Characteristic Tilting Modules over Quasi-hereditary Algebras

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1 Quasi–hereditary algebras

Tilting modules occur in a variety of contexts and we are fortunate to have a unifying language with which to describe them. This is provided by the language of *quasi-hereditary algebras* which for us will usually arise in the context of *highest weight categories.*

Definition 1.1 *(Cline, Parshall, Scott, [20]) Let k be a field, A a finite dimensional algebra over k, Λ an indexing set for the isomorphism classes of simple A-modules with correspondence $\lambda \leftrightarrow L(\lambda)$, and $\leq$ a partial order on Λ. We say that $(A, \leq)$ is a* quasi-hereditary algebra *if and only if for all $\lambda \in \Lambda$ there exists a left A-module, $\Delta(\lambda)$, called a* standard module *such that*

(I) *there is a surjection $\phi_\lambda : \Delta(\lambda) \to L(\lambda)$ and the composition factors, $L(\mu)$, of the kernel satisfy $\mu < \lambda$.*

(II) *the indecomposable projective cover, $P(\lambda)$, of $L(\lambda)$ maps onto $\Delta(\lambda)$ via a map $\psi_\lambda : P(\lambda) \to \Delta(\lambda)$ whose kernel is filtered by modules $\Delta(\mu)$ with $\mu > \lambda$.*

We now give a series of examples to both illustrate the concept and to demonstrate some of its characteristics. In these examples, and often in the sequel, we will use the following notation:

Notation 1.1 *In the situation that we have a filtered module*

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_i = M$$

with $M_j/M_{j-1} = N_j$ we will abbreviate this by the notation

$$M \cong \begin{matrix} N_i \\ \vdots \\ N_1 \end{matrix} .$$

Example

1. Put $A_1 = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$, the set of upper triangular matrices. Let $\Lambda = \{1, 2, 3\}$ and $e_i = e_{ii}$, the corresponding matrix idempotent. Order Λ by $1 < 2 < 3$. We then have

$$P(1) = A_1 \cdot e_1 = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}, \Delta(1) = P(1) = L(1),$$

$$P(2) = A_1 \cdot e_2 = \begin{pmatrix} k \\ k \\ 0 \end{pmatrix}, \Delta(2) = P(2) \cong \begin{matrix} L(2) \\ L(1) \end{matrix},$$

$$P(3) = A_1 \cdot e_3 = \begin{pmatrix} k \\ k \\ k \end{pmatrix}, \Delta(3) = P(3) \cong \begin{matrix} L(3) \\ L(2) \\ L(1) \end{matrix}.$$

So we have a quasi-hereditary algebra.

2. Put $A_2 = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$, $\Lambda = \{1, 2, 3\}$ and $e_i = e_{ii}$, but order Λ by $3 < 2 < 1$. We still have

$$\Delta(1) = P(1) = L(1).$$

However, if we try to set

$$\Delta(2) = P(2)$$

then we get that $L(1)$ is a composition factor of $\ker(\Delta(2) \to L(2))$ and this is not allowed. On the other hand, if we put

$$\Delta(2) = L(2),$$

we get the permitted

$$0 \to L(1) \cong \Delta(1) \to P(2) \to L(2) \cong \Delta(2) \to 0.$$

By similar reasoning, we must put

$$\Delta(3) = L(3).$$

In the end, we get $0 = P(0) \subset P(1) \subset P(2) \subset P(3)$ with $P(i)/P(i-1) = \Delta(i)$. Thus we again have a quasi-hereditary algebra.

3. Put $A_3 = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$. This time place the order $2 < 3 < 1$ on Λ. We then have

$$\Delta(1) = L(1),$$

$$\Delta(2) = L(2),$$

$$\Delta(3) \cong P(3)/L(1).$$

So

$$0 \to L(2) \to \Delta(3) \to L(3) \to 0,$$

and

$$0 \to L(1) \cong \Delta(1) \to P(3) \to \Delta(3) \to 0.$$

These three examples demonstrate the importance of the order on the quasi-hereditary structure.

4. Put $A_4 \subset \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$, with $a_{11} = a_{33}$. In this case, we have the two primitive idempotents

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We consider the two possible choices of orders

$1 < 2$ Here we get

$$\Delta(1) = L(1), \text{ and } \Delta(2) = P(2)$$

and so two short exact sequences

$$0 \to L(1) \to \Delta(2) \to L(2) \to 0,$$

$$0 \to \Delta(2) \to P(1) \to \Delta(1) \to 0.$$

In this case $(A_4, \leq)$ is a quasi-hereditary algebra.

$2 < 1$ We begin with the short exact sequence

$$0 \to L(1) \to P(2) \to L(2) \to 0.$$

From this we conclude that $\Delta(2) \neq P(2)$ since then L(1) is a composition factor of $\Delta(2)$ and this is not allowed. So we must try $\Delta(2) = L(2)$. Now by the filtration of $P(2)$, we must have that $\Delta(1) = L(1)$. However, we have the short exact sequence

$$0 \to ? \neq 0 \to P(1) \to L(1) = \Delta(1) \to 0$$

and there are no indices larger than 1. We conclude that no quasi-hereditary structure is possible with this ordering!

5. Put $A_5 = A_4 / \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We have two short exact sequences

$$0 \to L(2) \to P(1) \to L(1) \to 0,$$

and

$$0 \to L(1) \to P(2) \to L(2) \to 0.$$

If we try the filtration $1 < 2$, then $\Delta(1) = L(1)$ implying that $P(1)$ has no appropriate filtration. If we try $2 < 1$, then we get a symmetric failure. Thus no ordering of the simples allows a quasi-hereditary structure to be put on A_5.

We next want to consider the dual of this definition. This is motivated by the fact that any finite dimensional algebra, A, admits a duality: $-^* = \mathrm{Hom}_k(-, k) : A\text{-mod} \cong \text{mod-}A \cong A^{op}\text{-mod}$, that is, a contravariant equivalence between left and right modules. This duality sends simple modules to simple modules and sends projectives to injectives. With this in mind, we make the following dual

Definition 1.2 *(Cline, Parshall, Scott [20]) Let k be a field, A a finite dimensional algebra over k, Λ an indexing set for the set of isomorphism classes of simple A-modules with correspondence $\lambda \leftrightarrow L(\lambda)$, and $\leq$ a partial order on Λ. We say that $(A, \leq)$ is a* quasi-hereditary algebra *if and only if, for all $\lambda \in \Lambda$ there exists a left A-module $\nabla(\lambda)$, called a* costandard module*, such that*

(I) *there is an injection* $\phi_\lambda : L(\lambda) \to \nabla(\lambda)$ *and the composition factors,* $L(\mu)$, *of the cokernel satisfy* $\mu < \lambda$.

(II) *the indecomposable injective envelope of* $L(\lambda)$, $I(\lambda)$, *contains* $\nabla(\lambda)$ *as a submodule, and the inclusion* $\psi_\lambda : \nabla(\lambda) \to I(\lambda)$ *has a cokernel filtered by modules* $\nabla(\mu)$ *with* $\mu > \lambda$.

The two definitions are equivalent; in fact

Proposition 1.1 *(Parshall, Scott [75]) A is quasi-hereditary if and only if* A^{op} *is quasi-hereditary.*

We postpone the proof until we give another definition which is again equivalent to these two. But first, a few examples of this second definition.

Example

$$A_i = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

We know that the duality $\mathrm{Hom}_k(-, k)$ carries right projectives to left injectives. Hence we get

$$I(1) = \begin{pmatrix} k & k & k \end{pmatrix}^* = \begin{matrix} L(3) \\ L(2) \\ L(1) \end{matrix} ,$$

$$I(2) = \begin{pmatrix} 0 & k & k \end{pmatrix}^* = \begin{matrix} L(3) \\ L(2) \end{matrix} ,$$

$$I(3) = \begin{pmatrix} 0 & 0 & k \end{pmatrix}^* = L(3) .$$

$A_1 :\ 1 < 2 < 3$

$$\nabla(1) = L(1), \nabla(2) = L(2), \nabla(3) = L(3).$$

Here the required filtrations of the injectives by the ∇'s are given by the Jordan-Hölder series. Note that the Δ's are projective and the ∇'s are simple.

$A_2:\ 1 > 2 > 3$

$$\nabla(1) = I(1), \nabla(2) = I(2), \nabla(3) = I(3)$$

and here the Δ's are simple and the ∇'s are injective.

$A_3:\ 2 < 3 < 1$

$$\nabla(1) = \begin{array}{c} L(3) \\ L(2) \\ L(1) \end{array}, \nabla(2) = L(2), \nabla(3) = L(3)$$

and we have now

$$0 \to \nabla(2) \to I(2) \to L(3) = \nabla(3) \to 0.$$

Since $3 < 2$, all is well.

Example In our example, A_4, from earlier, we now have

$$I(1) = \begin{pmatrix} k & k & k \end{pmatrix}^* = \begin{array}{c} L(1) \\ L(2) \\ L(1) \end{array},$$

where the entries in the upper left hand and lower right hand corners of the matrix are equal; and

$$I(2) = \begin{pmatrix} 0 & k & k \end{pmatrix}^* = \begin{array}{c} L(1) \\ L(2) \end{array}.$$

In the case where the order is $1 < 2$, we may put $\nabla(2) = I(2)$ and $\nabla(1) = L(1)$. From the short exact sequence

$$0 \to \nabla(1) \to I(1) \to \nabla(2) \to 0$$

we see we have a quasi-hereditary algebra.

We leave it as an exercise to check the non quasi-hereditary examples.

A-mod is usually called a *highest weight category* if it satisfies either of the previous definitions. These definitions can be made in more generality. Call a poset, Λ, *interval finite* if for every $\mu \leq \lambda$ in Λ, $|\{\gamma | \mu \leq \gamma \leq \lambda\}|$ is finite. To be a *highest weight category*, an arbitrary abelian k-category, $\mathcal{C}$, in addition to satisfying the above conditions, must be closed under arbitrary direct limits, have either enough projectives or injectives, the endomorphism rings of simples must be finite dimensional, each object must be a direct limit (i.e., union) of finitely generated ones, and the simples must be indexed by an interval finite poset. These types of categories arise frequently in Lie theory. They give rise, locally, to finite dimensional algebras:

Definition 1.3 *A subset, $\Pi \subset \Lambda$, of a partially ordered set is called an* ideal *if for all pairs $x \leq y$ in Λ, $y \in \Pi$ implies $x \in \Pi$. Dually, a subset $\Pi' \subset \Lambda$, is called a* coideal *if for all pairs $x \leq y$ in Λ, $x \in \Pi'$ implies $y \in \Pi'$.*

Given a highest weight category $\mathcal{C}$, let $\Lambda' \subset \Lambda$ be the intersection of an ideal and a coideal containing a finite number of elements. Let $\mathcal{C}[\Lambda']$ be the full subcategory generated by $\{L(\lambda) | \lambda \in \Lambda'\}$, that is all those elements of $\mathcal{C}$ which have a filtration whose sections are $L(\lambda)$, $\lambda \in \Lambda'$. Then $\mathcal{C}[\Lambda']$ is the category A-mod for some quasi-hereditary algebra A. Under the situation where $\mathcal{C}$ has enough projectives one takes for the standard modules, $\Delta_{\Lambda'}(\lambda)$, the largest factor of $\Delta(\lambda)$ with filtration whose sections are $L(\mu)$, $\mu \in \Lambda'$. As projectives, $P_{\Lambda'}(\lambda)$, one takes the largest quotient of $P(\lambda)$ filtered by the $\Delta_{\Lambda'}(\mu)$, $\mu \in \Lambda$. Then $A = \operatorname{End}(\oplus_{\lambda \in \Lambda'} P_{\Lambda'}(\lambda)$ is a quasi-hereditary algebra whose poset is Λ'. The construction for the case that $\mathcal{C}$ has enough injectives is exactly dual. Full details, along with proofs can be found in Cline, Parshall, Scott [20].

Example

$$A_i = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix},$$

$(1 < 2 < 3)$. We can take $\{1,2\}$, $\{2,3\}$ or any singleton as Λ'. As an example, if we take $\{1,2\}$, we get that the projectives remain unchanged:

$$P(1) = L(1), P(2) = \begin{matrix} L(2) \\ L(1) \end{matrix}$$

but the injectives become truncated:

$$I(1) = \begin{matrix} L(3) \\ L(2) \\ L(1) \end{matrix} \text{ is truncated to } \begin{matrix} L(2) \\ L(1) \end{matrix}$$

and

$$I(2) = \begin{matrix} L(3) \\ L(2) \end{matrix} \text{ is truncated to } L(2).$$

As A we then get $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$; if we had taken a singleton then we would have gotten $\begin{pmatrix} k \end{pmatrix}$.

Using these ideas we can construct new quasi-hereditary algebras from old ones, in which the new one has an indexing set of smaller cardinality. The process, called *truncation*, allows us to set up an induction for proving facts about quasi-hereditary algebras. Explicitly, we may complete the partial order on the poset Λ to a total order $\{1 < 2 < \ldots < n\}$. Now we remove indices starting from the top, that is, first n, then $n-1$, etc.

In this case, we may say even more. Let e be the idempotent so that $P(n) = Ae$; then we claim that AeA is a direct sum of copies of $P(n)$ and that $\mathrm{End}_A(Ae) = eAe$ is semi-simple. To begin, we have $P(n) = \Delta(n)$ since we have no candidates to filter the kernel of $P(n) \to \Delta(n)$. $L(n)$ does not occur as a composition factor for any $\Delta(i)$, $i \neq n$. For any given $j < n$, $P(j)$ is filtered by $\Delta(n)$'s and others; we in fact have $\mathrm{Hom}_A(P(n), P(j)) = \mathrm{Hom}_A(P(n),$ sum of $\Delta(n)$'s in $P(j))$ since $L(n)$ does not occur as a composition factor for any $\Delta(i)$, $i < n$. It follows that each $\phi : P(n) \to P(j)$ is zero or injective, since the same holds true for $\mathrm{End}_A(P(n))$. We claim that trace of $P(n)$ in $P(j) = \sum_{\phi:P(n)\to P(j)} \phi(P(n))$ is a direct sum of copies of $P(n)$. This follows from the fact that for pair of maps, $\phi_1, \phi_2 : P(n) \to P(j)$, $\mathrm{im}\ \phi_1 \cap\ \mathrm{im}\ \phi_2$ is $P(n)$ or is 0. To see this, consider the short exact sequence

$$0 \to \mathrm{im}\ \phi_1 \to \mathrm{im}\ \phi_1 + \mathrm{im}\ \phi_2 \to \text{cokernel} \to 0.$$

Then ϕ_2 defines a map of $P(n)$ onto the cokernel and so must be isomorphic to it by a dimension count. Further $P(n)$ equals the cokernel, being projective, this map must lift to $\mathrm{im}\ \phi_1 + \mathrm{im}\ \phi_2$ and so the short exact sequence splits.

We can say even more. We have that the trace of $P(n)$ in $P(j)$ is equal to $P(j) \cap AeA$, where e is any idempotent associated to the projective $P(n)$. Carrying this a step further, we have that the trace of $P(n)$ in A is $AeA \cong P(n)^l$ for some l. Conversely, if $AeA \cong P(n)^l$, then we have submodules $\Delta(n)^l$ in A. By induction we then get a proof that the previous definitions are the same as the following.

Definition 1.4 *(Cline, Parshall, Scott [20]) Given a finite dimensional algebra A over a field k, we say*

(a) a two sided ideal, J, is called a heredity ideal *if and only if there is an idempotent e such that $J = AeA$ and $End_A(Ae) = eAe$ is semi-simple.*

(b) A is quasi-hereditary *if and only if there exists a chain of two sided ideals, called a* heredity chain,

$$0 \subset J_n \subset J_{n-1} \subset \ldots \subset J_1 = A$$

such that J_l/J_{l+1} is a heredity ideal in A/J_{i+1} for all l.

Remark 1.1 *This particular construction gives rise to a so-called* recollement *structure between the bounded derived categories of A/J, A and eAe allowing one to construct that of A from those of A/J and eAe.*

Proposition 1.2 *(Cline, Parshall, Scott [20]) The three definitions are equivalent.*

Proof: We have already shown how to build a heredity ideal from the top weight and so have passed from Definition 1.1. to Definiton 1.4. To go in the opposite direction, suppose we have a heredity chain:

$$0 \subset J_n \subset J_{n-1} \subset \ldots \subset J_1 = A.$$

We may assume, without loss of generality that e is primitive, where $J = AeA$. Put $\Delta(n) = Ae$. The semi-simplicity of $\mathrm{End}_A(Ae)$ implies, as above, that the trace of $Ae = \Delta(n)$ in A is a direct sum of copies of $\Delta(n)$. Further we clearly have $\mathrm{Hom}_A(\Delta(n), A/AeA) = 0$ since AeA is a trace ideal. Thus we may pass to A/J_n and proceed by induction. The last definition is gotten by a symmetric argument.

Corollary 1.3 *If A is a quasi-hereditary algebra with heredity chain*

$$J_n = Ae_nA \subset J_{n-1} = A(e_n + e_{n-1})A \subset \ldots$$

then all A/J_l are quasi-hereditary with respect to the same order as well as all eAe with $e = e_l + \cdots + e_n$ for any l.

Example

$$A_i = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

$(A_1 :\ 1 < 2 < 3)$ We get the heredity chain

$$\begin{pmatrix} k & k & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subset \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & 0 \end{pmatrix} \subset A_1$$

$(A_2 : \; 1 > 2 > 3)$ We get the heredity chain

$$\begin{pmatrix} 0 & 0 & k \\ 0 & 0 & k \\ 0 & 0 & k \end{pmatrix} \subset \begin{pmatrix} 0 & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} \subset A_2$$

$(\; A_3 : \; 2 < 3 < 1)$ We get the heredity chain

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & k & k \\ 0 & 0 & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} \subset \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix} \subset A_3$$

As an example of the usefulness of inductions constructed from such truncations we prove the following important property of quasi-hereditary algebras:

Theorem 1.4 *(Bernstein-Gelfand-Gelfand [16], Cline-Parshall-Scott [20], [75], Dlab-Ringel [32], [33]) If A is a quasi-hereditary algebra with poset $(\Lambda, \leq)$, then the global dimension of A is less than or equal to $2l(\Lambda) - 2$ where $l(\Lambda)$ is the length of the longest chain in Λ.*

Proof: Goes by induction on $l(\Lambda)$. If $l(\Lambda) = 1$, then there are no comparable elements and so A is semi-simple and its global dimension is 0. To build the induction recall that the global dimension is equal to the max $\{\text{pdim } L | L \text{ simple}\}$ by the long exact sequence in cohomology. Recall that if λ is a maximal weight in Λ then $\Delta(\lambda) = P(\lambda) = Ae_\lambda$. Let $e = \sum_{\lambda \text{ maximal in } \Lambda} e_\lambda$. Then $J = AeA$ is a heredity ideal and we have

$$0 \to J \to A \to A/J \to 0.$$

By induction gl dim $A/J \leq 2l(\Lambda) - 4$.

Now take a projective A/J-resolution of any $L \in A/J$-mod:

$$0 \to P_k \to P_{k-1} \to \ldots \to P_0 \to L \to 0.$$

Each projective A/J module, P_j, is surjected on by a projective A module, $\tilde{P}_j$, and the kernel is a direct sum of the various Δ's which are direct summands of J. Since these are themselves projective over A we get

$$\text{pdim}_A L \leq \; \text{pdim}_{A/J} L + 1 \text{ for all } A/J\text{-modules.}$$

All that remains are the simple A modules, L, which are not A/J modules. However, such a module is the homomorphic image of some Δ which is a direct summand of J. The kernel, K, of this surjection is an A/J-module and so has projective dimension (over A) at most $2l(\Lambda)-3$. Thus from the long exact sequence in cohomology we get that $\operatorname{pdim} L \leq \operatorname{pdim} K + 1 \leq 2l(\Lambda) - 2$.

2 Characteristic Tilting Modules

As part of our definitions for quasi-hereditary algebras we have given that projective modules are filtered by standard modules (Δ's) and injective modules by costandard ones (∇'s). We extend these characterizations to modules which are filtered by Δ's by putting $\mathcal{F}(\Delta)$ equal to the full subcategory of A-mod whose objects, X, have filtrations $0 \subset X_1 \subset X_2 \subset \ldots \subset X_k = X$, such that $X_j/X_{j-1} \cong \Delta(j)$, some standard module, for all j. Such modules are called Δ-*good* and the filtration a Δ-*filtration*. We define $\mathcal{F}(\nabla)$ analogously calling its objects ∇-*good* and the relevant filtrations ∇-*filtrations*. These categories have the key homological property

Proposition 2.1 *(Ringel [77], Auslander-Buchweitz [10], Auslander-Reiten [11])* $\mathcal{F}(\nabla) = \mathcal{F}(\Delta)^{\perp}$ *and* $\mathcal{F}(\Delta) = {}^{\perp}\mathcal{F}(\nabla)$, *in the following sense:*

$$\begin{aligned}
\mathcal{F}(\Delta) &= \{X \in A\text{-mod} | \mathrm{Ext}^1_A(X, \nabla(\lambda)) = 0, \text{ for all } \lambda \in \Lambda\} \\
&= \{X \in A\text{-mod} | \mathrm{Ext}^m_A(X, \nabla(\lambda)) = 0, \ m \geq 1, \text{ for all } \lambda \in \Lambda\} \\
&= \{X \in A\text{-mod} | \mathrm{Ext}^1_A(X, Y) = 0, \text{ for all } Y \in \mathcal{F}(\nabla)\} \\
&= \{X \in A\text{-mod} | \mathrm{Ext}^m_A(X, Y) = 0, \ m \geq 1, \text{ for all } Y \in \mathcal{F}(\nabla)\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}(\nabla) &= \{X \in A\text{-mod} | \mathrm{Ext}^1_A(\Delta(\lambda), X) = 0, \text{ for all } \lambda \in \Lambda\} \\
&= \{X \in A\text{-mod} | \mathrm{Ext}^m_A(\Delta(\lambda), X) = 0, \ m \geq 1, \text{ for all } \lambda \in \Lambda\} \\
&= \{X \in A\text{-mod} | \mathrm{Ext}^1_A(Y, X) = 0, \text{ for all } Y \in \mathcal{F}(\Delta)\} \\
&= \{X \in A\text{-mod} | \mathrm{Ext}^m_A(Y, X) = 0, \ m \geq 1, \text{ for all } Y \in \mathcal{F}(\Delta)\}.
\end{aligned}$$

Proof: We begin by showing

$$\mathrm{Hom}_A(\Delta(\lambda), \nabla(\mu)) = \left\{ \begin{array}{ll} \mathrm{End}_A L(\lambda) & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{array} \right.$$

Since we have a surjection $P(\lambda) \to \Delta(\lambda)$ from an indecomposable projective, it follows that $\Delta(\lambda)$ has a unique top, $L(\lambda)$. Similarly, since we have an injection $\nabla(\mu) \to I(\lambda)$ an indecomposable injective, $\nabla(\lambda)$ must have a unique socle, $L(\mu)$. So given a nonzero map $\Delta(\lambda) \to \nabla(\mu)$ we have that

$L(\mu)$ must occur in the composition series of $\Delta(\lambda)$ and $L(\lambda)$ must occur in a composition series of $\nabla(\mu)$. But the first requires $\mu \leq \lambda$ and the second $\lambda \leq \mu$. So $\mathrm{Hom}(\Delta(\lambda), \nabla(\mu)) = 0$, unless $\lambda = \mu$. But in this case, the map must factor $\Delta(\lambda) \to L(\lambda) \to \nabla(\lambda)$, inducing an isomorphism $L(\lambda) \to L(\lambda)$. Next

$$\mathrm{Ext}_A^m(\Delta(\lambda), \nabla(\mu)) = 0, \text{for all } \lambda, \mu.$$

First, if λ is maximal then $\Delta(\lambda)$ is projective so the result is clear. Next, if μ is maximal then $\nabla(\mu)$ is injective and the result is again clear. If neither is maximal, then let e be the idempotent associated to a maximal element of $\lambda \in \Lambda$. Then both $\Delta(\lambda)$ and $\nabla(\mu)$ are A/J modules for $J = AeA$ a heredity ideal. The statement now follows from:

Lemma 2.2 *(Dlab-Ringel [32]) For all $X, Y \in A/J$-mod and all $m \geq 0$, $Ext_{A/J}^m(X, Y) = Ext_A^m(X, Y)$.*

Proof: This is clear for $m = 0$. For $m = 1$ both Ext-groups are equivalence classes of short exact sequences

$$0 \to Y \to Z \to X \to 0.$$

Clearly, any such exact sequence over A/J is also one over A. On the other hand, since $e{\cdot}Y = e{\cdot}X = 0$ gives that $e^2{\cdot}Z = e{\cdot}Z = 0$. In general, though, for any $Z \in A$-mod we have $Z \in A/J$-mod if and only if $J \cdot Z = 0$, which occurs if and only if $e \cdot Z = 0$. Finally, we have $\mathrm{Hom}_{A/J}(Z_1, Z_2) \cong \mathrm{Hom}_A(Z_1, Z_2)$ for any pair of A/J modules and so any two short exact sequences are isomorphic over A/J if and only if they are isomorphic over A.

Now for $m > 1$, we do a dimension shift. Let P_A be a projective cover of X in A. Then we have

$$0 \to K_A \to P_A \to X \to 0$$

short exact. Multiplying by the idempotent e we get

$$0 \to eK_A = \Delta(\lambda)^l \to eP_A = \Delta(\lambda)^l \to eX = 0 \to 0$$

short exact. Then putting $K_{A/J} = K_A/eK_A$ and $P_{A/J} = P_A/eP_A$, this last projective. We derive the short exact sequence

$$0 \to K_{A/J} \to P_{A/J} \to X \to 0.$$

Putting all this together we get the diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \Delta(\lambda)^l \arrow[r] \arrow[d] & \Delta(\lambda)^l \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_A \arrow[r] \arrow[d] & P_A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K_{A/J} \arrow[r] \arrow[d] & P_{A/J} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

with the rows and columns exact. So from long exact sequences in cohomology, we get

$$\begin{aligned}
\mathrm{Ext}_A^m(X,-) &= \mathrm{Ext}_A^{m-1}(K_A,-) \\
&= \mathrm{Ext}_A^{m-1}(K_{A/J},-) \text{ since } \Delta(n) \text{ is projective and } m \geq 2 \\
&= \mathrm{Ext}_{A/J}^{m-1}(K_{A/J},-) \text{ by induction} \\
&= \mathrm{Ext}_{A/J}^{m}(X,-).
\end{aligned}$$

So we have $\mathrm{Ext}_{A/J}^m(\mathcal{F}(\Delta),\mathcal{F}(\nabla)) = 0$.

Now we show the reverse inclusion. So assume $X \in A$-mod with $\mathrm{Ext}_A^1(\Delta(\mu),X) = 0$ for all μ. We need to show $X \in \mathcal{F}(\nabla)$. Again, this goes by induction on the length of the partial ordering. If $l(\mu) = 1$ then A is semi-simple, all Ext-groups vanish and $\mathcal{F}(\Delta)^{\perp} = \mathcal{F}(\nabla)$, and the inclusion is clear.

For the induction step, we again let e be the idempotent associated to a maximal element of Λ and $J = AeA$. We let X' be the maximal A/J-submodule of X and let X'' be the cokernel of $X' \hookrightarrow X$. Since the socle of X'' is $\oplus L(\lambda)$ and the socle of a module must agree with the socle of its injective envelope we have that the injective envelope of X'', $I(X'')$, is $\oplus\nabla(\lambda)$. We get a short exact sequence

$$0 \to X'' \to \nabla(\lambda)^l \to C \to 0,$$

with C an A/J module. Applying $\mathrm{Hom}_A(\Delta(\mu), -)$ and passing to the long exact sequence in homology, we have

$$\begin{aligned} 0 \to \mathrm{Hom}_A(\Delta(\mu), X'') \to \quad & \mathrm{Hom}_A(\Delta(\mu), \nabla(\lambda)^l) \to \\ & \mathrm{Hom}_A(\Delta(\mu), C) \to \mathrm{Ext}^1_A(\Delta(\mu), X'') \to 0. \end{aligned}$$

For $\mu \neq \lambda$ there are no maps $\Delta(\mu) \to \nabla(\lambda)$ and so

$$\mathrm{Hom}_A(\Delta(\mu), X'') = 0.$$

(We will also need in the last step of the proof that this gives

$$\mathrm{Hom}^1_A(\Delta(\mu), C) \cong \mathrm{Ext}^1_A(\Delta(\mu), X'').)$$

Now applying $\mathrm{Hom}_A(\Delta(\mu), -)$ to

$$0 \to X' \to X \to X'' \to 0,$$

we get

$$\begin{aligned} 0 \to \mathrm{Hom}_A(\Delta(\mu), X') \to \quad & \mathrm{Hom}_A(\Delta(\mu), X) \to \\ & \mathrm{Hom}_A(\Delta(\mu), X'') \to \mathrm{Ext}^1_A(\Delta(\mu), X') \to 0. \end{aligned}$$

Now since $\mathrm{Hom}_A(\Delta(\mu), X'') = 0$ we get $\mathrm{Ext}^1_A(\Delta(\mu), X') = 0$. This Ext-group also vanishes over A/J and we conclude, by induction, that $X' \in \mathcal{F}_{A/J}(\nabla) \subset \mathcal{F}_A(\nabla)$.

Thus we will be done if we show $X'' \cong \nabla(\lambda)^l$, that is $C = 0$. In the proof, it is sufficient to show $\mathrm{Hom}_A(\Delta(\mu), C) = 0$ since every simple summand of the socle of C is surjected upon by some $\Delta(\mu)$. Now since C is an A/J module we have $\mathrm{Hom}_A(\Delta(\lambda), C) = 0$.

Now suppose $\mu \neq \lambda$; then the above sequence continues

$$\mathrm{Ext}^1_A(\Delta(\mu), X') \to \mathrm{Ext}^1_A(\Delta(\mu), X) \to \mathrm{Ext}^1_A(\Delta(\mu), X'') \to \mathrm{Ext}^2_A(\Delta(\mu), X').$$

Since $X, X' \in \mathcal{F}(\nabla)$ gives $\mathrm{Ext}^1_A(\Delta(\mu), X) = \mathrm{Ext}^2_A(\Delta(\mu), X') = 0$, in turn forcing $\mathrm{Ext}^1_A(\Delta(\mu), X'') = 0$. But as remarked earlier $\mathrm{Hom}^1_A(\Delta(\mu), C) \cong \mathrm{Ext}^1_A(\Delta(\mu), X'')$, and the proof is complete.

Now we look at our examples

Example

$$A_i = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}.$$

Then the indecomposables are given by the diagram:

$$\begin{tikzcd}
 & & \begin{pmatrix} k \\ k \\ k \end{pmatrix} \arrow[dr] & & \\
 & \begin{pmatrix} k \\ k \\ 0 \end{pmatrix} \arrow[ur] \arrow[dr] & & \begin{pmatrix} \tilde{0} \\ k \\ k \end{pmatrix} \arrow[dr] & \\
\begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix} \arrow[ur] & & \begin{pmatrix} \tilde{0} \\ k \\ 0 \end{pmatrix} \arrow[ur] & & \begin{pmatrix} \tilde{0} \\ \tilde{0} \\ k \end{pmatrix}
\end{tikzcd}$$

where the down arrows are surjections and the up arrows are injections. The notation $\tilde{0}$ indicates that this copy of k has been identified with 0 on passing to a factor module. Now for the various orderings on Λ, we indicate which modules belong to $\mathcal{F}(\Delta)$, (symbolized by Δ), $\mathcal{F}(\nabla)$, (symbolized by ∇), or both (symbolized by $\diamondsuit$)

- $A_1 :\ 1 < 2 < 3$ Here, the Δ's are projective and the ∇'s are simple.

$$\begin{tikzcd}
 & & \diamondsuit \arrow[dr] & & \\
 & \diamondsuit \arrow[ur] \arrow[dr] & & \nabla \arrow[dr] & \\
\diamondsuit \arrow[ur] & & \nabla \arrow[ur] & & \nabla.
\end{tikzcd}$$

- $A_2 :\ 1 > 2 > 3$ Here, the Δ's are simple and the ∇'s are injective.

$$
\begin{tikzcd}[column sep=small]
 & & \Diamond \arrow[dr] & & \\
 & \Delta \arrow[ur] \arrow[dr] & & \Diamond \arrow[dr] & \\
\Delta \arrow[ur] & & \Delta \arrow[ur] & & \Diamond.
\end{tikzcd}
$$

- $A_3 : \ 1 > 3 > 2$ Here we have

$$\Delta(1) = L(1), \Delta(2) = L(2), \Delta(3) = \left(\begin{array}{c} L(3) \\ L(2) \end{array} \right)$$

and

$$\nabla(1) = \left(\begin{array}{c} L(3) \\ L(2) \\ L(1) \end{array} \right), \nabla(2) = L(2), \nabla(3) = L(3).$$

$$
\begin{tikzcd}[column sep=small]
 & & \Diamond \arrow[dr] & & \\
 & \Delta \arrow[ur] \arrow[dr] & & \Diamond \arrow[dr] & \\
\Delta \arrow[ur] & & \Diamond \arrow[ur] & & \nabla.
\end{tikzcd}
$$

Theorem 2.3 *(Ringel [77]) Let $(A, \leq)$ be a quasi-hereditary algebra. Then $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = add\ T$, the full subcategory of A-mod consisting of all direct sums of direct summands of a module $T = \oplus_{\lambda \in \Lambda} T(\lambda)$, where T has, up to isomorphism, precisely card(Λ) indecomposable direct summands, $T(\lambda)$.*

T is called the *(characteristic) tilting module* of $(A, \leq)$. Often the $T(\lambda)$'s are also called *tilting modules*. The meaning of tilting will be clarified later on.

Proof: We begin by proving uniqueness. Let X and $Y \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$, both indecomposable. Both have Δ-good filtrations and assume that $\Delta(\lambda)$ embeds into both. Then we have short exact sequences:

$$0 \to \Delta(\lambda) \to X \to X' \to 0,$$

and

$$0 \to \Delta(\lambda) \to Y \to Y' \to 0,$$

with both X' and $Y' \in \mathcal{F}(\Delta)$. We wish to prove $X \cong Y$. To do so apply $\mathrm{Hom}(-, Y)$ to the first of the above exact sequences and consider the long exact sequence in cohomology:

$$0 \to \mathrm{Hom}(X', Y) \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}(\Delta(\lambda), Y) \to \mathrm{Ext}^1(X', Y) = 0.$$

This last equality since $X' \in \mathcal{F}(\Delta)$ and $Y \in \mathcal{F}(\nabla)$. So $\mathrm{Hom}(X, Y)$ surjects onto $\mathrm{Hom}(\Delta(\lambda), Y)$; let $\alpha \in \mathrm{Hom}(X, Y)$ be a lift of the injection of $\Delta(\lambda)$ into Y. We get a commutative diagram:

$$\begin{tikzcd}
\Delta(\lambda) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\alpha"'] & X' \\
\Delta(\lambda) \arrow[r] & Y \arrow[r] & Y'
\end{tikzcd}$$

We can repeat this applying $\mathrm{Hom}(-, X)$ to the second short exact sequence to get β lifting the embedding of $\Delta(\lambda)$ into X. Composing we get $\alpha \circ \beta : X \to Y \to X$ which sends $\Delta(\lambda)$ to $\Delta(\lambda)$. Because $\Delta(\lambda)$ has a unique simple top $\mathrm{End}_A(\Delta(\lambda)$ is a skew field. Therefore, since the restriction of $\alpha \circ \beta$ to $\Delta(\lambda)$ is not zero, the restriction must be an automorphism. In particular, $\alpha \circ \beta$ is non-nilpotent. But X is indecomposable implying that $\mathrm{End}_A X$ is a local artinian ring and so any non-nilpotent automorphism must be an isomorphism.

Next we show their existence. To begin, note that in the case that λ is a minimal weight, we have $L(\lambda) = \Delta(\lambda) = \nabla(\lambda) = T(\lambda)$, so this case presents no problem. Otherwise, given a non-minimal λ we want a module with $\Delta(\lambda)$ on the bottom, and so a module constructed by extensions. More precisely, we form universal extensions as follows. If $\mathrm{Ext}^1(\Delta(\mu), \Delta(\lambda)) \neq 0$ for some μ, then choose some such μ maximal. Since the kernel of $P(\mu) \to \Delta(\mu)$ is filtered by $\Delta(\nu)$'s with $\nu < \mu$ and $\Delta(\lambda)$ is filtered by $L(\nu)$'s with $\nu \leq \lambda$ we have that μ must be strictly less than λ. Suppose $\dim_{\mathrm{End}(\Delta(\mu))} \mathrm{Ext}^1(\Delta(\mu), \Delta(\lambda)) = l$. Pick a basis of l short exact sequences for this Ext group:

$$0 \to \Delta(\lambda) \to E_i \to \Delta(\mu) \to 0,$$

for $0 < i \leq l$. Add them together and form a pushout diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Delta(\lambda)^l \arrow[r] \arrow[d, "\text{codiag}"'] & \oplus_i E_i \arrow[r] \arrow[d] & \oplus \Delta(\mu) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & \Delta(\lambda) \arrow[r] & \text{Pushout} = X \arrow[r] & \Delta(\mu)^l \arrow[r] & 0.
\end{tikzcd}$$

Applying $\mathrm{Hom}(\Delta(\mu), -)$ and passing to the long exact sequence in homology, we get

$$\begin{aligned}
0 \to \quad & \mathrm{Hom}(\Delta(\mu), \Delta(\lambda)) \to \mathrm{Hom}(\Delta(\mu), X) \to \\
& \mathrm{Hom}(\Delta(\mu), \Delta(\mu)^l) \to \mathrm{Ext}^1(\Delta(\mu), \Delta(\lambda)) \to \mathrm{Ext}^1(\Delta(\mu), X) \to 0.
\end{aligned}$$

Since no $\Delta(\mu)$ is a direct summand of X, all maps $\Delta(\mu) \to X$ must lie in the image of $\Delta(\lambda)$, thus we have $\mathrm{Hom}(\Delta(\mu), \Delta(\lambda)) \cong \mathrm{Hom}(\Delta(\mu), X)$. Further, we have that both dim $\mathrm{Hom}(\Delta(\mu), \Delta(\mu)^l)$ and dim $\mathrm{Ext}^1(\Delta(\mu), \Delta(\lambda))$ equal l and so $\mathrm{Hom}(\Delta(\mu), \Delta(\mu)^l) \cong \mathrm{Ext}^1(\Delta(\mu), \Delta(\lambda))$. So $\mathrm{Ext}^1(\Delta(\mu), X) = 0$. Thus there is no extension

$$0 \to X \to E \to \Delta(\mu) \to 0,$$

and $X \in \mathcal{F}(\Delta)$.

The construction continues recursively. If there is a ν with $\mathrm{Ext}^1(\Delta(\nu), X) \neq 0$, then choose ν maximal. We have $\nu < \lambda$ and $\nu \not\geq \mu$. We then form a universal extension

$$0 \to X \to X' \to \Delta(\nu)^m \to 0.$$

Arguing as above we have that $\mathrm{Ext}^1(\Delta(\nu), X') = 0$. Further, since $\nu \not\geq \mu$ both $\mathrm{Hom}(\Delta(\lambda), \Delta(\nu)^m) = 0$ and $\mathrm{Ext}^1(\Delta(\lambda), \Delta(\nu)^m) = 0$. So $\mathrm{Ext}^1(\Delta(\lambda), X') = \mathrm{Ext}^1(\Delta(\lambda), X) = 0$. Continuing in this manner we eventually exhaust the set $\Lambda' = \{\mu \in \Lambda | \mu < \lambda\}$ and so obtain a module, $X \in \mathcal{F}(\Delta)$, such that $\Delta(\lambda) \subset X$ is the bottom factor of the Δ-good filtration of X and such that $\mathrm{Ext}^1(\Delta(\mu), X) = 0$ for all $\mu \in \Lambda$. Hence $X \in \mathcal{F}(\Delta)^\perp = \mathcal{F}(\nabla)$ by 2.1 implying that $X \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.

It remains to show that X is indecomposable. We show that End X is local, doing so by induction. To begin, $\Delta(\lambda)$ is clearly indecomposable. The induction step relies on the short exact sequence:

$$0 \to X' \to X \to \Delta(\mu)^l \to 0,$$

where X is the relevant universal extension and X' is indecomposable by induction. Now let $0 \neq e : X \to X$ be an idempotent in $\mathrm{End}(X)$. We want to show that $e = 1$. Now X' is filtered by Δ's with indices not smaller than or equal to μ. This implies that $\mathrm{Hom}(X', \Delta(\mu)) = 0$ which, in turn, implies that $e|_{X'}$ maps X' to X'. Now if $e|_{X'} = 0$ the sequence splits and we have a contradiction. Thus by induction $e|_{X'} = 1|_{X'}$. But then $(e-1)|_{X'} = 0$. But then if $e \neq 1$ then $e - 1$ provides a splitting of the above exact sequence and we again get a contradiction.

Let us follow the construction in some of the examples:

Example

$$A_i = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

$A_1 :$ $1 < 2 < 3$ Here the Δ's are projective and so there are no non-trivial extensions; $T(\lambda) = P(\lambda)$.

$A_2 :$ $3 < 2 < 1$ Here, the Δ's are simple. $\Delta(3) = L(3) = \nabla(3) = T(3)$. We construct $T(2)$: $\Delta(2) = L(2)$ extends with $L(3) = \Delta(3)$ with

$$0 \to \Delta(2) \to ? \to \Delta(3) \to 0.$$

Here $\mathrm{Ext}^1(\Delta(3), \Delta(2))$ is one dimensional and so we get the universal extension

$$0 \to \Delta(2) \to I(2) \cong \begin{matrix} L(3) \\ L(2) \end{matrix} \to \Delta(3) \to 0$$

and no further extension is possible. So $T(2) = I(2)$. Analogously, we get $T(1) = I(1)$ after the formation of two universal extensions.

$A_3 :$ $2 < 3 < 1$ We have

$$\Delta(1) = L(1), \nabla(1) = \begin{matrix} L(3) \\ L(2) \\ L(1) \end{matrix}$$

$$\Delta(2) = \nabla(2) = L(2)$$

$$\Delta(3) = \begin{matrix} L(3) \\ L(2) \end{matrix}, \nabla(3) = L(3).$$

We read immediately that $T(2) = \Delta(2) = \nabla(2) = L(2)$ and $T(3) = \Delta(3) = \begin{matrix} \nabla(3) \\ \nabla(2) \end{matrix}$. We see that the number of factors in the Δ- good and in the ∇-good filtrations may vary.

Now $\Delta(1)$ extends with $\Delta(3)$:

$$0 \to \Delta(1) \to I(1) = P(1) \to \Delta(3) \to 0,$$

and here the process stops; so $T(1) = \begin{array}{c}\Delta(3)\\ \Delta(1)\end{array}$.

At this point it is important to note that the order with which one takes the universal extensions is critical. For example, in this case, $\Delta(1)$ extends with $\Delta(2)$ to give $\begin{array}{c}L(2)\\ L(1)\end{array}$ with no further extensions. However, this is not a correct characterization of $T(1)$. Note that these T have already been seen in Example 2 where the tilting modules are there indicated by the $\diamondsuit$.

For the last example: $A_4 \subset \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$, with $a_{11} = a_{33}$. with $1 < 2$.

Here we have

$$\Delta(1) = L(1), \text{ and } \Delta(2) = P(2) = \begin{array}{c}L(2)\\ L(1)\end{array}$$

and

$$\nabla(1) = L(1), \text{ and } \nabla(2) = I(2) = \begin{array}{c}L(1)\\ L(2)\end{array}.$$

As usual, we have $\Delta(1) = \nabla(1) = T(1)$. also $\Delta(2)$ extends with $\Delta(1)$ to $P(1) = I(1) = \begin{array}{c}L(1)\\ L(2)\\ L(1)\end{array}$.

Notice that the homomorphism $\Delta(\lambda) \to T(\lambda)$ played a crucial role in the proof of the previous theorem. In fact, this is the link to general Auslander-Reiten theory as expressed in the upcoming corollary:

Definition 2.1 *Let $\mathcal{C}'$ be a subcategory of $\mathcal{C}$. We call $M \to N$, with $M \in \mathcal{C}$ and $N \in \mathcal{C}'$, a* left $\mathcal{C}'$-approximation *of M, if for all $X \in \mathcal{C}'$ all maps $M \to X$ factor through N. Dually, $N' \to M$ is a* right $\mathcal{C}'$-approximation *if for all $X \in \mathcal{C}'$ all maps $X \to M$ factor through N.*

And we arrive at the

Corollary 2.4 *For each $\lambda \in \Lambda$, there is a short exact sequence*

$$0 \to \Delta(\lambda) \to T(\lambda) \to U(\lambda) \to 0,$$

with $U(\lambda)$ filtered by $\Delta(\mu)$'s with $\mu < \lambda$. In particular, the composition factor $L(\lambda)$ occurs uniquely in a composition series for $T(\lambda)$; that is, $T(\lambda)$ is determined by its highest weight. Moreover, $\Delta(\lambda) \to T(\lambda)$ is a left $\mathcal{F}(\nabla)$ approximation of $\Delta(\lambda)$.

Dually, there is a short exact sequence

$$0 \to V(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0,$$

with $V(\lambda)$ filtered by $\nabla(\mu)$'s with $\mu < \lambda$. Further $T(\lambda) \to \nabla(\lambda)$ is a right $\mathcal{F}(\Delta)$ approximation of $\nabla(\lambda)$.

Proof: The veracity of the short exact sequence is clear. To substantiate the claim that $\Delta(\lambda) \to T(\lambda)$ is a right approximation, apply $\mathrm{Hom}(-, X)$, $X \in \mathcal{F}(\nabla)$, to $0 \to \Delta(\lambda) \to T(\lambda) \to U(\lambda) \to 0$ to get a long exact sequence in cohomology:

$$0 \to \mathrm{Hom}(U(\lambda), X) \to \mathrm{Hom}(T(\lambda), X) \to \mathrm{Hom}(\Delta(\lambda), X) \to \mathrm{Ext}^1(U(\lambda), X) = 0.$$

This last equality holds since $U(\lambda) \in \mathcal{F}(\nabla)$ and so $\mathrm{Hom}(T(\lambda), X)$ surjects onto $\mathrm{Hom}(\Delta(\lambda), X)$ and the second claim is clear.

The situation for $\nabla(\lambda)$ is dual.

Remark 2.1 *Note that the highest weight composition factor, $L(\lambda)$, determines $T(\lambda)$ uniquely and so the dual construction starting with $\nabla(\lambda)$ produces the same $T(\lambda)$.*

In fact, one can copy the universal construction method to produce such approximations for all objects in $\mathcal{F}(\nabla)$ or $\mathcal{F}(\Delta)$, (see Ringel [77] or Auslander-Reiten [11]). Both categories are closed under extensions or direct summands. In the language of that theory, we have

Theorem 2.5 *(Ringel [77], Auslander-Reiten [11], Auslander-Smalø[13], [14]) The categories $\mathcal{F}(\nabla)$ and $\mathcal{F}(\Delta)$ are functorially finite. In particular, they have almost split sequences.*

Proof: For a discussion of the terms and a proof see Auslander-Smalø[14].

One particular consequence of the above construction is :

Corollary 2.6 *1.* $T = \oplus T(\lambda)$ *has no self-extensions, i.e.,* $Ext^i(T,T) = 0$ *for all* i.

2. There exists a long exact sequence

$$0 \to A \to T_1 \to \ldots \to T_l \to 0,$$

for some l *less than or equal to the global dimension of* A *and all* $T_i \in$ *add* T.

3. The number of direct summands equals $|\Lambda|$ = *the number of isomorphism classes of simples.*

Example A_4. Here

$$T(1) = L(1),\ T(2) = P(1) = I(1) = \begin{array}{c} L(1) \\ L(2) \\ L(1) \end{array}$$

and

$$A = \begin{array}{c} L(1) \\ L(2) \\ L(1) \end{array} \oplus \begin{array}{c} L(2) \\ L(1) \end{array}.$$

So we get a short exact sequence

$$0 \to A \to T_1 = \begin{array}{c} L(1) \\ L(2) \\ L(1) \end{array} \oplus \begin{array}{c} L(1) \\ L(2) \\ L(1) \end{array} \to T_2 = L(1) \to 0.$$

Corollary 2.5 says that T is a *generalized tilting module* and Happel's theorem [53] says that:

Corollary 2.7 *There is a derived equivalence*

$$D^b(A\text{-}mod) \cong D^b(R\text{-}mod)$$

where $R = End_A(T)$ *is called the* Ringel dual *of* A.

We will take a closer look at the Ringel dual in the next section.

We finish the chapter with one last consequence of the approximation property:

Corollary 2.8 *Let $X \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. Then $T(\lambda)$ occurs as a direct summand of X if and only if the natural homomorphism $\Delta(\lambda) \to L(\lambda) \to \nabla(\lambda)$ factors through X.*

Proof: Assume first that $T(\lambda)$ occurs as a summand of X. We then have

$$\begin{tikzcd}
 & T(\lambda) \arrow[d, dashed] & \\
\Delta(\lambda) \arrow[ur] \arrow[r] & X \arrow[r] \arrow[d, dashed] & \nabla(\lambda) \\
 & T(\lambda) \arrow[ur] &
\end{tikzcd}$$

and so $\Delta(\lambda) \to L(\lambda) \to \nabla(\lambda)$ factors through X.

On the other hand, suppose $\Delta(\lambda) \to L(\lambda) \to \nabla(\lambda)$ factors through X. Then we have the diagram and the map $T(\lambda) \to X \to T(\lambda)$ cannot carry $L(\lambda)$ to 0 and so must be an isomorphism as above. Thus $T(\lambda)$ must be a direct summand of X.

3 Ringel Duals of quasi-hereditary algebras

The major goal of this section is to show that the Ringel dual, R, of a quasi-hereditary algebra, A, is again a quasi-hereditary algebra. But first we compute a few examples

Example

$$A_i = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

A_1 : $1 < 2 < 3$ Here the tilting modules are projective and so we get $R = A$.

A_2 : $1 > 2 > 3$ Here the tilting modules are injective and so, again, we have $R = A$.

A_3 : $1 > 3 > 2$ Here

$$T = \begin{matrix} L(3) \\ L(2) \\ L(1) \end{matrix} \oplus L(2) \oplus \begin{matrix} L(3) \\ L(2) \end{matrix} .$$

This gives us a new algebra

$$R = \begin{pmatrix} k & 0 & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix},$$

where "R_{ij}"$= \mathrm{Hom}_A(T(i), T(j))$.

A_4 : Here

$$T = \begin{matrix} L(1) \\ L(2) \\ L(1) \end{matrix} \oplus \begin{matrix} L(2) \\ L(1) \end{matrix} .$$

In this case R is the same algebra except that the indices are reversed.

Theorem 3.1 *(Ringel [77]) Let $(A, \leq)$ be a quasi-hereditary algebra and let $T = \oplus T(\lambda)$ be its characteristic tilting module. Then $R = End_A(T)$ is a quasi-hereditary algebra with respect to the opposite order on the poset. Moreover, the Ringel dual of R is Morita equivalent to A.*

Proof: The functor $F = \mathrm{Hom}_A(T, -) : A\text{-mod} \to \mathrm{End}_A(T)\text{-mod}$ behaves well with respect to direct sums. In particular, the indecomposable direct summands of T, $T(\lambda)$, are carried to the projectives, say $P_R(\lambda) = \mathrm{Hom}_A(T, T(\lambda))$. Since $T \in \mathcal{F}(\Delta)$, F is exact on $\mathcal{F}(\nabla) = \mathcal{F}(\Delta)^{\perp}$. Hence a ∇-filtration of any $M \in A$-mod is sent to a $F(\nabla)$-filtration of $F(M)$. Thus we try $F(\nabla(\lambda)) = \mathrm{Hom}_A(T, \nabla(\lambda))$ as the new $\Delta_R(\lambda)$. At least they filter the R projectives and the sequence

$$0 \to V(\lambda) \to T(\lambda) \to \nabla(\lambda) \to 0$$

goes to

$$0 \to \mathrm{Hom}_A(T, V(\lambda)) \to \mathrm{Hom}_A(T, T(\lambda)) \to \mathrm{Hom}_A(T, \nabla(\lambda)) \to 0.$$

So, $\Delta_R(\lambda)$ is a quotient of $P(\lambda)$ and the kernel is filtered by $\Delta_R(\mu)$, $\mu >_R \lambda$, by the construction of $V(\lambda)$, where $>_R$ is the opposite order on the poset defining the quasi-hereditary structure on A.

What are the composition factors of $\Delta_R(\lambda)$? We have $\Delta_R(\lambda) = \mathrm{Hom}_A(T, \nabla(\lambda)) = \mathrm{Hom}_A(\oplus_\nu T(\nu), \nabla(\lambda)) = \oplus_\nu \mathrm{Hom}_A(T(\nu), \nabla(\lambda))$. Now $\mathrm{Hom}_A(T(\nu), \nabla(\lambda))$ will certainly be 0 unless $L(\lambda) = \mathrm{soc}(\nabla(\lambda))$ occurs as a composition factor of $T(\nu)$. So the composition factors are $L(\nu)$, $\nu \leq_R \lambda$. Further, $\nu = \lambda$ occurs precisely once since $\mathrm{Hom}_A(T(\lambda), \nabla(\lambda))$ is one dimensional over $\mathrm{End}_A(\nabla(\lambda))$. This becomes clear on applying $\mathrm{Hom}_A(-, \nabla(\lambda))$ to the short exact sequence

$$0 \to \Delta(\lambda) \to T(\lambda) \to U(\lambda) \to 0.$$

The result is the short exact sequence

$$0 \to \mathrm{Hom}_A(U(\lambda), \nabla(\lambda)) \to \mathrm{Hom}_A(T(\lambda), \nabla(\lambda)) \to \mathrm{Hom}_A(\Delta(\lambda), \nabla(\lambda)) \to 0,$$

with $(U(\lambda), \nabla(\lambda) = 0$. So all the axioms for a quasi-hereditary algebra are satisfied with respect to the reverse order on the poset.

To prove the statement about the second Ringel dual, we have to find the characteristic tilting module of R. We already know that F turns a ∇_A-filtration into a Δ_R-filtration. So put $I = \oplus_{\lambda \in \Lambda} I_A(\lambda)$. Then $F(I)$ has a Δ_R

filtration. From the injectiveness of I we conclude $\mathrm{Ext}_A^{\geq 1}(\mathcal{F}(\nabla_A), I) = 0$. But F is an exact functor on $\mathcal{F}(\nabla_A)$ and so we get

$$\mathrm{Ext}_A^{\geq 1}(F\mathcal{F}(\nabla_A), F(I)) = \mathrm{Ext}_A^{\geq 1}(F\mathcal{F}(\Delta_R), F(I)) = 0.$$

So $F(I) \in \mathcal{F}(\Delta_R)$. By construction it must be, up to multiplicity, T_R. But then $\mathrm{End}_R(F(I)) = \mathrm{End}_A(I) = A$ and the proof is complete.

Now we check it on the examples.

Example

$$A_i = \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

$A_1:$ $1 < 2 < 3$ As we have already seen the fact that the tilting modules are projective implies that the underlying algebra of the Ringel dual is the original algebra. So, in this case, $\mathrm{Hom}_A(T, \nabla(\lambda))$ must take the Δ's to the ∇'s. Further, since the projectives of the Ringel dual must be the image of the direct summands of the characteristic tilting module we have that the projectives are carried to the projectives.

$A_2:$ $1 > 2 > 3$ In this case, the tilting modules are the ∇'s which are also the injectives, the Δ's are simple and the ∇'s are injective. Thus we get that the Δ_R's are projective and the same order reversing correspondence we got in the above case.

$A_3:$ $1 > 3 > 2$ Here

$$T = T(1) \oplus T(2) \oplus T(3) = \begin{matrix} L(3) \\ L(2) \\ L(1) \end{matrix} \oplus L(2) \oplus \begin{matrix} L(3) \\ L(2) \end{matrix}.$$

We have the following identifications

$$\Delta(2) = \nabla(2) = L(2)$$

$$\Delta(3) = \begin{matrix} L(3) \\ L(2) \end{matrix}, \nabla(3) = L(3)$$

$$\Delta(1) = L(1), \nabla(1) = \begin{matrix} L(3) \\ L(2) \\ L(1) \end{matrix}.$$

We then get

$\mathrm{Hom}_A(T, \nabla(1)) = \mathrm{Hom}_A(T(1), \nabla(1)) = \mathrm{Hom}_A(\nabla(1), \nabla(1))$ is simple

$\mathrm{Hom}_A(T, \nabla(2)) = \mathrm{Hom}_A(T(2), \nabla(2)) = \mathrm{Hom}_A(\nabla(2), \nabla(2))$ is simple

and

$$\mathrm{Hom}_A(T, \nabla(3)) \text{ has two factors: } \begin{matrix} L(3) \\ L(1) \end{matrix} .$$

We can conclude

$$\Delta_R(1) = L(1), \Delta_R(2) = L(2), \text{ and } \Delta_R(3) = \begin{matrix} L(3) \\ L(1) \end{matrix} .$$

and

$$R = \begin{pmatrix} k & 0 & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix},$$

with the ordered poset $1 < 3 < 2$.

A_4 Given that $(1 < 2)$

$$\Delta(1) = L(1), \ \Delta(2) = \begin{matrix} L(2) \\ L(1) \end{matrix}$$

$$\nabla(1) = L(1), \ \nabla(2) = \begin{matrix} L(1) \\ L(2) \end{matrix}$$

$$T(1) = L(1), \ T(2) = \begin{matrix} L(1) \\ L(2) \\ L(1) \end{matrix} = P(1) = I(1).$$

Applying F, we have

$$\mathrm{Hom}_A(T, \nabla(1)) = L(1)$$

$$\mathrm{Hom}_A(T, \nabla(2)) = \mathrm{Hom}_A(T(2), \nabla(2)) \text{ is simple}$$

Again, R is the same algebra except that the indices are reversed.

4 The Chief Examples

Chief among the examples of quasi-hereditary algebras are category $\mathcal{O}$, Kac-Moody algebras, the Frobenius kernels, and the Schur algebras, including the q-Schur algebras. In this chapter, we will describe some of these naturally occurring algebras. We begin with category $\mathcal{O}$.

4.1 Category $\mathcal{O}$

Let $\mathfrak{g}$ be a finite dimensional semisimple complex Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ and a Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_- \oplus \mathfrak{n}_+$; then $\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$ and $\mathfrak{b}_- = \mathfrak{n}_- \oplus \mathfrak{h}$ are Borel subalgebras. The corresponding universal enveloping algebras are denoted $\mathcal{U}(\mathfrak{g})$, $\mathcal{U}(\mathfrak{b}_+)$ and so on. The objects of the *BGG-category* $\boldsymbol{\mathcal{O}}$, as originally defined by J.Bernstein, I.M.Gelfand and S.I.Gelfand in [16], are left $\mathfrak{g}$-modules M with the following properties:

1. M is finitely generated as $\mathcal{U}(\mathfrak{g})$-module,
2. M is $\mathfrak{h}$ diagonalizable (that is: $M = \oplus_{\mu\in\mathfrak{h}^*} M_\mu$, where M_μ is the μ-weight space $\{m \in M : h \cdot m = \mu(h)m \text{ for all } h \in \mathfrak{h}\}$),
3. M is $\mathfrak{n}_+$-finite (that is, for each $m \in M, \mathcal{U}(\mathfrak{n}_+) \cdot m$ is finite dimensional over $\mathbb{C}$).

The morphisms in $\mathcal{O}$ are arbitrary $\mathfrak{g}$-module homomorphisms. (Note that this definition, and all that follows, depends on the choice of the Cartan decomposition of $\mathfrak{g}$.)

Important objects of category $\mathcal{O}$ are the highest weight modules. Given any $\mathfrak{g}$-module, M, an element $m \in M_\lambda$ (for some λ) which is annihilated by $\mathfrak{n}_+$ is called a *highest weight vector* of weight λ. If m generates M, then M is called a *highest weight module* of highest weight λ. All finite dimensional and many other simple $\mathfrak{g}$-modules (in particular, all simple objects in $\mathcal{O}$) are highest weight modules.

As a special case of highest weight modules, the category $\mathcal{O}$ contains the *Verma modules*. Given a fixed weight $\lambda \in \mathfrak{h}^*$, the Verma module associated to λ (i.e. having highest weight λ) is defined as

$$\Delta(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_+)} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the one-dimensional representation on which each $h \in \mathfrak{h} \subset \mathfrak{b}_+$ acts by multiplication by $\lambda(h)$ and $\mathfrak{b}_+$ acts via the quotient map $\mathfrak{b}_+ \to \mathfrak{h}$. The $\Delta(\lambda)$ also have another definition. Let $I(\lambda)$ be the left ideal in $\mathcal{U}(\mathfrak{g})$ which is generated by $\mathfrak{n}_+$ together with all elements $h - \lambda(h)$ for $h \in \mathfrak{h}$; then $\Delta(\lambda)$ is the quotient $\mathcal{U}(\mathfrak{g})/I(\lambda)$.

As $\mathcal{U}(\mathfrak{n}_-)$-modules, there is an isomorphism $\mathcal{U}(\mathfrak{n}_-) \cdot v_\lambda \cong \Delta(\lambda)$, sending $u \cdot v_\lambda$ to $u \otimes 1$, where v_λ is a generator of the free $\mathcal{U}(\mathfrak{n}_-)$-module $\mathcal{U}(\mathfrak{n}_-) \cdot v_\lambda$. This is a consequence of the Poincaré-Birkhoff-Witt theorem, which implies that $\mathcal{U}(\mathfrak{g})$ is free over $\mathcal{U}(\mathfrak{b}_+)$ (on each side). Hence $\Delta(\lambda)$ is a free $\mathcal{U}(\mathfrak{n}_-)$-module of rank one. It follows that the Verma module $\Delta(\lambda)$ is the universal highest weight module corresponding to the weight λ; it maps onto any other highest weight module corresponding to the same weight λ.

Our next step is to show that these Δ's have a composition series which makes them appropriate choices for the standard objects for some quasi-hereditary algebra. First, we have that $\Delta(\lambda)$ has a unique one dimensional λ-weight space. As such it surjects onto the simple $L(\lambda)$ with multiplicity 1. In addition, we have the following classical theorem usually referred to as the *BGG theorem*.

Theorem 4.1 *(Bernstein-Gelfand-Gelfand [16]) Let λ and μ be elements of $\mathfrak{h}^*$.*

1. *Then the $\mathbb{C}$-dimension of $Hom_{\mathcal{O}}(\Delta(\lambda), \Delta(\mu))$ is either 0 or 1. All non-zero homomorphisms are injective.*

2. *The following assertions are equivalent:*

   (a) *There is an inclusion $\Delta(\lambda) \subset \Delta(\mu)$;*

   (b) *the simple module $L(\lambda)$ is a composition factor of $\Delta(\mu)$;*

   (c) *there are positive roots $\gamma_1, \ldots, \gamma_n$ such that there is a chain of inequalities $\mu \geq s_{\gamma_1}(\mu) \geq \ldots \geq s_{\gamma_n} \cdots s_{\gamma_1}(\mu) = \lambda$ (where s_γ denotes the reflection associated with γ).*

Remark 4.1 *Recall that $\mathfrak{h}^*$ has a basis of* simple roots*, say $\lambda_1, \ldots, \lambda_l$. Let s_{λ_i} be reflection in the hyperplane perpendicular to λ_i and passing through the origin. Then these reflections generate the Weyl group.*

The category $\mathcal{O}$ decomposes into a direct sum of categories $\mathcal{O}_\theta$, indexed over central characters θ of $\mathcal{U}(\mathfrak{g})$. To define these *blocks* of category $\mathcal{O}$, begin by fixing a central character θ, that is a homomorphism from the center $Z(\mathfrak{g})$

of $\mathcal{U}(\mathfrak{g})$ to $\mathbb{C}$. Given a module M in $\mathcal{O}$, the subset of M containing all m which are annihilated by a power of $z-\theta(z)$, $z \in Z(\mathfrak{g})$, is the direct summand of M which lies in $\mathcal{O}_\theta$. Under this construction each block has but a finite number of simples and the corresponding weights correspond to Weyl group orbits acting on $\mathfrak{h}^*$. Further, as we will later need, the respective Hom sets are finite dimensional.

The categories, $\mathcal{O}_\theta$, have enough projective and injective objects. We will give an explicit construction of the projective objects. Fix a central character θ and a weight $\lambda \in \mathfrak{h}^*$. The goal is to construct an object Q in $\mathcal{O}$ (depending on λ and θ) which has a generator q with the property that for each M in $\mathcal{O}_\theta$

$$\mathrm{Hom}_{\mathcal{O}}(Q, M) \simeq M_\lambda$$

given by sending each $\phi \in \mathrm{Hom}_{\mathcal{O}}(Q, M)$ to $\phi(q) \in M$. Since taking weight spaces, M_λ, is exact, the functor $\mathrm{Hom}_{\mathcal{O}}(Q, -)$ is exact and so the direct summand of such a Q, lying in $\mathcal{O}_\theta$, must be a projective module and is the projective for which we search. How then is such a Q constructed?

Note first that multiplication by elements of $\mathfrak{n}_-$ moves one down in the weight lattice while multiplication by elements of $\mathfrak{n}_+$ moves one up. Since $\mathcal{O}_\theta$ has only finitely many simple modules with highest weights $\lambda_1, \ldots, \lambda_\nu$, it is possible to find an integer $N >> 0$ such that $(\mathfrak{n}_+)^N$ annihilates all vectors of weight λ of objects in the block $\mathcal{O}_\theta$. Let I be the left ideal in $\mathcal{U}(\mathfrak{g})$ which is generated by all elements $h - \lambda(h)$ (for $h \in \mathfrak{h}$) and by the Nth powers of $\mathfrak{n}_+$. Define Q as the quotient of $\mathcal{U}(\mathfrak{g})$ modulo the left ideal I and q as the image of 1. Then q is a generator of Q of weight λ; hence the evaluation map $\mathrm{Hom}_{\mathcal{O}}(Q, M) \to M_\lambda$ is well-defined and injective. Conversely, choose a vector v in M_λ. To see that there is a map $Q \to M$ sending $q \mapsto v$, note that there is a map $\mathcal{U}(\mathfrak{g}) \to M$ sending 1 to v. The kernel must contain $(\mathfrak{n}_+)^N$ by the choice of N and so must factor through Q.

So now we have that each block $\mathcal{O}_\theta$ has only finitely many simples indexed by a finite set of weights and that the category has enough projectives. How should the weights be ordered? The last assertion in the BGG-theorem defines a partial order on the set of weights. It turns out that this ordering corresponds to the Bruhat ordering on the Weyl group in the following way. If κ is a regular dominant weight and $\lambda = w_1 \cdot \kappa$ and $\mu = w_2 \cdot \kappa$ are in the orbit of κ under the dot action of the Weyl group, then $\lambda \leq \mu$ in the above ordering if and only if $w_1 \geq w_2$ in the Bruhat ordering.

The next step in constructing a quasi-hereditary algebra is to see how a finite dimensional algebra might arise. The key is the earlier cited property that the Hom sets are finite dimensional. Begin by fixing a finitely generated projective, P in $\mathcal{O}_\theta$, whose direct summands include a representative of each isomorphism class of indecomposable projective in the block. Let A be the endomorphism ring $\mathrm{End}_{\mathcal{O}}(P)$. Then the functor $\mathrm{Hom}_{\mathcal{O}}(P,-)$ defines an equivalence between $\mathcal{O}_\theta$ and mod-A. We call this algebra A 'the' algebra associated with the block $\mathcal{O}_\theta$. Up to Morita equivalence, it is uniquely defined.

To see that A is quasi-hereditary we need to know that each projective module has a finite filtration, the subquotients of which are Verma modules. To see this, consider the universal enveloping algebra $\mathcal{U}(\mathfrak{n}_+)$ and factor out the left ideal generated by $(\mathfrak{n}_+)^N$ (where N is the above integer). Since $\mathfrak{h}$ acts diagonally on $\mathcal{U}$, we may choose a basis of this vector space, $\bar{x}_1, \ldots, \bar{x}_s$ such that the elements $x_i \cdot q$, have weight λ_i in Q and $\lambda_i < \lambda_j$ implies $i > j$. Now since $x_1 \cdot q$ is anihilated by $\mathfrak{n}_+$ it is a highest weight vector of weight λ_1. So there is a mapping of $\Delta(\lambda_1)$ into Q whose image is the submodule of Q generated by $x_1 \cdot q$. That this submodule is isomorphic to the Verma module $\Delta(\lambda_1)$ follows from the PBW basis theorem. The quotient of Q modulo this submodule contains a submodule (generated by the image of $x_2 \cdot q$) which is isomorphic to $\Delta(\lambda_2)$ and so on. This gives not only the desired filtration, but also an ordering condition on the subquotients; the first subquotient has highest weight λ_1 being maximal among the λ_i, and so on. Direct summands inherit such filtrations.

In summary then, we have associated to each block of category $\mathcal{O}$, a quasi-hereditary algebra, $A = \mathrm{End}_{\mathcal{O}}(P_{\mathcal{O}_\theta})$. Each such block has a finite number of simples indexed by highest weights corresponding to some factor group of the Weyl group and ordered according to the Bruhat order on the Weyl group. The standard objects are given by the Verma modules.

4.2 Schur algebras

We next consider the classical Schur algebras (associated with the group GL_n); these are much better known than the other Schur algebras, and their elementary definition gives us the possibility to compute some explicit examples for later use. The generalized Schur algebras (associated with other reductive groups) are shortly discussed at the end of this subsection.

There are many ways to define these algebras. We will begin with a definition via the category of polynomial representations of GL_n. Let k be an infinite field and $G = GL_n(k)$, the general linear group over k. A finite dimensional representation V of G is given by sending the elements $g \in G$

to matrices in $\mathrm{End}_k(V)$. The entries of these matrices are functions f of the entries of g. The representation V is called a *polynomial representation* of G if all these functions f are polynomials in the *coordinate functions*, $c_{i,j}$, which sends g to its (i,j)th entry. Among the polynomial representations are the trivial representation, det sending g to $det(g)$, the natural n-dimensional representation, and the symmetric, divided powers and exterior algebras. Tensor products of polynomial representations are again polynomial.

Schur showed in his 1901 thesis that each polynomial representation is a direct sum of homogeneous representations (where the polynomials are homogeneous of a fixed degree). The category of polynomial representations of G of fixed homogeneous degree turn out to be a highest weight category; the resulting quasi-hereditary algebras are called *Schur algebras* (see Green [51] and Martin [63]).

There are other definitions of these algebras. The Schur algebra $S_k(n,r)$ depends on the field k and two integers n (fixed by G) and r (which will correspond to the degree of the homogeneous polynomials). It turns out that all these definitions give the same algebra:

1. Let V be the natural n-dimensional representation of G and $E = V^{\otimes r}$ its r-fold tensor product. Let G act diagonally by $g \cdot (e_1 \otimes \ldots \otimes e_r) = g \cdot e_1 \otimes \ldots \otimes g \cdot e_r$. This representation defines a ring homomorphism $\psi : kG \to End_k(E)$. The span of the image of ψ is $S_k(n,r)$.

2. The symmetric group S_r acts on E by place permutation from the right: $(e_1 \otimes \ldots \otimes e_r) \cdot \sigma = e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(r)}$. Then the Schur algebra is the centralizer of this action: $S_k(n,r) = \mathrm{End}_{kS_r}(E)$.

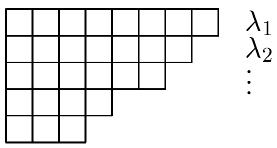
3. Since the base field k is infinite, the coefficient functions $c_{i,j}$ are algebraically independent. Hence they generate a polynomial ring $A_k(n)$. Its subspace $A_k(n,r)$ with basis the homogeneous polynomials of degree r is not an algebra, of course, but a coalgebra. Let $c_{i,j}$ be such a homogeneous polynomial (which is given by a pair of multi-indices $i = (i_1, \ldots, i_r)$ and $j = (j_1, \ldots, j_r)$, so $c_{i,j} = c_{i_1,j_1} \cdot \ldots \cdot c_{i_r,j_r}$). Then one defines the comultiplication by $\Delta(c_{i,j}) = \sum_l c_{i,l} \otimes c_{l,j}$ (which is the natural comultiplication induced by the multiplication of matrices by going from $G \times G \to G$ to $k[G] \to k[G] \otimes k[G]$, which gives $c_{i,j}(g \cdot h) = \sum_l c_{i,l}(g) \cdot c_{l,j}(h)$). For the counit, put $\epsilon(c_{i,j}) = \delta_{i,j}$ (the generalized Kronecker symbol). Then $S_k(n,r) = A_k(n,r)^*$ is the algebra which is dual to this coalgebra.

It is the ∇'s which are most naturally defined in this set up. Let $H \subset B \subset G$ be the diagonal matrices lying within a Borel subgroup of G. For any $\lambda \in H^*$, put k_λ equal to the one-dimensional representation where each $h \in H$ acts by multiplication by $\lambda(h)$ and each $b \in B$ acts via its image under the natural map $B \to H$. Such a representation can be lifted to G via induction in the sense of algebraic groups to

$$\nabla(\lambda) = \{\text{algebraic } f : G \to k_\lambda | f(xb) = \lambda(b)^{-1} f(x), \text{ for all } x \in G \text{ and for all } b \in B\}.$$

These modules sometimes go under the name of *Schur modules.* Their duals are the $\Delta(\lambda)$'s and are called the *Weyl modules.*

With these, $S_k(n,r)$ becomes a quasi-hereditary algebra. The simples are indexed by the partitions of r into at most n parts; thus $\lambda \in \Lambda(n,r)$ is given as $\lambda = (\lambda_1, \ldots, \lambda_n)$, the λ_i nonnegative integers satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\sum \lambda_i = r$. These are ordered by the so-called *dominance order*: $\trianglelefteq$. We have that $\lambda \trianglelefteq \mu$, if and only if, for all i, $\sum_1^i \lambda_j \leq \sum_1^i \mu_j$. The combinatorics and the characters, that is, the formal characters in terms of the dimensions of the weight spaces, of the Δ's and ∇'s are well known. To describe these we consider the *Young diagram* associated to each partition. That is, the subset of $\mathbb{N}_0 \times \mathbb{N}_0$, consisting of those ordered pairs (i,j), such that $0 < i \leq n$ and $0 < j \leq \lambda_i$. We picture such a diagram as:



(Fortunately, the geometry of the situation allows for only eight different conventions on how these are drawn. Unfortunately, each such convention is used by at least one person in the field.) These diagrams can then be filled by the integers $1, \ldots, n$. Such a filling is called *semi-standard*, sometimes *costandard*, if the numbers are weakly increasing along the rows and strictly increasing along the columns. It turns out that the standard module $\Delta(\lambda)$ and under the conventions implicit from the first section, the costandard module, $\nabla(\lambda)$ as well, has a basis indexed by the semi-standard fillings of the diagram for λ.

When char $k = 0$, these statements are shown by considering an idempotent $e \in S(n,r)$ such that $eS(n,r)e \cong S_r$, at least when $r \leq n$. Then the *Schur functor:* $M \mapsto eM$ maps representations of the Schur algebra to representations of the symmetric group. These are already known to be indexed by the appropriate partitions. To proof the case for general k, it turns out that the Schur algebra, $S(n,r)$ can be defined over the integers and then the Schur algebra over k is given by tensoring with k. This allows the structure of the Δ's and ∇'s to be given over the integers and then transported to the algebras over arbitrary k.

These algebras may be generalized to arbitrary algebraic groups. Let G be a semisimple algebraic group over an algebraically closed field, k, of arbitrary characteristic. In this situation, we have a coalgebra of regular functions on the underlying variety of the group given by rational functions $G \to k$. These have finite dimensional coalgebras analogous to the $A_k(n)$ above. which can be dualized to give one brand of the generalized Schur algebras. A twist is that these algebras give the *rational* representation theory of the underlying group ([35] [36]).

Recently Doty [43] has given another method for constructing generalizations that capture a polynomial representation theory. In this case, the arbitrary algebraic group, G, is embedded into some Gl_n and hence into some $M_n(k)$, the set of all matrices. Then Doty considers the regular functions on the Zariski closure of the embedding. This is again a coalgebra and has finite dimensional subcoalgebras whose duals give finite dimensional algebras.

4.3 Other Examples

We finish this section with a brief list of some other quasi-hereditary algebras:

1. One may replace $\mathfrak{g}$ by a *Kac-Moody algebra.* In this case there are analogs of category $\mathcal{O}$, except that the resulting categories are infinite highest weight categories and contain no projectives. Nevertheless, they may be described "locally" as the category of modules for a quasi-hereditary algebra.

2. Given semisimple $\mathfrak{g}$, we can replace $U(\mathfrak{g})$ by $U_q(\mathfrak{g})$, the quantized enveloping algebra or *quantum group* of which there are several versions. These are usually specified by generators and relations; details we will here omit. In these cases, we have Verma modules, highest weight representations and a form of category $\mathcal{O}$.

3. One may also form *q-Schur algebras*. As in the classical definition, there are many equivalent ways to define them. One is to place an action of $U_q(\mathfrak{gl}_\mathfrak{n})$ on $V^{\otimes r}$. The algebra $S_q(n, r)$ may then be taken as the image of $U_q(\mathfrak{gl}_\mathfrak{n})$ in End $V^{\otimes r}$. This turns out to be the endomorphism ring for $V^{\otimes r}$ considered as a particular module over the Hecke algebra associated to the symmetric group. These are quasi-hereditary algebras closely analogous to there classical cousins which, in fact, is what one obtains for $q = 1$. The interest in these algebras stems from the fact that on specialization of q to a root of unity their module category is related to that of the finite general linear groups in non-describing characteristic.

4. TG_1 the first Frobenius kernel is also a highest weight category, see, for example, Cline [18].

5 Fusion Rules over the Schur Algebras

For the current section we work over the Schur algebras. Our goal is to characterize the tilting modules and the Ringel duals. To do so we will need the fact that the tensor product of two tilting modules, taken over the ground field, is again a tilting module. This involves a little sleight of hand since the modules occurring in this product are modules over different Schur algebras. This result follows from the

Theorem 5.1 *If $M_1, \ldots, M_t$ are all in $\mathcal{F}(\nabla)$ (respectively, $\mathcal{F}(\Delta)$) then their tensor product, $M_1 \otimes_R \ldots \otimes_R M_t$ is also in $\mathcal{F}(\nabla)$ (respectively, $\mathcal{F}(\Delta)$).*

This is the work of many people. For algebraic groups, J.-p. Wang [84] worked out a special case and S. Donkin [34] settled most cases; Mathieu [64] settled the general case in an elegant way that relied on the previous work of Joseph and Polo (see van der Kallen [58]). Littelmann [62] gave a different proof using standard monomials. Padarowski [70] proved it for quantum groups using Lusztig's canonical basis, following a suggestion of Donkin. Unfortunately all these proofs are too complicated to go into here, but we do provide a few remarks.

We have already seen that $\mathcal{F}(\nabla) = \mathcal{F}(\Delta)^{\perp}$, so to show that a module has a ∇-good filtration, it is enough to check that $\mathrm{Ext}^1_A(\mathcal{F}(\nabla), M)$ vanishes. This is more or less what has been known in this context, even before the result of Ringel, as *Donkin's criterion.* This is used in all the proofs. Mathieu, and most of the others use geometric methods, in particular, the so-called *Frobenius splittings.*

An easy, but very nice, consequence of 5.1 is the following description of tilting modules for the Schur algebras.

Proposition 5.2 *(Donkin [40], though see Green's book [51]). The indecomposable tilting modules over the Schur algebra $S_k(n,r)$ are precisely the indecomposable direct summands of the modules $\Lambda^{\tilde{\alpha}}V := \Lambda^{\tilde{\alpha_1}}V \otimes \ldots \otimes \Lambda^{\tilde{\alpha_m}}V$ for $\alpha \in \Lambda^+(n,r)$ where $\tilde{\alpha}$ is the conjugate partition to the partition α.*

Proof: The proof begins with the fact that $\Lambda^0 = k$ and $\Lambda^i = \Delta(1^i) = \nabla(1^i)$ for $1 \le i \le n$. Each of these modules is simple and each of the weight spaces are one-dimensional. In fact, an exterior power $\Lambda^j(V)$ has a basis $e_{i_1} \wedge \ldots \wedge e_{i_j}$ for $i_1 < \ldots < i_j$ with dimension $\begin{pmatrix} n \\ j \end{pmatrix}$. It is the jth fundamental representation and is simple. $\Delta(1^j)$ has a basis consisting of the semistandard fillings of

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$$

Here the rows are trivial and the columns are strictly increasing. Since the two modules have the same dimension and $\Delta(1^j)$ is universal among highest weight modules, we may conclude that in fact $\Delta(1^j) \cong \Lambda^j(V)$. Then, by Theorem 5.1, we get that all Λ^α are tilting.

The converse is: given an indecomposable tilting module, T, it is characterized by its highest weight α. But the highest weight of $\Lambda^{\tilde{\alpha}}(V)$ corresponds to the unique vector

$$(e_1 \wedge \ldots \wedge e_{\tilde{\alpha}_1}) \otimes (e_1 \wedge \ldots \wedge e_{\tilde{\alpha}_2}) \otimes \ldots \otimes (e_1 \wedge \ldots \wedge e_{\tilde{\alpha}_{\alpha_1}}).$$

Now a given basis element e_i of V has weight ϵ_i, where $\epsilon_i(\mathrm{diag}(a_{11}, \ldots, a_{nn}) = a_{ii}$. So this highest weight vector has weight

$$(\epsilon_1 + \ldots + \epsilon_{\tilde{\alpha}_1}) + (\epsilon_1 + \ldots + \epsilon_{\tilde{\alpha}_2}) + \ldots + (\epsilon_1 + \ldots + \epsilon_{\tilde{\alpha}_{\alpha_1}})$$

which is equal to α and the proof is complete.

Remark 5.1 *For those readers familiar with Green's book you will note that this is how he constructs the simple modules. That is, he proves that there is a unique $L(\lambda)$ in $T(\lambda)$.*

Our next aim, the main result of the section, is to compute the Ringel dual of $S_K(n,r)$ in the case $n \geq r$.

Theorem 5.3 *(Donkin) For $S_K(n,r)$, $T = \oplus_{\alpha \in \Lambda^+(n,r)} T(\alpha)$, $S_K(n,r) = End_{S_K(n,r)}(T)$; that is, $S_K(n,r)$ is its own Ringel dual.*

Remark 5.2 *This also holds for the q-Schur algebras.*

Proof (sketch): Donkin has given two proofs. We will sketch the earlier one here and mention the latter in the context of Howe duality later. To consider the proof, we will need to collect some information on the Schur algebras. First, the indecomposable injective $S_K(n,r)$-modules are precisely the indecomposable direct summands of

$$S^{\alpha}(V) = S^{\alpha_1}(V) \otimes S^{\alpha_2}(V) \otimes \ldots \otimes S^{\alpha_n}(V),$$

where $\alpha \in \Lambda^+(n,r)$ and $S(V)$ is the symmetric powers of V. We will also need the fact that $S_K(n,r)$ is the dual of $A_K(n,r)$, the coalgebra of polynomials of degree r in n^2 variables.

As mentioned in the previous section, whenever $n \geq r$ there is an idempotent $e = e^2 \in S_K(n,r)$, such that $eS_K(n,r)e \cong k\Sigma_r$, the symmetric group algebra over k. In this situation, we have the functor, called the *Schur functor*, $\mathrm{Hom}_{S_K(n,r)}(S_K(n,r)e, -)$ which is the same as multiplying the given module on the left by e. This functor sends $S_K(n,r)$-modules to $k\Sigma_r$-modules. It is an equivalence of categories in the cases that the char $k = 0$ or is greater than r.

One checks that $eS^{\alpha}(E) \cong \mathrm{Ind}_{\Sigma_\alpha}^{\Sigma_r} k$, where k is the trivial module for the Young subgroup $\Sigma_\alpha = \Sigma_{\alpha_1} \times \Sigma_{\alpha_2} \times \ldots \times \Sigma_{\alpha_n}$ and that $e\Lambda^{\alpha}(E) \cong \mathrm{Ind}_{\Sigma_\alpha}^{\Sigma_r} \tilde{k}$, where $\tilde{k}$ is the sign representation for the Young subgroup. One accomplishes this by explicit computation.

One then verifies that the Schur functor induces an isomorphism

$$\mathrm{Hom}_{S_K(n,r)}(S^{\alpha}(E), S^{\beta}(E)) \cong \mathrm{Hom}_{k\Sigma_r}(eS^{\alpha}(E), eS^{\beta}(E))$$

and

$$\mathrm{Hom}_{S_K(n,r)}(\Lambda^{\alpha}(E), \Lambda^{\beta}(E)) \cong \mathrm{Hom}_{k\Sigma_r}(e\Lambda^{\alpha}(E), e\Lambda^{\beta}(E))$$

for all $\alpha, \beta \in \Lambda^+(n,r)$. In each case one shows that the dimensions over k are the same and that they are independent of k. Then one confirms injectivity by hand.

One then constructs the isomorphism explicitly:

$$\begin{array}{rcll}
S_K(n,r) & \cong & \mathrm{End}_{S_K(n,r)}(\oplus_\alpha S^{\alpha}(E)) & \\
& \cong & \mathrm{End}_{k\Sigma_r}(\oplus_\alpha eS^{\alpha}(E)) & \text{Schur functor} \\
& \cong & \mathrm{End}_{k\Sigma_r}(\oplus_\alpha \mathrm{Ind}_{\Sigma_\alpha}^{\Sigma_r} k) & \text{first claim} \\
& \cong & \mathrm{End}_{k\Sigma_r}(\oplus_\alpha (\tilde{k} \otimes \mathrm{Ind}_{\Sigma_\alpha}^{\Sigma_r} k)) & \tilde{k} \otimes _ \\
& \cong & \mathrm{End}_{k\Sigma_r}(\oplus_\alpha e\Lambda^{\alpha}(E)) & \\
& \cong & \mathrm{End}_{S_K(n,r)}(\oplus_\alpha \Lambda^{\alpha}(E)) & \text{first claim} \\
& = & R & \text{the Ringel dual.}
\end{array}$$

This has several nice consequences (See Donkin and Erdmann [42]).

Corollary 5.4 *If* $n \geq r$, *then* $[T(\alpha) : \nabla(\beta)] = [\Delta(\tilde{\beta}) : L(\tilde{\alpha})]$.

Proof: $\mathrm{Hom}_{S_K(n,r)}(T, -)$ is exact on $\mathcal{F}(\nabla)$, hence $[T(\alpha) : \nabla(\beta)] = [P_R(\tilde{\alpha}) : L(\tilde{\beta})]$, since the order is changed and the α exchanges with the $\tilde{\alpha}$. In particular, tilting modules 'contain' the decomposition numbers of GL_n.

Theorem 5.5 *(Donkin, [39]) Let* $X = M(n)^m$ *over* $k = \bar{k}$, *where* $M(n)$ *is the* $n \times n$ *matrices, and* $G = GL_n(k)$. *Then the algebra of invariants* $k[X]^G$ *is generated by the functions* $x_1, \ldots, x_m \mapsto \chi_s(x_{i_1}, \ldots, x_{i_r})$ *where* $r, s \in \mathbb{N}$, $i_1, \ldots, i_r \in \{1, \ldots, m\}$ *and* $x_{i_1}, \ldots, x_{i_r}$ *are elements of* $M(n)$ *with their induced actions on* $\Lambda^s(k^n)$ *and* χ_s *is its trace.*

In characteristic 0, this result is due to Sibirski and Procesi. Donkin has further generalized it to arbitrary quivers. In order to prove the statement, Donkin first reduces the problem to computing relative class functions of $G \hookrightarrow G^{\times m}$, that is the regular functions on $G \times \ldots \times G$ which are invariant under conjugation by G. These differ from invariants by a localization at the determinant. He then computes class functions via rational modules, namely the tilting modules. The actual proof uses many of the above arguments and others.

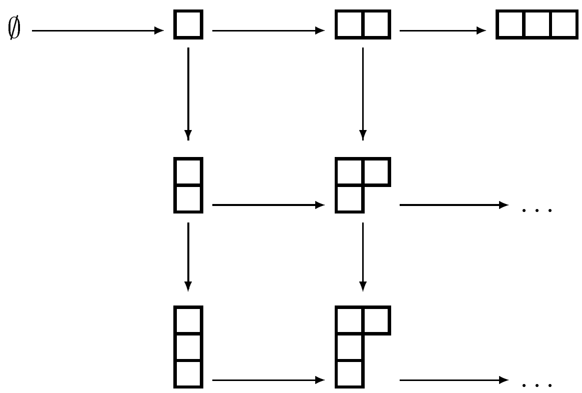
6 Tilting Modules and Representations of Symmetric Groups

In section 4 we saw that the Schur algebra could be defined as the endomorphism ring of $V^{\otimes r}$ for the action of Σ_r, acting by the permutation of the factors. In fact these algebras satisfy a classical double centralizer property called *Schur-Weyl Duality* which we now describe. Assume $k = \bar{k}$ and that $V = k^n$ is the natural module for the action of GL_n. Then GL_n also acts diagonally on $V^{\otimes r}$. We have as well a right action of Σ_r on $V^{\otimes r}$ which is permutation of the places. Schur-Weyl duality gives that $\mathrm{End}_{\Sigma_r}(V^{\otimes r})$ is the image of kGL_n in $\mathrm{End}_k(V^{\otimes r}) = S_k(n,r)$ and $\mathrm{End}_{S_k(n,r)}(V^{\otimes r})$ is the image of $k\Sigma_r$ in $\mathrm{End}_k(V^{\otimes r})$. When $n \geq r$ this image is just $k\Sigma_r$ and is a cellular quotient, otherwise. The basic idea of this section will be to use Schur-Weyl duality to transfer information about the representation theory of the general linear group GL_n to information about the representation theory of the symmetric group Σ_r.

We begin by noting that $V = \Lambda^1 V$ is simple and tilting since $S_k(n,1) \cong \mathrm{GL}_n$ is a full matrix ring. Therefore, $V^{\otimes r}$ is tilting as well. We then have that $V^{\otimes r} = \oplus_{\lambda \in \Lambda} T(\lambda)^{\eta_\lambda}$. The maximal semisimple quotient of $\mathrm{End}_{S_k(n,r)}(V^{\otimes r})$ has the form $\oplus_\lambda \mathrm{Mat}_{\eta_\lambda}(k)$. So the $\mathrm{End}_{S_k(n,r)}(V^{\otimes r})$ simples (which are simple $k\Sigma_r$ modules, as well) have dimension η_λ, this being a general fact about endomorphism algebras. We thus get the principle that decomposing the tilting module $V^{\otimes r}$ into indecomposables gives the simple representations of $k\Sigma_r$.

This principle has been used by several people including Erdmann [46]. Now we look at the results of Mathieu, Georgiev and Papadopoulo ([65], [49], [50], [67], [68]). We will get a nice explicit dimension formula for a class of simple $k\Sigma_r$ representations. The involved combinatorics is in terms of paths in a certain graph. To understand this, we first recall the situation in characteristic zero. In this case, $k\Sigma_r$ is semisimple and its representations are parametrized by the partitions of r, that is, those $\lambda = (\lambda_1, \dots \lambda_n)$ with $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$ and $\Sigma \lambda_i = r$, written $\lambda \vdash r$. Then a basis for the simple $L(\lambda)$ is indexed by the standard tableaux of shape λ, that is, bijective fillings of the Young diagram for λ with the set $\{1, \ldots, r\}$ which increases along rows and columns.

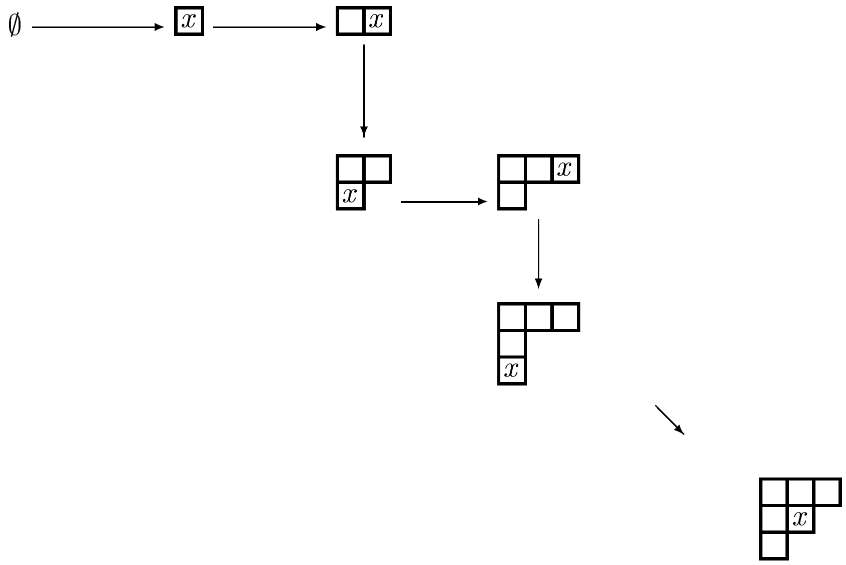
Now let $\mathcal{Y} = \{\text{Young diagrams}\}$ and let us associate to $\mathcal{Y}$ a graph whose vertices consist of the elements of $\mathcal{Y}$ and whose edges are given by the rule that there is an arrow $Y \to Y$' if $Y' = Y$ plus one box:



Now a standard tableau of shape λ corresponds precisely to a path from $\emptyset$ to λ ; reading the filling in order tells you which box to add. So, for example we associate to the tableau

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 5 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

the path



In this set-up the *Frobenius Theorem* says that the dim of $L(\lambda)$ equals the number of paths from $\emptyset$ to λ .

Next, let us return to char $k = p$; we still have Schur-Weyl duality and that the simple $S_k(n,r)$ modules are parametrized by the partitions of r. Then the Schur functor, that is multiplying by e sends a simple either to 0 or to a simple. Now define an infinite subclass of partitions as follows. Fix $l < p$, $\mathcal{Y}_l(p) = \{$Young diagrams associated to partitions $(m_1, \ldots, m_l)$ such that $m_l - m_1 \leq p - l\}$.

Theorem 6.1 *(Mathieu [64]) Suppose $\lambda \in \mathcal{Y}_l(p)$, then the $k\Sigma_{|\lambda|}$ simple $L_k(\lambda)$ has dimension equal to the number of paths from $\emptyset$ to λ inside the subgraph spanned by vertices lying in $\mathcal{Y}_l(p)$.*

We will sketch Mathieu's proof using the above principle. An alternative proof can be given using the results of Kleschev (see Brundan-Kleschev-Suprunenko). This last works for Hecke algebras of type A as well (See Wenzl [85]).

Example Consider the hook $(a+1, 1^b)$ in the case where $l = b+1$ and $a+b+1 = p$ is prime. In char 0, dim $L(\lambda)$ is the number of standard tableaux $= \begin{pmatrix} a+b \\ a \end{pmatrix}$. What is the dimension in characteristic p? Note that this satisfies the hypothesis: $m_1 - m_l = a+1-1 < a+b+1-(b+1) = p-l$. What are the possible paths? It's actually easier to think of traversing the path in the reverse direction. So to begin, we must remove a box from the top row if we are to remain in $\mathcal{Y}_l(p)$. After the removal of this box however, we are free to proceed in any order. Thus dim $L_p(\lambda) = \begin{pmatrix} a+b-1 \\ a-1 \end{pmatrix}$.

Proof (sketch): Now for the proof. The idea is to produce tilting modules via tensoring. An induction is started with a class of simple tilting modules like $V : T(\lambda) = L(\lambda)$ if and only if $L(\lambda) = \nabla(\lambda) = \Delta(\lambda)$. We will find such things in the so-called fundamental alcove. Given $G = Gl_l(k)$ for $k = \bar{\mathbb{F}}_p$, we let H be the diagonal matrices and $P = H^*$ the character group. Then $P = \mathbb{Z}\epsilon_1 + \ldots + \mathbb{Z}\epsilon_l$ where ϵ_i sends an $l \times l$ matrix to its (i,i)th entry. Then let $P^* = \text{Hom }(P, \mathbb{Z})$. Make the following identifications:

$$\begin{array}{rcl} \alpha_i & = & \epsilon_i - \epsilon_{i+1} \\ \alpha_0 & = & \epsilon_1 - \epsilon_l = \sum_i \alpha_i \\ h_i & = & \epsilon_i^* - \epsilon_{i+1}^* \\ h_0 & = & \epsilon_1^* - \epsilon_l^* \end{array} .$$

If $p \in P$ is in the weight lattice, define $s_i : p \mapsto p - h_i(p)\alpha_i$. Then the $s_i \in GL(P)$ generate the Weyl group $W \cong S_l$ via the permutation representation on $\mathbb{Z}^l$. Let s_0 be the affine reflection on P: $s_0 : \lambda \mapsto (h_o(\lambda) - p)\alpha_0$. Then $\langle W, s_0 \rangle$ generates the affine Weyl group W_{aff}.

Now put $P^+ = \{\lambda \in P | \lambda(h_i) \geq 0 \text{ for all } 1 \leq i \leq l\}$. So $\epsilon_i(\lambda) = \lambda(\epsilon_i^*) \geq \lambda(\epsilon_{i+1}^*) = \epsilon_{i+1}(\lambda)$ and so the elements of P^+ are partitions.; in other words, P^+ is the index set for Λ of the highest weight category we are here considering. Let $C = \{\lambda \in P^+ | \lambda(h_0) \leq p - l + 1\}$. Since $\lambda(h_0) = \epsilon_1(\lambda) - \epsilon_l(\lambda)$ these are exactly the weights which satisfy the condition of the theorem. Let $C^0 = \{\lambda \in P^+ | \lambda(h_0) < p - l + 1\}$. This is the *fundamental alcove.*

Definition 6.1 *For any rational representation, M, let*

$$ch\ M = \sum_{\mu \in P} (dim\ M_\mu) e^\mu \in \mathbb{Z}[P]$$

be the formal character *of M.*

Lemma 6.2 *1. If $\lambda \in C$, then $T(\lambda) = \Delta(\lambda) = \nabla(\lambda) = L(\lambda)$.*

2. For $\lambda, \mu \in C$, $\lambda \neq \mu$, then the $L(\lambda)$ and $L(\mu)$ are in different blocks.

Proof: We use the ordering given by the usual dominance order and then block decomposition is given by the strong linkage principle (Andersen [3]): $L(\lambda)$ and $L(\mu)$ are in the same block if there exists a $w \in W_{\text{aff}}$ such that $\lambda + \rho = w(\mu + \rho)$, that is, if λ and μ are in the same orbit under the dot action. Now $\lambda \in C$ is minimal in its block and the conclusions follow.

Now we want to form tensor products by $V = k^l$ in order to produce new tilting modules. In order to distinguish $T(\lambda)$ with $\lambda \in C^0$, from all the other ones, we look at dimensions:

Proposition 6.3 *$p | dim_k T(\lambda)$ if and only if $\lambda \notin C^0$.*

Before we proceed to the proof, we collect some general facts on the divisibility by p of dimensions of group representations. For this G can be any group (finite or infinite). Let M be a finitely dimensional G-module. Then let M^G be the G invariants of M, that is $\{m \in M | gm = m \text{ for all } g \in G\}$. Let M_G be the G-coinvariants, that is $M_G = ((M^*)^G)^*$. We then have natural maps $M^G \hookrightarrow M$ and $(M^*)^G \hookrightarrow M^*$, giving $M \cong M^{**} \to M_G$ and so $M^G \to M \to M_G$. Denote the image of this map by $T(M)$ and its kernel by $U(M)$.

Now given any pair of G-modules $\text{Hom}_k(M, N)$ is also a a G-module. Put $T(M, N)$ equal to $T\text{Hom}\ (M, N))$ and $U(M, N) = U(\text{Hom}\ (M, N))$. Now the identity map induces a natural embedding $\pi : M^G \otimes N^G \to (M \otimes N)^G$ since G acts diagonally, $g(m \otimes n) = (gm \otimes gn)$.. We then have $\pi(M^G \otimes U(N)) \subset U(M \otimes N)$ and $\pi(U(M) \otimes N^G) \subset U(M \otimes N)$ inducing a map $T(M) \otimes T(N) \to T(M \otimes N)$. In particular, $T(M, M)$ is a quotient of $\text{End}_G(M)$ and since the standard composition $\text{End}_G(M)$ and $\text{End}_G(M^*)$ preserves G invariance, $T(M, M)$ becomes a k-algebra via $T(M, M) \otimes T(M, M) \to T(\text{End}_G(M) \otimes \text{End}_G(M)) \to T(\text{End}_G(M)) = T(M, M)$. $T(M, M)$ has a unit element 1_M, which is 0 if and only if $T(M, M) = 0$ if and only if $T(M, N) = T(N, M) = 0$ for all N.

Our need is for the

Lemma 6.4 *Given M, N indecomposable G modules then $T(M, N) \cong k$ if $M \cong N$ and p does not divide dim M and is 0 otherwise.*

Thus the divisibility of the dim M by p is decided by $T(M, M)$.

Proof: We give a sketch of the proof. We have that $U(M, N)$ is the kernel of the map

$$\text{Hom}_G(M, N) \to \text{Hom}_k(M, N) \to (\text{Hom}_G(N, M)).$$

So $\phi : M \to N$ is in $U(M, N)$ if and only if, for all $\psi : N \to M$, the composition $\phi \circ \psi$ has trace 0, by definition. Given an indecomposable M, a G invariant endomorphism, α, of M has a unique eigenvalue, say $\lambda(\alpha)$ with multiplicity dim M. Thus the trace of $\alpha = \dim M \cdot \lambda(\alpha)$. Thus if $\alpha \neq 0$ then trace$(\alpha) = 0$ only if $p|\dim M$. If $M \ncong N$, then trace$(\alpha) = 0$.

As an immediate consequence, we have that if X is any indecomposable with $p|\dim X$, then $p|\dim Z$ for any direct summand Z of $X \otimes Y$ for any module Y. One uses the fact that if $M = \oplus m_N N$ then $T(M, M) = \oplus M_{m_N}(k)$ summed over those N whose dimension is not a multiple of p. One then reasons that $p|\dim X$ gives $T(X \otimes Y, X \otimes Y) = 0$ since $T(X \otimes Y, X \otimes Y) = 0$ is a right $T(X, X)$ module. Last, this gives $T(Z, Z) = 0$.

Returning to the proposition, $p|\dim T(\lambda)$ if and only if $\lambda \notin C^0$. For $\lambda \in C$, $T(\lambda) = \Delta(\lambda)$, but $p|\dim \Delta(\lambda)$ is known; in fact, $\Delta(\lambda)$ is obtained as $k \otimes_{\mathbb{Z}} L_{\mathbb{Z}}(\lambda)$ where $L_{\mathbb{Z}}(\lambda)$ is a free $\mathbb{Z}$ lattice with $\mathbb{C} \otimes_{\mathbb{Z}} \mathbb{L}_{\mathbb{Z}}(\lambda) = \mathbb{L}_{\mathbb{C}}(\lambda)$, the finite dimensional simple module with highest weight λ in char 0, that is for the finite dimensional module of the corresponding Lie algebra. Now this, in turn, implies that $\dim_k \Delta(\lambda) = \dim_{\mathbb{C}} L_{\mathbb{C}}(\lambda)$ which is given by the Weyl character formula:

$$\text{ch } L(\lambda) = \prod_{\alpha \in \{\text{positive roots}\}} \frac{(\lambda + \rho)(h_\alpha)}{\rho(h_\alpha)}.$$

Recall that $\lambda \in C$ is defined by $(\lambda + \rho)(h_0) \leq p$ and $\lambda \in C^0$ by $(\lambda + \rho)(h_0) < p$. Hence, the same is true for all h_α since $h_0 = \sum h_i$ is the longest root. This implies that the denominator is not divisible by p and the numerator is so if and only if $\lambda \notin C^0$. So the proposition is true for $\lambda \notin C$.

We continue by induction. Let $\lambda \in P^+$ and $\lambda \notin C$. Then there is a fundamental weight, ω, such that $\lambda - \omega \in P$. This implies that $T(\lambda) | T(\lambda - \omega) \otimes T(\omega)$, the last being a tilting module of highest weight λ. Next we have that $(\lambda - \omega)(h_0) = \lambda(h_0) - 1$ implying that $\lambda - \omega \notin C^0$. so we may conclude that $p|\dim T(\lambda - \omega)$ and so, by the lemma, that $p|\dim T(\lambda)$. Thus the proposition is OK.

Now we tensor tilting modules:

Lemma 6.5 *Given $l < p$, ω a fundamental weight:*

1. *If $\lambda \in C^0$ then $T(\omega) \otimes T(\lambda) = \oplus T(\lambda + \nu)$, the sum being over those ν which are weights occurring in $\Delta(\omega)$; that is, over those $\nu \in W \cdot \omega$ with $\lambda + \nu \in P^+$.*

2. *If $\lambda \notin C^0$ then $T(\omega) \otimes T(\lambda) = \oplus T(\nu)^{\eta_\nu}$ with $\eta_\nu \neq 0$ only if $\nu \notin C^0$.*

Proof:

1. $T(\omega) = \Delta(\omega)$, $T(\lambda) = \Delta(\lambda)$. A classical formula, *the Littlewood-Richardson decomposition* gives that, up to filtration $\Delta(\omega) \otimes \Delta(\lambda) = \sum \Delta(\lambda + \nu)$. One then observes that, in this case, each summand occuring in this composition series belongs to a different block. Since there are no non-trivial extensions between modules in different blocks, this must, in fact, be a direct sum. Last, since direct summands of tilting modules are tilting modules, we get that $\Delta(\lambda + \nu) = T(\lambda + \nu)$ and the conclusion follows.

2. $\lambda \notin C^0$ implies that $p|\dim T(\lambda)$ implying that $p|\dim T(\omega) \otimes T(\lambda)$ and all its direct summands. We apply the previous proposition.

The theorem now follows: we know that $V^{\otimes r} = \oplus_\lambda T(\lambda)^{m_\lambda}$ for some indecomposable tilting modules. We want to determine m_λ for $\lambda \in C^0$. We proceed by induction on r. The result is trivial for $r = 1$. Assume it true for $r - 1$. Then $V^{\otimes r} = \oplus_\lambda T(\lambda - \epsilon_i)^{m_{\lambda - \epsilon_i}} \oplus$ other tilting modules lying outside of C^0 and $m_{\lambda - \epsilon_i}$ is given by the appropriate number of paths. Tensoring with $V = \Delta(1, 0, \ldots)$ and using the above lemma $m_\lambda = \sum(m_{\lambda - \epsilon_i})$, the sum over those $\lambda - \epsilon_i$ that lie in C^0. The point being that for those weights lying outside of C^0, tensoring by V can produce no summands lying in C^0.

Example Another example shows that the Young diagrams can be quite large. For example, given $p = l + 1$, $1 \leq b \leq p$ and a any natural number, then if Y is the diagram for the partition $((a+1)^b, a^{(l-b)})$, then there is but one path from $\emptyset$ to Y. (In fact, Mathieu states this is the sign representation for Σ_{la+b}).

Remark 6.1 *In the same paper, Mathieu shows that dim $L(\lambda)$ can generally be described by paths. Let m_μ be defined by $V \otimes T(\lambda) \cong T(\mu)^{m_\mu} \oplus$ rest. Then form a graph as above with the number of $\lambda \to \mu$. equal to m_μ. The reasoning above then tells us that dim $L_k(\lambda) =$ the number of paths. Unfortunately, the graph depends on p and is not known. (However, this may be a convenient language.)*

If G is any algebraic group, then there is an analogue to this result decomposing tensor products.

Theorem 6.6 *(Goergiev-Mathieu [49]): either $p|$dim $T(\lambda)$ or $\lambda \in C^0$ and $T(\lambda)$ is simple.*

So again, if we leave C^0 we never come back.

Their interest in this topic is in understanding fusion rules. That is rules that give the decomposition of tensor products of modules. To understand this form a category whose objects are tilting modules for G and whose morphisms are $T(M, N)$ with tensor product as usual. Note that, in particular, this kills everything outside of C^0!

There is a close relationship to Moore and Seiberg's tensor products (see Kazhdan and Lusztig [61]) and also to those of Gel'fand and Kazhdan. Moore and Seiberg are working with category $\mathcal{O}$ of a fixed level l for affine Kac-Moody algebras. The normal tensor product, however, increases the level. So they define a new tensor product that stays in the category. Their multiplicities remain the same.

Remark 6.2 *Georgiev and Mathieu [50] believe that this is explained by lifting to the quantum group. See paper by Finkelberg [47] and a recent paper by Ostrik [69]. See also a paper by Andersen and Paradowski [8].*

We have now seen how to use tilting modules for computing the decomposition numbers of the symmetric groups and GL_n. In fact, one can also use tilting modules to show that both problems are equivalent. James had already shown that the decomposition numbers for Σ_r are the decomposition numbers for GL_n. This is, in fact, easily reproved using the Schur functors (see Green [52]).

The converse is due to Erdmann [46]. Precisely, let λ, μ be elements of $\Lambda^+(n, r)$, then $[\Delta(\mu) : L(\lambda)] = [S^{t(\mu')} : D^{t(\lambda')}]$, where $t(\mu) = p\mu + (p-1)(\mu - 1, \mu - 2, \ldots, 1, 0) \in \Lambda^+(n, r)$. The proof is based on tilting modules, using arguments similar to those we have already seen.

We finish with a conjecture of Mathieu announced at Reims in the Summer of '95 [65]:

Conjecture 6.7 *Let Y be a Young diagram and $Y_0 = Y$. Let $Y_n = (n^l) + Y$. Then $Q_Y(z) = \sum$ dim $L_p(Y_n)$ is a rational function.*

7 Character formulae–known or conjectured

We have seen that knowledge of the tilting modules implies character formulae in various settings. Thus it is natural to ask for the characters of the tilting modules themselves, either over algebraic groups or over quantum groups. There is a plethora of conjectures by Andersen and Soergel and a proof (by Soergel based on the work of Sergei Arkhipov) for one of these. One should note that, like the Kazhdan-Lusztig and the Lusztig conjectures, these conjectures give the formulas in terms of the so-called Kazhdan-Lusztig polynomials. We will not go into combinatorial details here, but refer you to Soergel's recent paper; for now we discuss only his method of proof.

From the work of Kazhdan-Lusztig (or Moore and Seiberg) certain equivalences are known between certain categories of representations over quantum groups and over affine Kac-Moody algebras. Hence, we start with the latter. Let $\mathfrak{g} = \sum \mathfrak{g}_i$ be a $\mathbb{Z}$-graded Lie algebra over a field $k = \bar{k}$. Assume that $\mathfrak{g}$ satisfies:

- all the graded pieces, $\mathfrak{g}_i$, are finite dimensional,
- $\mathfrak{g}$ is generated by $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$, and $\mathfrak{g}_1$,
- there exists a *semi-infinite character*, $tr : \mathfrak{g}_0 \to k$, such that $tr(adX\, adY : \mathfrak{g}_0 \to \mathfrak{g}_0)$ equals $tr([X, Y])$ for all $X \in \mathfrak{g}_1$ and all $Y \in \mathfrak{g}_{-1}$.

We then have a *triangular decomposition*, $\mathcal{N} = \mathcal{N}_- = \mathfrak{g}_{<0}$, $\mathcal{B} = \mathfrak{g}_{\geq 0}$ and enveloping algebras U of $\mathfrak{g}$, N of $\mathcal{N}$ and B of $\mathcal{B}$. Put $\mathcal{M}$ equal to the category of $\mathbb{Z}$-graded $\mathfrak{g}$-modules which are graded free of finite rank over N.

Theorem 7.1 *(Arkhipov [9], see also Soergel [80], [81], [82]). $\mathcal{M}$ is self-dual. More precisely, there is an equivalence $\mathcal{M} \to \mathcal{M}^{op}$ which is exact and which sends $U \otimes_B E$ to $U \otimes_B (k_{-\mu} \otimes E^*)$ for every finite dimensional $\mathbb{Z}$-graded $\mathfrak{g}_0$-representation E.*

More general results are due to Arkhipov and, earlier, to Voronov. The equivalence is a tensor product with a certain bimodule followed by duality. The bimodule is constructed by direct computation.

In any case, we can restrict to char $k = 0$ and $\mathfrak{g}$ semisimple as a $\mathfrak{g}_0$ module under the adjoint action. Let Θ be the category of $\mathbb{Z}$-graded $\mathfrak{g}$-modules which are semisimple over $\mathfrak{g}_0$ and locally finite over $\mathfrak{g}_{\geq 0}$. Put $\Lambda = \{\text{finite dimensional simple } \mathbb{Z}\text{-graded } \mathfrak{g}_0\text{-modules}\}/\cong$. Then $E \in \Lambda$ is concentrated in just one degree. For $E \in \Lambda$ define a Verma module $\Delta(E) = U \otimes_B E$ and put $L(E)$ equal to its unique simple quotient. In this way one gets all the simples for Θ. Note that Δ-filtrations may have infinite lengths; however, the multiplicities are finite and so the finite parts may be described by quasi-hereditary algebras. Define $\nabla(E)$ to be the graded dual of $U \otimes_{U(g_{\leq 0})} E$ by which we mean that we dualize each finite dimensional graded piece: $(\mathcal{M}^*)_i = (\mathcal{M}_i)^*$. Then, by the Ringel formalism, we have a unique indecomposable tilting module $T(E)$ for each $E \in \Lambda$. It turns out that Θ does not have enough projectives but does have enough injectives. Further, after truncating to get $\Theta_{\leq n}$, there are projective covers. If we assume $L = L(k_{-\mu} \otimes E^*)$ has the projective cover $P(k_{-\mu} \otimes E^*)$ in Θ, this will have a Δ-filtration. Here, we have a *Brauer-Humphreys reciprocity:* $[P(k_{-\mu} \otimes E^*) : \Delta(F)]. = [\nabla(F) : L(k_{-\mu} \otimes E^*)]$.

Now we use the equivalence $\mathcal{M} \cong \mathcal{M}^{op}$, in particular, the fact that it sends Δ's to Δ's, to get more information. Consider the image of P. The image, call it T, has a filtration by the images of the Δ's filtering P, where the order is turned around. So we get a bottom factor of $\Delta(k_{-\mu} \otimes E^*) = \Delta(E)$ .Then from the fact that $\mathrm{Ext}^1(P, \Delta) = 0$ we conclude that $\mathrm{Ext}^1(\Delta(F), T) = 0$ for all F. This implies that T is the tilting module $T(E)$. Now compare the multiplicities in Δ-filtrations:

$$[T(E) : \Delta(F)] = [P(k_{-\mu} \otimes E^*) : \Delta(k_{-\mu} \otimes F^*)] = [\nabla(k_{-\mu} \otimes E^*) : L(k_{-\mu} \otimes F^*)].$$

Conjecture and Theorem 7.2 *(Soergel) Characters of the tilting modules are given by the characters of the Δ's together with the decompositions of the ∇'s; that is, by a "Kostant function" plus a "Lusztig conjecture" .*

This covers the Kac-Moody algebras since μ exists. More accurately, it covers the affine case, negative level (the needed character formulas for simples is then in the positive level). Here, Kashiwara and Tanisaki proved the character formulas for simples in the simply laced cases (the other cases are yet missing in the literature.) Kazhdan and Lusztig then relate the Lusztig conjecture to the Lusztig conjecture for quantum groups at roots of unity. Still much remains to be done.

We next consider another related open problem. Suppose U is quantum GL_n, then there is Jimbo's or Du's quantized Schur duality ${}_{U_q(gl_n)}(V^{\otimes r})_{\mathcal{H}}$, where $\mathcal{H}$ is the Hecke algebra (in effect, the quantized $k\Sigma_r$). Now , as we've already seen, the character formulas of the tilting modules determine the dimensions of the simple $\mathcal{H}$-modules at roots of unity. These are known by the so-called Lascoux-Leclerc-Thibidon conjecture, proven recently by Ariki. These are determined from a canonical basis and their combinatorics look very different from the Kazhdan-Lusztig combinatorics. This, in turn, is related to the representation theory of the symmetric groups via a conjecture of James. Putting all these together, we see that the characters of the tilting modules are, at least conjecturally, more or less known. A series of conjectures by Andersen (and Soergel) then relates these characters to those for the algebraic groups.

Somewhat more specifically, suppose G is an algebraic group and U its quantized enveloping algebra $U_q(\mathfrak{g})$ over $\mathbb{Q}(v)$ and $U_{\mathcal{A}}$ its Lusztig form over $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. Suppose further that q is a pth root of unity in $\mathbb{C}$; then $\mathbb{C}$ becomes an $\mathcal{A}$ module by specializing v to q. Put $U_q = U_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}$. Then the simple (type 1) U_q-modules are parametrized by their highest weights, $\lambda \in X^+$. Call them $L_q(\lambda)$. These imbed into a set of costandard objects $\nabla(\lambda)$ and are surjected upon by a set of standard objects $\Delta(\lambda)$. The category also has tilting modules $T(\lambda)$. Let C_{p^2} be the bottom p^2 alcove; that is, $C_{p^2} = \{\lambda \in X^+ | \langle \lambda + \rho, \alpha^\vee \rangle < p^2,$ for all roots $\alpha\}$. Then we have

Conjecture 7.3 *(Andersen and Soergel): Suppose $p \geq h$, the Coxeter number (i.e., order of (ω_0)). Then $\lambda \in C_{p^2}$ implies that char $T(\lambda) =$ char $T_q(\lambda)$.*

The right side is what is more of less known by Soergel.

What is the evidence? First, according to Andersen, Jantzen and Soergel [7], the statement is true for $\lambda \in C_{p^2} \cap ((p-1)\rho + X_1$ and $p >> 0$, where X_1 is the set of restricted weights, that is, those λ such that $\langle \lambda, \alpha^\vee \rangle < p$ for all p. The conjecture is also true in the cases A_1 and A_2 where Paradowski [70] has worked it all out by hand. They are, as well, consistent with other results and with the following more precise conjectures.

Let $B = \mathbb{Z}[v]_{(v-1,p)}$, a local ring. The map $v \to 1 \in k$ makes B into a local ring whose residue field is k. This implies that the U_k modules are the finite dimensional G-modules (Andersen-Polo-Wen). Now one can lift all relevant data from k to B: that is, all Δ's, ∇'s and $T(\lambda)$'s. This last, by using Ringel's construction and comparing Ext's under base change. But also $\mathbb{C}$ is a B-algebra via $v \to q$, q a primitive lth root of unity. Denote $T_B(\lambda)$ the lifted tilting module. Clearly $T_B(\lambda) \otimes_B \mathbb{C}$ is a tilting module for U_q (via filtrations) and has highest weight λ. This gives that $T_B(\lambda) \otimes_B \mathbb{C} =$

$T_q(\lambda) \oplus$ some others. The content of the previous conjecture is that in the bottom p^2-alcove there are "no others" and so here we have the equivalent

Conjecture 7.4 $T_B(\lambda) \otimes_B \mathbb{C} = T_q(\lambda)$ *and* $T_B(\lambda) \otimes_B k = T(\lambda)$.

8 Howe duality in positive characteristic

Recall that if A is a quasi-hereditary algebra and T is a full tilting module then $R = \mathrm{End}_A(T)$ is called the Ringel dual of A. If $A = S_K(n,r)$ is a Schur algebra then $T = \sum_{\alpha\in\Lambda(n,r)} \Lambda^\alpha V$ is a full tilting module and if $n \geq r$ $S_K(n,r)$ is its own Ringel dual. In [38] Donkin gave a second proof, later extended in his book on q-Schur algebras, which we now describe.

Fix $n \geq r$ and let V be the natural left module for GL_n. Let W be the natural right GL_n module. Then $V \otimes W$ is a bimodule and $\Lambda(V \otimes W)$ is also a module for GL_n (although this requires some work in the quantum case). We consider the rth component $\Lambda^r(V \otimes W)$. We can work out a specific basis: If V has the basis $\{v_1, \ldots, v_n\}$ and W has the basis $\{w_1, \ldots, w_n\}$, then $V \otimes W$ has a basis of the form $\{b_1 = v_1 \otimes w_1, \ldots, b_{n^2} = v_n \otimes w_m\}$. This implies that $\Lambda^r(V \otimes W)$ has a basis $\{b_{i_1} \wedge b_{i_2} \wedge \ldots \wedge b_{i_r} | 1 \leq i_1 < i_2 < \ldots < i_r\}$, which again requires some work in the quantum case.

Now for each $\alpha \in \Lambda(n,r)$, define a map $\phi_\alpha : V^{\otimes r} \to \Lambda^r(V \otimes W)$ by

$$e_i = v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_r} \mapsto$$

$$(v_{i_1} \otimes w_1) \wedge \ldots \wedge (v_{i_{\alpha_1}} \otimes w_1) \wedge (v_{i_{\alpha_1+1}} \otimes w_2) \wedge \ldots \wedge (v_{i_{\alpha_1+\alpha_2}} \otimes w_2) \wedge \ldots \wedge (v_{i_r} \otimes w_{l(\alpha)}).$$

It then follows that ϕ_α is a left module homomorphism and, by checking the relations directly, one can confirm that it factors through $\Lambda^\alpha V$. So one gets an induced map, with abuse of notation, $\phi_\alpha : \Lambda^\alpha V \to \Lambda^r(V \otimes W)$. By direct computation on basis elements, one confirms that this map is in fact injective. Doing a similar check one then gets the

Proposition 8.1 *(Donkin [40])* $\Lambda^r(V \otimes W) \cong \oplus_{\alpha\in\Lambda(n,r)} \Lambda^\alpha V$ *as left* GL_n *modules and* $\Lambda^r(V \otimes W) \cong \oplus_{\alpha\in\Lambda(n,r)} \Lambda^\alpha W$ *as right* GL_n *modules.*

We can thus conclude that $\Lambda^r(V \otimes W)$ is a full tilting module on both sides (provided that $n \geq r$.) We know that the Ringel dual $R = \mathrm{End}_{S_K(n,r)} \sum_{\alpha\in\Lambda(n,r)} \Lambda^\alpha V \cong \mathrm{End}_{A=S_K(n,r)}(\Lambda^r(V \otimes W))$. But $\Lambda^r(V \otimes W)$ is a right $S_K(n,r)$-module, that is, a left $S_K(n,r)^{op}$ module. It follows that there is a homomorphism of algebras $S_K(n,r)^{op} \to R$. Now $W^{\otimes r}$ is a tilting module and one can verify by direct computation that it is a direct summand of $\oplus \Lambda^\alpha W$. However both $W^{\otimes r}$ and $\oplus \Lambda^\alpha W$ are faithful; this gives that $S_K(n,r)^{op} \to R$ is injective. But both $S_K(n,r)^{op}$ and R have dimensions which are independent of the ground ring. (This follows from the fact that the Δ's and ∇'s, as well as the filtration multiplicities are, themselves, independent of the ground ring.) It follows that we need only check the equality in char 0. and so we get

Corollary 8.2 *$S_K(n,r)^{op}$ is the Ringel dual of R. Moreover, $S_K(n,r) \cong S_K(n,r)^{op}$ by contravariant duality via the map sending a standard basis element $\xi_{i,j} \mapsto \xi_{j,i}$.*

Note that in the above set up we could replace W by the natural right GL_m module for some m and the theorem 8.1 would continue to hold. This is a particular case of *Howe duality* as studied by Adamovich and Rybnikov [1]. The name is motivated by the results of Howe [55] in characteristic 0, relating certain Lie algebras with each other via actions on modules familiar from invariant theory. In particular, Adamovich and Rybnikov extend the results of Donkin to additional pairs of groups.

Let $k = \bar{k}$ be any field in the first two cases and any field not of char 2 in the last four. Let $m, n \in \mathbb{N}_0$ and (G_1, G_2) be any of the following pairs of groups

- (GL_m, GL_n)
- (Sp_{2m}, Sp_{2n})
- (O_{2m}, SO_{2n})
- $(O_{2m+1}, Spin_{2n})$
- (Pin_{2m}, SO_{2n+1})
- $(Spin_{2m+1}, Spin_{2m+1})$

where the spinor groups are the two-fold coverings

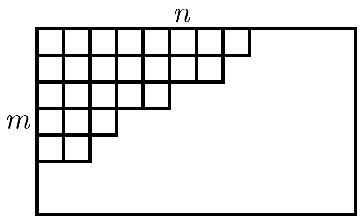
$$\begin{tikzcd} Pin_{2m} \arrow[r] & O_{2m} \\ Spin_{2m} \arrow[r] \arrow[u, "\subset"'] & SO_{2m} \arrow[u, "\subset"'] \end{tikzcd}$$

Consider the following module categories, $\mathcal{C}$:

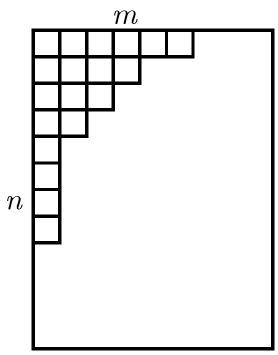
- finite dimensional polynomial for GL_n,
- finite dimensional rational for O, SO, or Sp,
- finite dimensional rational with weights in $\mathbb{Z}+\frac{1}{2}$ for Pin or $Spin$ (over $\mathbb{Z}$, $\mathcal{C}$ is a category over the orthogonal group).

In each case, one may attach to a standard (or costandard) representation a Young diagram. These diagrams are related to the action of a subgroup of diagonal elements. In the case, of GL_n this is given by the action of the diagonal elements on the *highest weight*. For the other groups, the rules are more complex, for example, for SO_{2m}, one attaches not only a diagram, but a sign indicating the sign of the last coordinate of the weight. Full details can be found in Adamovich and Rybnikov's paper and in the classic work by Weyl [86].

Each of these subcategories is a limit of subcategories depending on $j \in \mathbb{N}$ with coordinates smaller than $j+1$. These are modules over a generalized Schur algebra. If this corresponds to the group G_1, then $j=n$ and we get a Schur algebra we may designate as $S_1(m,n)$. The corresponding dominant weights are those whose Young diagram fits in an $m \times n$ box.



Each pair comes, then, with two Schur algebras, $S_1(m,n)$ and $S_2(n,m)$, where the weights for the latter fit in an $n \times m$ box.



The claim is that there always exist a $G_1(m) \times G_2(n)$ module, $M(m,n)$, such that

- M is a full tilting module both for S_1 and S_2,
- the natural morphisms $S_1 \to \mathrm{End}_{S_2}M$ and $S_2 \to \mathrm{End}_{S_1}M$ are isomorphisms.

Remark 8.1 *Changing the order, $\leq$, is given by transposing positions within the rectangle. When $G_1 = O$ then a sign change is also needed.*

In this situation, one then gets the usual consequences: these include a derived equivalence and a formulae:

$$\begin{aligned} \Delta_1(\lambda) &\cong \mathrm{Hom}_{S_2}(M, \nabla(\lambda^*)) \text{ and} \\ \Delta_2(\lambda) &\cong \mathrm{Hom}_{S_1}(M, \nabla(\lambda^*)) \end{aligned}$$

$$\begin{aligned} [T_1(\lambda) : \Delta_1(\mu)] &= [\nabla_2(\mu^*) : L_2(\lambda^*)] \text{ and} \\ [T_2(\mu^*) : \Delta_2(\lambda^*)] &= [\nabla_1(\lambda) : L_1(\mu)] \end{aligned}$$

$$\begin{aligned} \mathrm{Ext}^i_{G_2}(\Delta_2(\lambda^*), \Delta_2(\mu^*) &= \mathrm{Ext}^i_{G_1}(\nabla_1(\lambda), \nabla_1(\mu) \text{ and} \\ \mathrm{Ext}^i_{G_1}(\Delta_1(\lambda), \Delta_1(\mu) &= \mathrm{Ext}^i_{G_2}(\nabla_2(\lambda^*), \nabla_2(\mu^*) \text{ for all } i \end{aligned}$$

The construction of M generalizes that of Donkin: find natural modules, tensor them and form exterior powers. In some sense, this is characteristic free classical invariant theory.

We return to $GL_n(k)$, $k = \bar{\mathbb{F}}_p$. Mathieu and Papadopoulo have given a reinterpretation of the Georgiev-Mathieu and Mathieu results in the set-up of Howe duality. They have obtained a character formula for a class of simple GL_n-modules. By stabilization, they get a formula for a family of simple GL_∞-modules as well.

In another paper, the same authors lifted this result to char 0 to obtain a combinatorial formula for the weight multiplicities of some infinite dimensional highest weight $\mathfrak{gl}_\mathfrak{n}$ modules without the use of Kazhdan-Lusztig. Essentially, these characters are limits over the above characters in char p.

Remark 8.2 *There is no known counterpart in char 0, that is for Category $\mathcal{O}$, of Howe duality. Also there is no quantized version except for Donkin's result on q-Schur algebras in Type A.*

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Index

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