem 14.6 *Let $\mathcal{G}$ be a semisimple infinitesimal group of characteristic $p \geq 3$. Then the following statements are equivalent:*

(1) *$\mathcal{B}_0(\mathcal{G})$ is tame.*

(2) *There exists $n \geq 1$ such that $\mathcal{G} \cong \mathcal{Q}_{[n]}$.*

(3) *$H(\mathcal{G})$ is special biserial.*

(4) *$H(\mathcal{G})$ is tame.* $\square$

The verification of (2) $\Rightarrow$ (3) entails the determination of the Gabriel quiver and the relations of a non-simple block $\mathcal{B} \subset H(\mathcal{Q}_{[n]})$. The Gabriel quiver Q of $\mathcal{B}$ has vertex set $V = \mathbb{Z}/(p^{n-1}) \times \mathbb{Z}/(2)$ and arrows

$$\alpha_{(i,j)} : (i,j) \to (i+1, j+1) \quad ; \quad \beta_{(i,j)} : (i,j) \to (i-1, j+1).$$

From the Loewy structure of the principal indecomposables we obtain the following relations

$$\alpha_{(i+1,j+1)} \circ \alpha_{(i,j)} = 0 = \beta_{(i-1,j+1)} \circ \beta_{(i,j)} \ \ ; \ \ \beta_{(i+1,j+1)} \circ \alpha_{(i,j)} = \alpha_{(i-1,j+1)} \circ \beta_{(i,j)}.$$

For infinitesimal groups of height ≤ 1, that is, for restricted Lie algebras, we have better control over the nilpotent radical of the group. Here one obtains:

Theorem 14.7 *Let $\mathcal{G}$ be an infinitesimal group of height ≤ 1 such that $\mathcal{B}_0(\mathcal{G})$ is tame. Then its Lie algebra L satisfies $L/C(L) \cong s\ell(2)$.* $\square$

Since the center $C(L)$ has α_p-rank ≤ 1, its structure is completely understood. The Gabriel quiver of $\mathcal{B}_0(\mathcal{G})$ is that of $\mathcal{B}_0(\mathrm{SL}(2)_1)$ with possibly other relations.

15. The Stable Auslander-Reiten Quiver

By the results of the preceding sections we now have a fairly good understanding of distribution algebras of finite and tame representation type. In our discussion one important invariant of the Morita equivalence class of a self-injective algebra Λ has been left out of the account: its *stable Auslander-Reiten quiver* $\Gamma_s(\Lambda)$. By definition, $\Gamma_s(\Lambda)$ is a directed graph with set of vertices given by the isoclasses $[M]$ of the non-projective indecomposable modules of $\mathrm{mod}(\Lambda)$. The arrows roughly correspond to the irreducible maps. The quiver comes fitted with a certain quiver automorphism τ, the so-called *Auslander-Reiten translation.* For a self-injective algebra this is just the composite of the square of the Heller operator with the Nakayama functor. Thus, if Λ is a Frobenius algebra with Nakayama automorphism μ, then we have

$$\tau([M]) = [\Omega_\Lambda^2(M^{(\mu)})],$$

where $M^{(\mu)}$ denotes the space M with action twisted by μ^{-1}. For this and other basic facts on AR-theory we refer the reader to [4, 6].

For $\Lambda = k[G]$, the group algebra of a finite group G, the study of $\Gamma_s(\Lambda)$ was initiated by Reiten [66] some 23 years ago. She showed that finite AR-components of $k[G]$ always have tree class A_n. A few years later, Webb [82] provided a list of the possible tree classes for the infinite AR-components: these are either infinite Dynkin diagrams or Euclidean diagrams. Okuyama [58] proved that Euclidean tree classes only occur at characteristic 2, in fact only $\tilde{A}_{12}$ appears [7]. In 1995 Erdmann completed the classification of the AR-quiver of $k[G]$ by showing that components belonging to wild blocks have tree class A_∞ (cf. [17]).

The AR-theory of the distribution algebras of infinitesimal group schemes is not nearly as well understood. In fact, the only class of such groups where we have a complete classification are the unipotent infinitesimal groups. This was shown in [18] in case the underlying groups have height ≤ 1, yet the methods also work in the general case [26]. In default of a theory of vertices and sources, we again employ rank varieties to obtain control of the AR-quiver.

Throughout, we will be working over an algebraically closed field k. Given an infinitesimal k-group $\mathcal{G}$, we let $\Gamma_s(\mathcal{G})$ denote the stable Auslander-Reiten quiver of $H(\mathcal{G})$.

Lemma 15.1 *Let $\Theta \subset \Gamma_s(\mathcal{G})$ be a component. Then we have $\hat{\mathcal{V}}_{\mathcal{G}_r}(M) = \hat{\mathcal{V}}_{\mathcal{G}_r}(N)$ for $[M]$, $[N] \in \Theta$.* $\square$

In other words, rank varieties are invariants of the AR-components of $H(\mathcal{G})$, so that we can speak of the variety $\hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta)$ of the component $\Theta \subset \Gamma_s(\mathcal{G})$. Let $[M] \in \Theta$. Since Θ does not contain any projective modules, we have $\hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) \neq \{0\}$. Given $\varphi \in \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) \setminus \{0\}$, we thus have

$$(0) \neq \operatorname{Ext}^1_{A_r}(k, M),$$

where $A_r \cong k[X]/(X^p) \subset H(\alpha_{p^r})$ is the algebra occurring in the definition of rank varieties. Thus, we can define subadditive functions and obtain the following analogue of Webb's result for finite groups.

Theorem 15.2 ([26]) *Let $\Theta \subset \Gamma_s(\mathcal{G})$ be a component. The tree class of Θ is either*

(a) *a finite Dynkin diagram* $A_n,\ D_n,\ E_6,\ E_7,\ E_8,$

(b) *an infinite Dynkin diagram* $A_\infty,\ A_\infty^\infty,\ D_\infty,$

(c) *a Euclidean diagram* $\tilde{A}_{12},\ \tilde{D}_n,\ \tilde{E}_6,\ \tilde{E}_7,\ \tilde{E}_8.$ $\square$

For infinitesimal groups of height ≤ 1 one has more information (cf. [18, 21]). Since the representation-finite blocks of their distribution algebras are Nakayama algebras [21], A_{p^ℓ} is the only finite Dynkin diagram that actually occurs. If $p \geq 3$ and $\mathcal{G}$ is supersolvable, this result continues to hold. Moreover, the stable AR-quivers does not possess components of Euclidean tree class in that case (cf. [26]). For arbitrary groups, rank varieties give the following information:

Theorem 15.3 ([26]) *Let $\mathcal{G}$ be an infinitesimal k-group of height r, $\Theta \subset \Gamma_s(\mathcal{G})$ a component.*

(1) $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) = 1$ *if and only if Θ is either finite or an infinite tube.*

(2) *If* $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) \geq 3$*, then* $\Theta \cong \mathbb{Z}[A_\infty]$. $\square$

The preceding result can be used to shorten the list of (15.2) for Frobenius kernels of smooth reductive groups.

Example. Let $\mathcal{G} = \mathrm{SL}(2)$, $\mathcal{B} \subset H(\mathrm{SL}(2)_1)$ a non-simple block. Recall from §11 that $\mathcal{B}$ has two simple modules. One can show that the AR-quiver $\Gamma_s(\mathcal{B})$ consists of 2 components of type $\mathbb{Z}[\tilde{A}_{12}]$ and infinitely many components of type $\mathbb{Z}[A_\infty]/(\tau)$ (cf. [22] for the details).

Theorem 15.4 ([26]) *Let $\mathcal{G}$ be a smooth reductive group of characteristic $p \geq 5$. A component $\Theta \subset \Gamma_s(\mathcal{G}_r)$ belongs to one of the following types: $\mathbb{Z}[A_\infty]$, $\mathbb{Z}[A_\infty]/(\tau^\ell)$, $\mathbb{Z}[A_\infty^\infty]$, $\mathbb{Z}[D_\infty]$, or $\mathbb{Z}[\tilde{A}_{12}]$.*

Proof. In view of (15.2) our task mainly is to rule out finite Dynkin diagrams and to determine the possible diagrams of Euclidean tree class. With regard to the former, a theorem by Auslander states that finite Dynkin diagrams correspond to non-simple blocks of finite representation type. We have seen in (11.1) that distribution algebras of Frobenius kernels of smooth reductive groups do not admit such blocks.

If Θ has Euclidean tree class, then we have $\dim \hat{\mathcal{V}}_{\mathcal{G}_r}(\Theta) = 2$. By general theory, Θ is attached to a principal indecomposable module. Accordingly, the component $\Omega_{H(\mathcal{G})}(\Theta)$ contains a simple module S with a two-dimensional rank variety. The arguments of (11.1) now imply the existence of a subgroup $\mathcal{H} \subset \mathcal{G}$, a simple $\mathrm{SL}(2)_r$-module $L(\lambda)$, and a projective $\mathcal{H}_r$-module P such that

(a) $\mathcal{G}_r \cong \mathrm{SL}(2)_r \times \mathcal{H}_r$, and

(b) $S \cong L(\lambda) \otimes_k P$.

The structure of the simple and projective modules of Frobenius kernels of smooth groups is given by Steinberg's (twisted) tensor product theorem (see [48]). One can use this result to see that the projective cover $P(\lambda)$ of $L(\lambda)$ has a semisimple heart with two isomorphic constituents. This implies that $\Theta \cong \mathbb{Z}[\tilde{A}_{12}]$. □

We continue by extending our example concerning $\mathrm{SL}(2)_1$. In the previous section (see (14.6)) we observed that the blocks of the distribution algebra $H(\mathcal{Q}_{[n]})$ are special biserial. Thanks to work by Erdmann-Skowroński [19] the AR-theory of such algebras is well understood. To formulate our final result, we recall the definition of a special class of tame algebras:

Definition. An algebra Λ is called *domestic* if it does not have finite representation type, and if there exist $(\Lambda, k[X])$-bimodules $Q_1, \ldots, Q_n$ such that

(a) Q_i is a finitely generated free right $k[X]$-module, and

(b) for each natural number $d > 0$, all but a finite number of isoclasses of indecomposable Λ-modules of dimension d are of the form $[Q_i \otimes_{k[X]} V]$ for some indecomposable $k[X]$-module V.

Theorem 15.5 ([32]) *Let $n \geq 2$. The stable Auslander-Reiten quiver $\Gamma_s(\mathcal{Q}_{[n]})$ is the disjoint union of* p-1 *components of type $\mathbb{Z}[\tilde{A}_{2p^{n-1}-1}]$, and infinitely many components of type $\mathbb{Z}[A_\infty]/(\tau)$. Moreover, the algebra $H(\mathcal{Q}_{[n]})$ is of domestic representation type.* □

The components $\mathbb{Z}[\tilde{A}_{2p^{n-1}-1}]$ have tree class A_∞^∞. Accordingly, the Auslander-Reiten theory of distribution algebras differs from that for finite groups, where components of tree class A_∞^∞ only occur for $p = 2$ and when the defect group of the relevant block is dihedral or semidihedral (see [17]). Moreover, by work of Skowroński [70] such a group algebra is domestic if and only if its Sylow-2-subgroups are Klein four groups.

Since most algebras are wild, a full classification of the indecomposable modules is usually not possible. Aside from classifying the AR-components, one is thus also interested in determining the position of certain classes of indecomposable modules within the Auslander-Reiten quiver. We say that an indecomposable module M *lies at an end* if its isoclass $[M]$ has exactly one successor and exactly one predecessor in the stable Auslander-Reiten quiver. By general theory, this amounts to saying that the middle term of the almost split sequence terminating in M has exactly one non-projective indecomposable summand.

For the group algebra of a finite group G there are a number of results concerning the position of simple modules within $\Gamma_s(G)$. If such a module S belongs to a wild block $\mathcal{B} \subset k[G]$, then it is located at an end of a component of tree class A_∞ if either G is p-solvable [51], or if G is of Lie type [52], or if G is the symmetric or alternating group [8]. An example showing that simple modules of wild blocks may have two predecessors can be found in [52].

Let $\mathcal{G}$ be an infinitesimal k-group of characteristic $p \geq 3$. If $\mathcal{G}$ is supersolvable, then a component of type $\mathbb{Z}[A_\infty]$ contains at most one simple vertex, which lies at an end [26]. Thus, simple modules of complexity ≥ 3 will have this property. If $\mathcal{G}$ has height ≤ 1 this result holds for arbitrary $\mathcal{G}$ (cf. [24]). In particular, a simple module of the first Frobenius kernel of a smooth, reductive group of characterstic $p \geq 5$ lies at an end if and only if it belongs to a wild block. The baby Verma modules of such groups are also located at ends [27].

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