

Preface

At the 6th International Conference on Aperiodic Crystals [1] in Liverpool in September 2009, Roger Penrose gave a public lecture *Simple sets of shapes that tile the plane but cannot ever repeat*. In his lecture, he not only talked about the discovery of his famous tiling in the 1970s [2], but also discussed his later attempts at finding a single prototile that tiles the plane only nonperiodically [3], concluding that it is still an important open problem he would like to see solved.

Just a few months later, this wish seems to have come true. Recently, Joan Taylor, who lives in northern Tasmania and has been fascinated by tilings for many years, discovered a single hexagonal tile with local rules that tiles the plane only nonperiodically. Hence, this is an example of an aperiodic monotile or ‘einstein’. In collaboration with Joshua Socolar from Duke University, important aspects were worked out, including clarification of the necessary and sufficient local rules and an alternative strategy for the proof, as well as versions of the prototile that satisfy various constraints on the form of the local rules. A manuscript describing their results, entitled *An aperiodic hexagonal tile*, is now available on the arXiv [4]; it has been submitted for publication and is currently being reviewed.

By chance, we had the opportunity to visit Joan Taylor at her home, and discuss the history of her discovery with her. We believe that it is useful to make her original strategy and drawings accessible to the public as a supplement to the preprint [4], and it is with her permission that we make them available here. After our initial posting, we learnt that this work is expected to be included in a revised version of the submitted article. Until the article appears, the submitted manuscript [4] should be considered and cited as the primary source for presentation of the result.

Michael Baake & Uwe Grimm
Hobart, March 2010

References

- [1] <http://www.aperiodic09.org/>
- [2] R. Penrose, The rôle of aesthetics in pure and applied mathematical research, *Bull. Inst. Math. Appl.* **10** (1974) 266–271.
- [3] R. Penrose, Remarks on tiling: Details of a $(1 + \varepsilon + \varepsilon^2)$ -aperiodic set, in *The Mathematics of Long-Range Aperiodic Order*, ed. R.V. Moody, NATO ASI Series C 489 (Kluwer, Dordrecht, 1997) pp. 467–497.
- [4] J.E.S. Socolar and J.M. Taylor, An aperiodic hexagonal tile, preprint arXiv:1003.4279v1 [math.CO] (2010).

APERIODICITY OF A FUNCTIONAL MONOTILE

J. M. TAYLOR

ABSTRACT. The subject monotile does not possess local matching rules expressible through geometric deformations of the sides of the tile (what might be called 'classical' matching rules) yet is functionally equivalent to an aperiodic classical three prototile set. This is possible because two of the three tiles are auxiliary, merely embodying rules for the juxtapositions of the other tile to itself.

The fourteen half-hexagonal prototiles with which we begin the proof group themselves into seven hexagonal prototiles and then into three clusters of hexagonal tiles. The three clusters determine the tiling structure and confirm the unique composition of the fourteen prototile set. Aperiodicity for the fourteen, seven and three prototile sets follows immediately. Functional equivalence to the three prototile set establishes the aperiodicity of the monotile with its non-classical rules.

Tilings by this functional monotile have some unusual features. Not least among them is the non-unique decomposition (seven ways), which exists together with scale-invariance, and is fundamental to the proof of aperiodicity. The novelty of tilings by the monotile is readily appreciated when they are coloured to distinguish the monotile from its mirror-image tile (not shown or discussed further here).

SUMMARY

Fourteen half-hexagonal tiles (and their mirror-images) with matching expressed via geometric deformations of their sides comprise a prototile set. The 'left' and 'right' half-hexagons (they have an up/down orientation) necessarily match to make nine hexagonal tiles, two of which are aberrant and will not tile. This leaves seven hexagonal tiles, named A to G, and their mirror-images, named \bar{A} to \bar{G} .

Arrangements around a particular half-hexagonal left and right pair, the C_L and C_R tiles, are studied. (This choice of study is later justified.) Sixteen possible arrangements, again half-hexagonal in shape, are found. They too must join in left and right halves to form bigger arrangements. Two of these half-hexagonal arrangements will not mate with any other and so cannot exist in a tiling, leaving fourteen available. The fourteen are shown to be composed versions of the fourteen half-hexagonal prototiles with equivalent matching conditions.

A sequence of observations leads to the knowledge that in any tiling a C tile (a C_L tile + a C_R tile) or a \bar{C} tile must appear on a triangular grid and such that only one other hexagonal tile interposes between each pair. First it is found that in any tiling there must be at least one of A, \bar{A} , B, \bar{B} , G or \bar{G} tiles. Then that these tiles only appear in particular abutments: the A+B+ \bar{G} grouping and its mirror-image \bar{A} + \bar{B} +G grouping, called glugons. The three hexagons are 'glued' together at their tops and cannot appear in any other arrangement. So any tiling contains at least one glugon.

A glugon must have three C (or \bar{C}) tiles around it and each C (or \bar{C}) tile must have another glugon at its opposite end. That these abutments must continue shows that any tiling is made up of hexagonal cells with glugons on the vertices and C (or \bar{C}) tiles on the sides and, so far, with 'holes' in the middle. These holes can admit seven hexagonal tiles: a central one surrounded by six others. No glugon can reside in the hole. It is found that the central tile can only be a C (or \bar{C}) tile and that it is surrounded by D, \bar{D} , E, \bar{E} , F and \bar{F} tiles in the arrangement of a composed $C_L + C_R$ combination with their outer half-hexagons completed and called a C cluster. This establishes the C (or \bar{C}) tiles on their triangular grid and a three prototile set consisting of glugon, C tile and C cluster.

Several results follow. The arrangements of half-hexagonal tiles in any tiling are completely and uniquely described by the arrangements around the C_L and C_R tiles (and their mirror-images), that is by composed versions of the fourteen half-hexagonal prototiles. This necessarily unique composition implies that only non-periodic tilings are possible with the fourteen prototile set. Because the seven hexagons, A to G, and the three prototile set consisting of glugon, C tile and C cluster, are merely collections of the fourteen prototiles they too can only produce non-periodic tilings. The fourteen, seven and three prototile sets are aperiodic.

Lastly the three prototile set is shown to be functionally equivalent to a single tile, albeit the matching is no longer by geometric deformation of the sides of the prototile. The matching rules for the three prototile set are simplified and codified and the hexagonal cells that they form become the template for the hexagonal single tile. The function of the glugon becomes the 'vertex rule', the function of the C tile becomes the 'side rule' and the orientation of the tile is described the 'continuous stripe rule'. By virtue of the equivalence of the three prototile set the single tile, with its matching rules, is aperiodic.

The single tile matching conditions apply equally to the individual hexagons that make up the glugon, C tile and C cluster, or simply to the seven hexagons, A to G, so that the single tile version is also scale invariant. The simultaneous seven, three and single tile interpretations of a tiling is curious as are the unusual tilings that result from colouring distinctly the single tile and its mirror-image; tilings that, although they possess long-range order, are much more irregular than the now familiar quasicrystalline tilings.

The body of this proof is divided into the following sections.

1. The Fourteen and Seven Prototile Sets,
2. Arrangements around C_L and C_R tiles,
3. Matching of Composed Half-Hexagonal Tiles,
4. Glugons,
5. Tiling Structure,
6. Fourteen, Seven and Three Prototile Aperiodicity and
7. Functional Equivalence of the Monotile.

1. THE FOURTEEN AND SEVEN PROTOTILE SETS

Image files JMTSA1A.JPG and JMTSA1B.JPG belong to this section.

We assume a fourteen half-hexagonal prototile set, which is depicted in the first image file, JMTSA1A.JPG. The tiles have an up/down orientation evident by the three black curved stripes, the intermediate stripe crossing above the middle of the long side of the tile. These stripes cross the short sides towards one end of a side, each cutting its side in the same proportion. Because we will be concerned much with the hexagonal tiles formed by the left-right pairs of these half-hexagonal prototiles we number the short sides from top to bottom on the right tile, side 1, 2 and 3 and those on the left tile from bottom to top, side 4, 5 and 6. Considering only the stripes (which must continue across sides of abutting tiles) a side 1 can match only to a side 6, 5 or 3.

The letter-labels match A only to A, \bar{A} only to \bar{A} , etc. and possess clocksense. They can be enacted as the notches depicted below the tiles; one clockwise notch for A, two clockwise notches for B, etc. and one anti-clockwise notch for \bar{A} , etc. so that \bar{A} denotes the mirror-image of A. The letter-labels could be moved towards one end of a side and so perform the work of a curved stripe at the same time, but for the sake of clarity we leave them separate. Thus the matching conditions can be expressed by geometric deformations of the sides of the tiles. The tiles themselves also possess mirror-images, which may appear in a tiling. The mirror-image of an A_L tile is a \bar{A}_R tile, etc.

In most cases the left tile will only match, on its long side, its same named right mate, e.g. A_L matches only A_R . The exception is that C_L will match to E_R and E_L will match to C_R as well as to their namesakes (likewise the mirror-image pairs). Here see JMTSA1B.JPG. In this way nine hexagonal tiles can be made out of the fourteen half-hexagonal prototiles (and the mirror-images make nine more). Two of the nine, the exceptions just mentioned, will not tile because none of the tiles that will match to their sides 1 and sides 6 will match with each other. We are left with seven hexagonal tiles, which are assumed as a second prototile set.

2. ARRANGEMENTS AROUND C_L AND C_R TILES

Image file JMTSA2.JPG belongs to this section.

We now turn our attention to discovering which combinations of three of the fourteen half-hexagonal prototiles will match to the short sides of C_L and C_R tiles. These are the only arrangements of tiles we shall need to study exhaustively, however we won't be able to justify this selectivity until the establishment of the tiling structure in section 5.

The following observations may be confirmed by referring to the possible abutments among the prototiles.

A C_L tile may have on its side 6 either G_R , \bar{B}_R , \bar{F}_R or B_R .

If 6 has G_R then 5 has D_R and then 4 has \bar{B}_L ...Fig. 2.1.

or 4 has \bar{G}_L ...Fig. 2.2.

If 6 has \bar{B}_R then 5 has F_R and then 4 has \bar{G}_L ...Fig. 2.3.

or 4 has \bar{B}_L ...Fig. 2.4.

or 5 has E_R and then 4 has B_L ...Fig 2.5.

former belongs to a C_L and the latter to an E_R prototile (or C_R and E_L) and this abutment is not possible in a tiling (see section 1).

Because in all other respects the exchange of labelling is one-to-one and the labels are then the same for the fourteen composed arrangements and the fourteen prototiles the matching conditions are equivalent. The obvious mirror-image observations apply to the mirror-image prototiles and the composed arrangements around the \bar{C}_L and \bar{C}_R tiles.

4. GLUGONS

Image file JMTSA4.JPG belongs to this section.

Here we establish that three particular tiles, A, B and \bar{G} , (or their mirror-images, \bar{A} , \bar{B} and G) out of the seven hexagonal prototiles (and their mirror-images) appear in any tiling and only in abutment to one another in a unique way. Thus there exist 'glugons' made up of these three hexagonal tiles 'glued' together.

First we note that one of A, \bar{A} , B, \bar{B} , G or \bar{G} necessarily appears in any tiling by the fourteen prototiles (which is equivalent to saying, by the seven hexagonal prototiles). Of course to begin tiling with any one of these six tiles is to include that one in a tiling. To begin with a D tile implies one of A, \bar{A} , B or G (see Fig. 4.1(a)), because only one of these will abut side 1 of the D tile. Similarly to begin with an E tile or an F tile implies one of A, \bar{A} , \bar{B} or \bar{G} (Figs. 4.1(b) and 4.1(c)). Mirror-image observations apply to side 6 of \bar{D} , \bar{E} and \bar{F} tiles. From section 1 it can be seen that a C tile or \bar{C} tile cannot tile without other tile types, and then we have that any tiling has at least one of A, \bar{A} , B, \bar{B} , G or \bar{G} .

Next we show that given an A tile it must abut a \bar{G} tile on side 1 and B tile on side 6, the three tiles meeting at their top vertices (Fig. 4.2(a)). The A tile must have C or \bar{C} tiles on sides 2 and 5. The C/ \bar{C} tile on side 2 implies the \bar{G} tile on side 1; and the C/ \bar{C} tile on side 5 implies the B tile on side 6.

Given a B tile (Fig. 4.2(b)) it must abut a C/ \bar{C} tile on side 2, which implies an A tile on side 1 of the B tile. The A tile in turn must abut, on its side 2, another C/ \bar{C} tile. Side 6 of the B tile can now only abut a \bar{G} tile. The A+B+ \bar{G} arrangement formed is the same as that in Fig. 4.2(a).

Similarly, given a \bar{G} tile (Fig. 4.2(c)) it must abut, on its side 5, a C/ \bar{C} tile, which implies an A tile on side 6 of the \bar{G} tile. The A tile must abut, on its side 5, another C/ \bar{C} tile. Now the only tile that will abut side 1 of the given \bar{G} tile is a B tile. The A+B+ \bar{G} arrangement has again the same form and it is named a glugon (Fig. 4.3). By symmetric arguments the mirror-image \bar{A} - \bar{B} -G glugon follows the presence in a tiling of any of \bar{A} , \bar{B} or G tiles.

5. TILING STRUCTURE

Image file JMTSA5.JPG belongs to this section.

In this section we find that C or \bar{C} tiles must appear on a regular triangular grid admitting only one hexagonal tile between any two of them. Thus, in terms of half-hexagonal tiles, any tiling can be described fully by arrangements around C_L or C_R tiles (and their mirror-images).

A glugon must exist in any tiling. Any glugon must have C or \bar{C} tiles around it because no other tile will match the F and \bar{D} or \bar{F} and D side labels (Fig. 5.1). Such an attached C or \bar{C} tile can, at its opposite end, only abut another glugon: From section 2 the tiles which abut the top and bottom of a C or \bar{C} tile are one of

A, \bar{A} , B, \bar{B} , G or \bar{G} unless they form a composed C_L+C_R tile (or call it a composed C tile), which the presence of the first glugon prohibits.

With the existence of a glugon, glugons being surrounded by three C tiles and the C tiles abutting more glugons, any tiling possesses hexagonal cells with walls made of these tiles and 'holes' in the middle (Fig. 5.2). Obviously no glugon can be found in the hole. The hole has the shape of a cluster of seven hexagons, six around a central one. The central tile cannot be one of A, \bar{A} , B, \bar{B} , G or \bar{G} because no glugon can exist in the hole. Nor can the central tile be one of D, E or F because it could not find a match for its side 1, or if \bar{D} , \bar{E} or \bar{F} for its side 6, from among the remaining C, \bar{C} , D, \bar{D} , E, \bar{E} , F or \bar{F} (see section 4, Figs. 4.1(a), (b) and (c)). Therefore the central tile, if any tile, is a C or \bar{C} tile, and these tiles are established on the triangular grid.

Now it follows from the absence in the hole of any of A, \bar{A} , B, \bar{B} , G or \bar{G} tiles and the results of section 2 that it can only be filled in, if at all, by a composed C_L+C_R tile with its outer hexagons completed to form what we shall call a 'C cluster' (or the mirror-image, \bar{C} cluster). So if tiling is possible, it is possible by a three prototile set: the glugon, the C tile and the C cluster, and their mirror-images (Fig. 5.1).

6. FOURTEEN, SEVEN AND THREE PROTOTILE APERIODICITY

Image file JMTSA6.JPG belongs to this section.

With the triangular grid of C or \bar{C} tiles established any tiling can be completely and uniquely described in terms of arrangements around C_L and C_R tiles (and their mirror-images). We have seen that these arrangements are composed versions of the fourteen half-hexagonal prototiles with equivalent matching conditions. Such necessarily unique composition implies that any tiling by copies of these prototiles is non-periodic and therefore the fourteen prototile set is aperiodic. That the seven and three prototile sets are merely patches of tiling by the fourteen prototile set implies that they too are aperiodic.

It only remains to show that some tiling is possible. This is achieved by repeatedly enlarging a tile or patch of tiles and decomposing them, that is by inflation. See the accompanying image file. This decomposition is the reverse of the composition of section 3. A half-hexagonal tile is doubled in linear dimensions and divided into four half-hexagonal tiles in the previously established manner and the equivalent stripes and labels applied. Repeated application of this process will admit of a tiling of the plane.

7. FUNCTIONAL EQUIVALENCE OF THE MONOTILE

Image files JMTSA7A.JPG and JMTSA7B.JPG belong to this section.

The matching rules of the three prototile set may be simplified and yet give equivalent matching conditions. Where a glugon meets a C cluster the G and \bar{A} side labels can be replaced with a single clockwise notch intermediate to the two labels and the A and \bar{G} labels with a single anticlockwise notch. Where the glugon meets the C tile a similar replacement may be used: The \bar{D} and F side labels becomes a clockwise notch and the D and \bar{F} side labels becomes an anti-clockwise notch. To prevent cross-matching two styles of notch are needed. Here we used round and square notches (Fig. 7.1). Those curved stripes that are not now redundant can

be transmitted by triangular lugs and notches. These new matching rules are thus still classical, i.e. transmitted by geometric deformations of the sides of the tiles.

We can codify these changes entirely by stripes if we employ a mnemonic of blue stripe for clockwise and red stripe for anti-clockwise. Thick and thin stripes can be used to distinguish round and square clocksense notches. The coloured stripes continue across abutting sides where the notches engaged before. The hexagonal cells of the tiling structure (section 5) are now picked out by the thin, coloured stripes (Fig. 7.2).

These hexagonal cells may be styled single tiles and are marked with black curved stripes and thick blue and red stripes on their faces. The glugon becomes a 'vertex rule' which says that not three same coloured stripes can meet there and the C tile becomes a 'side rule' which says that a side of a tile has at one end a red stripe and at the other a blue stripe co-linear with it. The black curved stripe must be continued across abutting sides. Now that we no longer have notches, the blue and red colours can represent arrays of little clocksense curved arrows. With all these conditions a tiling can be described by a single tile and its mirror-image tile together with its matching rules (Fig. 7.3).

Furthermore the seven hexagonal tiles, A to G, can each take on the single tile attributes and \bar{A} to \bar{G} those of the mirror-image single tile: The single tile interpretation is scale-invariant (Fig. 7.4). It is aperiodic (though not classical) by virtue of its functional equivalence to the three prototile set. Its unique composition (forming larger single tiles from collections of smaller ones) is achieved on seven different tile arrangements and its decomposition is non-unique, the larger tile being transformed into one of seven different arrangements of smaller tiles, the choice of arrangement depending on the surroundings.

Of course the single tile cannot tile in any way other than the three prototile set to which it is equivalent. But the tilings by the three prototile set are rather cluttered and the unusual 'growth' or ordering pattern inherent in these tilings is striking when described by a single hexagonal tile coloured to contrast it with its mirror-image tile.

TASMANIA

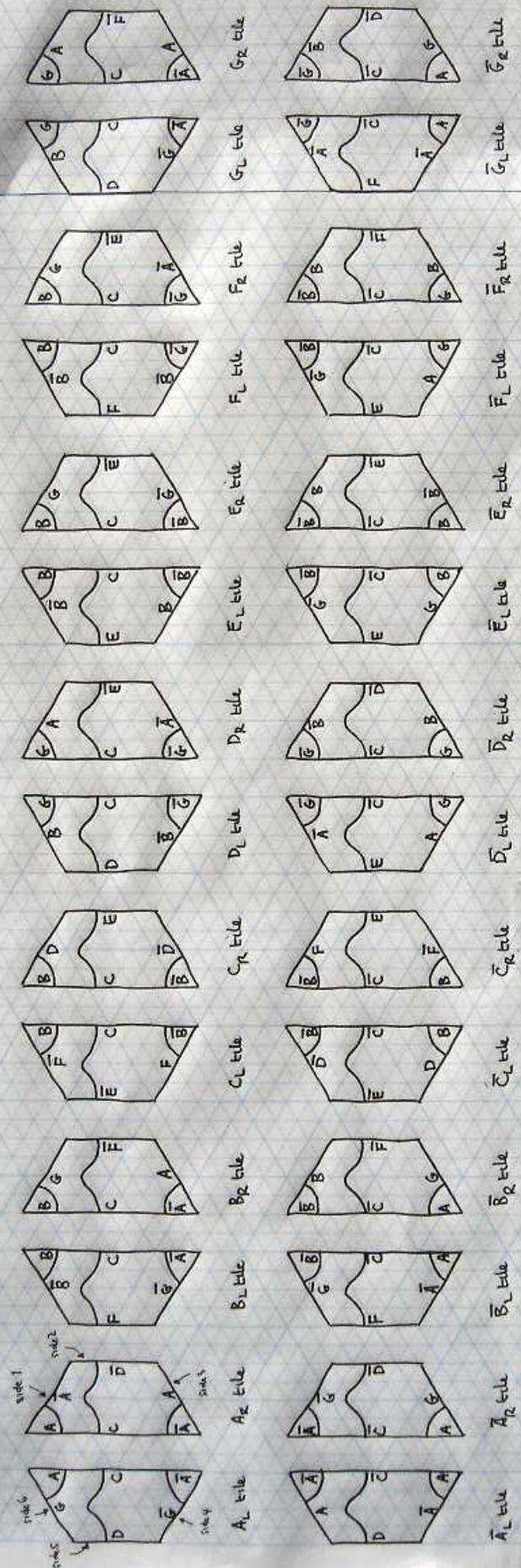
J. TAYLOR

Section 1

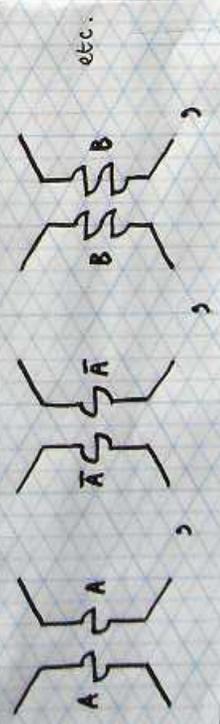
16 June, 2009

The Fourteen Half-hexagonal Prototiles and their Mirror-image tiles.

Some-labelled sides about and curved scribe is continuous.



Matching Rules: The letter labels are shorthand for notches which possess clockwise and self-matching, e.g.



If the notches were to coincide with the stripes in position they could serve both purposes. However, for clarity of argument they are better separate.

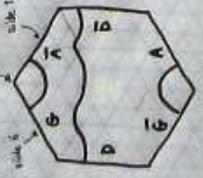
J. TAYLOR Section 1

16 June, 2009

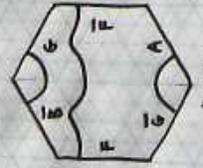
The seven hexagonal tiles and their mirror-image tiles.

The four been half hexagonal prototiles will form just nine hexagonal tiles, two of which are aberrant and cannot partake of tiling.

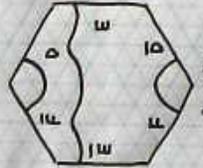
top vertex
side 6
side 1



A-tile



B-tile



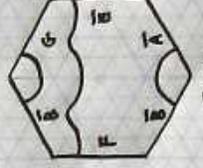
C-tile



D-tile



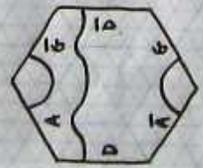
E-tile



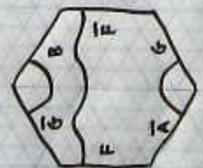
F-tile



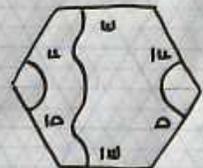
G-tile



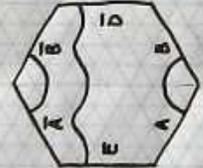
\bar{A} -tile



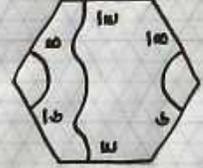
\bar{B} -tile



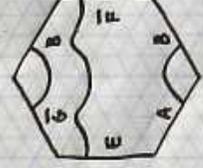
\bar{C} -tile



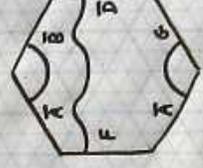
\bar{D} -tile



\bar{E} -tile

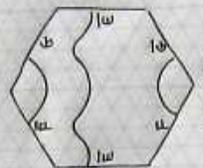


\bar{F} -tile

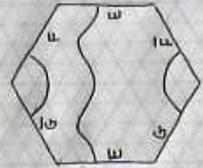


\bar{G} -tile

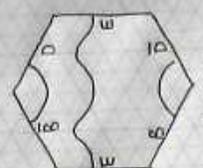
The two aberrant tiles and their mirror-image tiles.



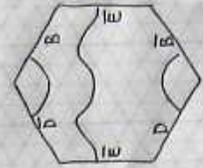
H-tile



\bar{H} -tile

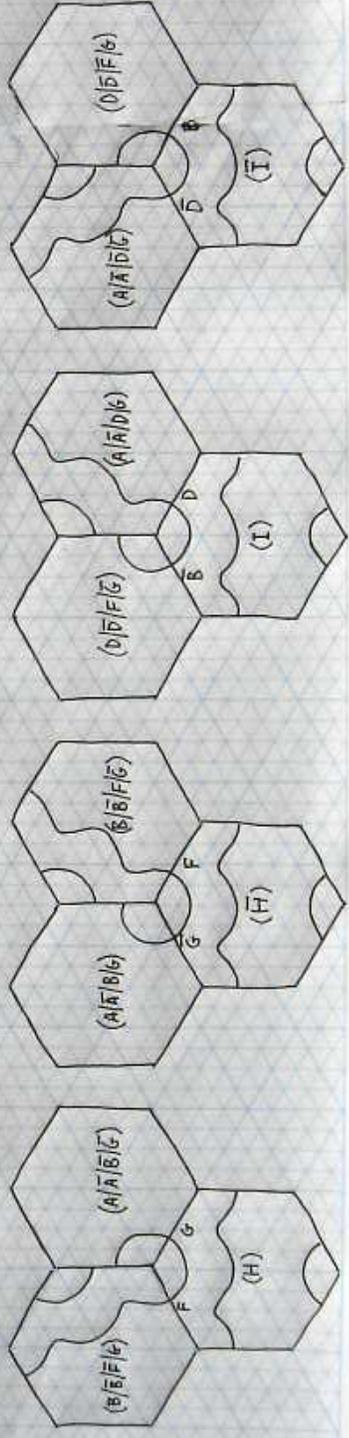


I-tile



\bar{I} -tile

The H, \bar{H} , I, and \bar{I} tiles cannot appear in a tiling. They imply tiling contradictions.



19 June, 2009

Section 2.

Arrangements around a C_L tile and a C_R tile
 Minor-image arrangements exist for E_L and E_R tiles.

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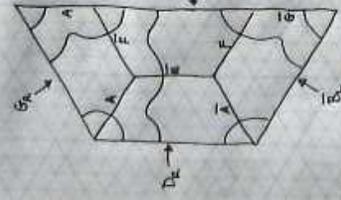


Fig 2.1 *

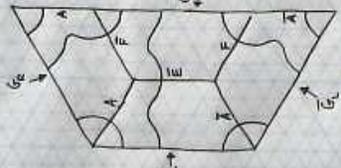


Fig 2.2 A_L



Fig 2.3 B_L

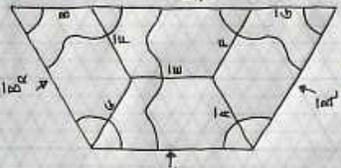


Fig 2.4 F_L

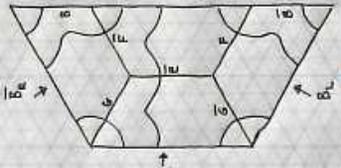


Fig 2.5 E_L

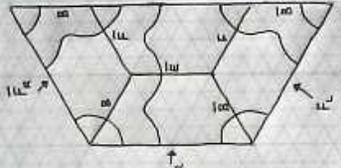


Fig 2.6 G_L

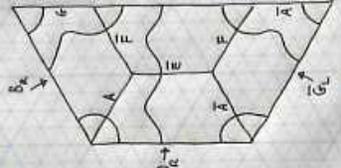


Fig 2.7 G_L

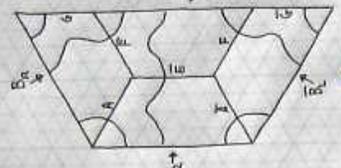


Fig 2.8 R_L

Because a C_L tile on its long side can only match a C_R and the starred arrangements here do not, on their long sides, match any other arrangement here they cannot exist in any tiling of the plane. The blue labels given to these arrangements will be justified in the next section.

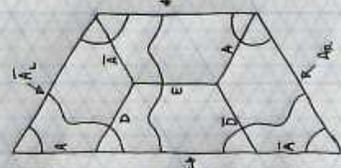


Fig 2.9 A_R

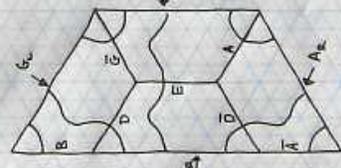


Fig 2.10 B_R

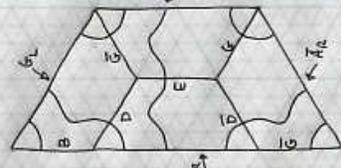


Fig 2.11 F_R

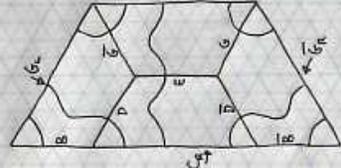


Fig 2.12 E_R



Fig 2.13 G_R



Fig 2.14 G_R

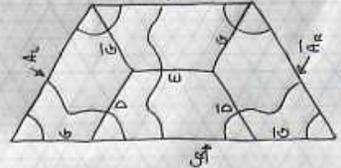


Fig 2.15 R_R

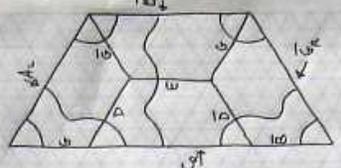
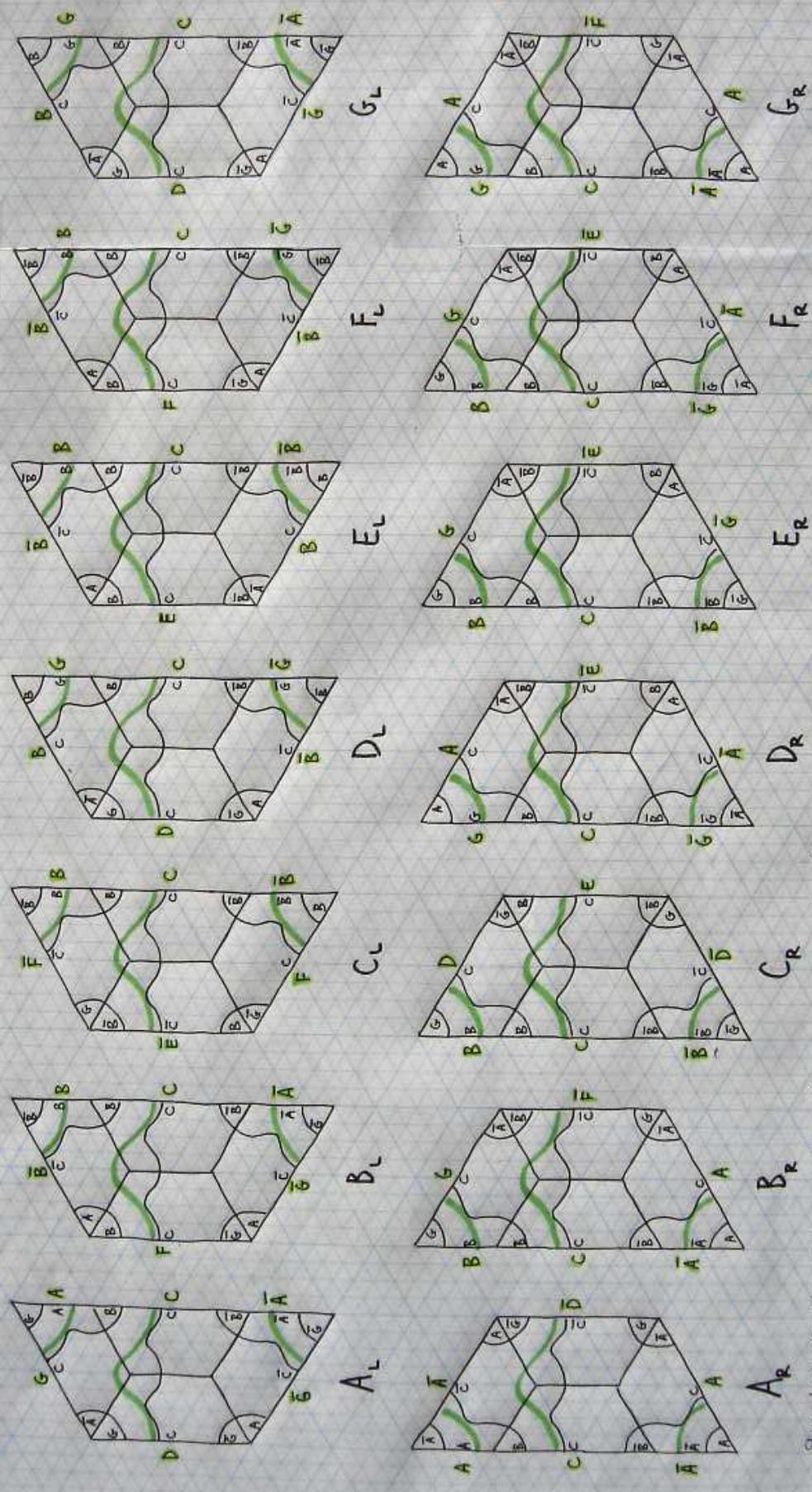


Fig 2.16 *

J. TAYLOR
19 June, 2009

Matching conditions of the composed tiles are equivalent to those of the prototiles.
The mirror-image letter-labels are found on the arrangements around \bar{C}_L and \bar{C}_R tiles and the mirror-image prototiles.



Shown here are the matching conditions of the viable composed tiles from section 2 (in black) and of the equivalent prototile (in green).
Recall from section 1 that the tile pairings C_L - E_R and E_L - C_R cause tiling failure. The two sets of matching conditions are equivalent.

J. TAYLOR
24 June, 2009

11 7 2009

Tiling Structure

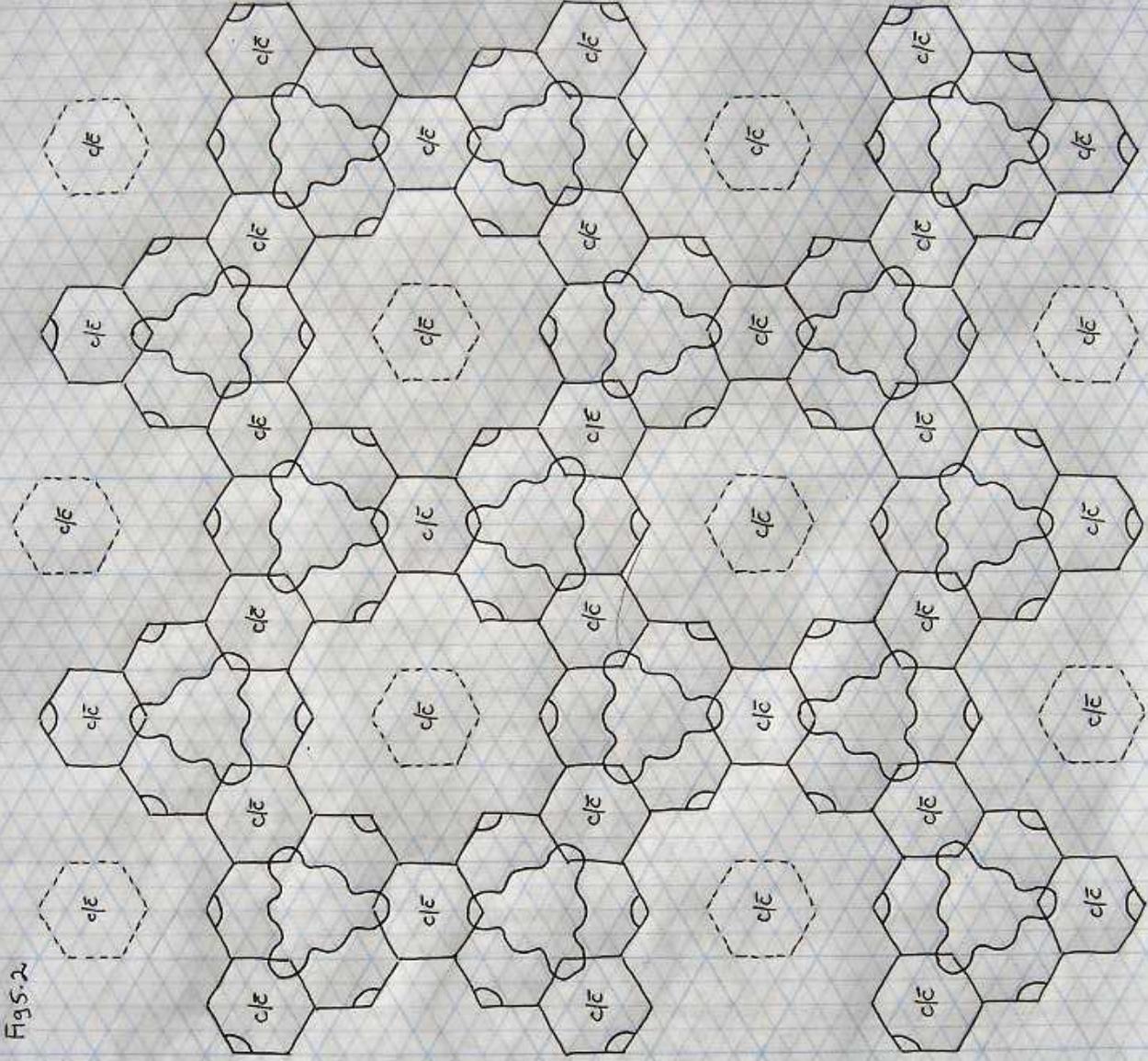
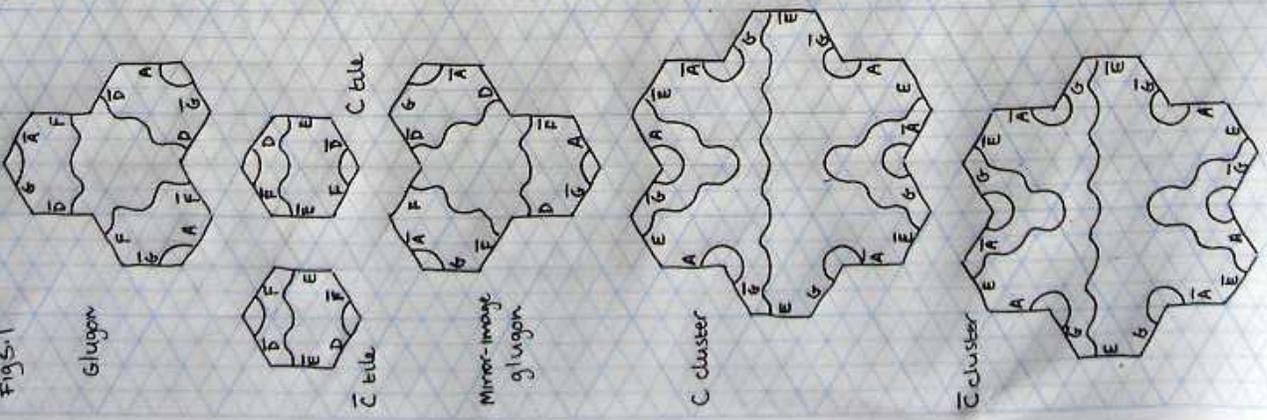


Fig 5.2

Section 5

Fig 5.1



Glugon

C tile

Mirror-image
glugon

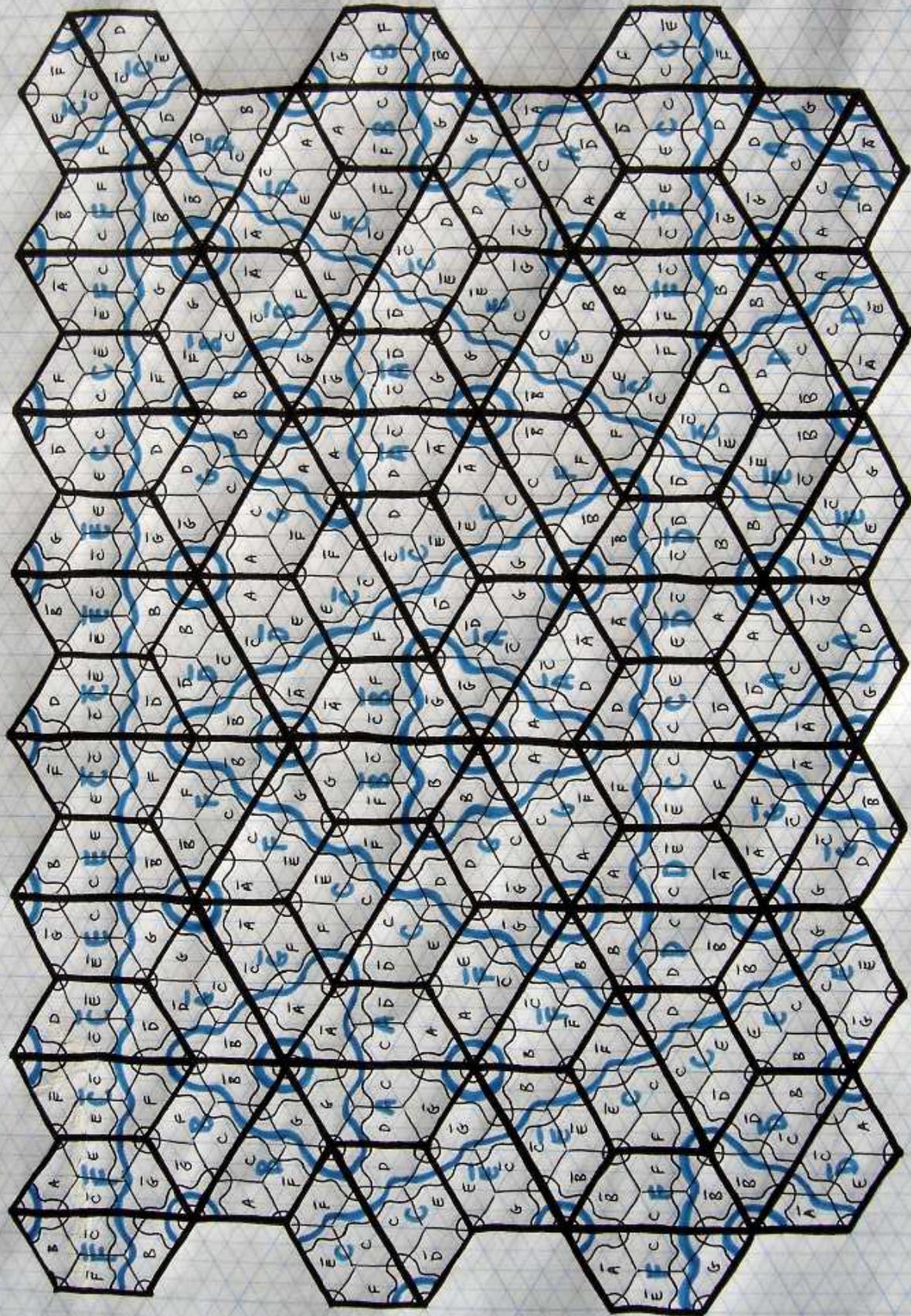
C cluster

C cluster

Section 6

Tiling is possible by enlargement and decomposition, i.e. inflation.

J. TAYLOR
26 June, 2009



Section 7.

Matching rules for the 3 prototile set can be simplified.

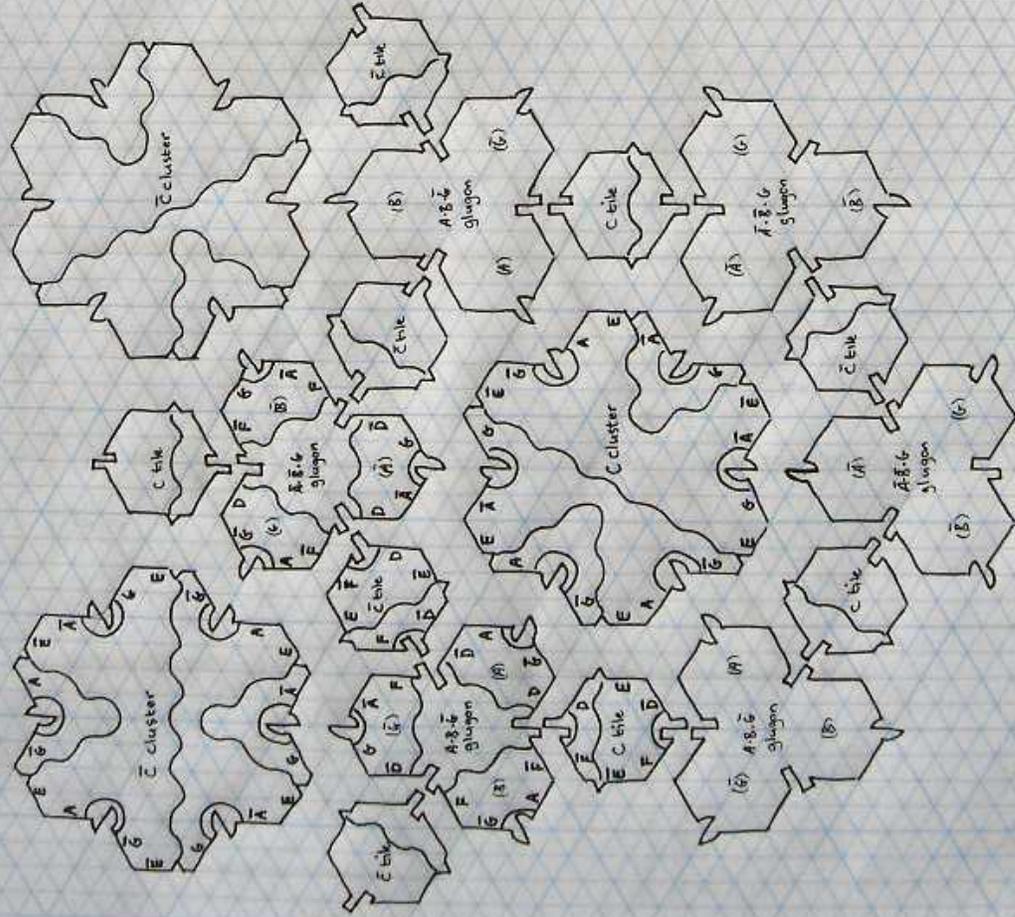


Fig. 7.1 The letter-labels can be replaced by clock-sense matches to equivalent effect. The necessary black curved stripes can be implemented as triangular legs and notches.

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30 June, 2009.

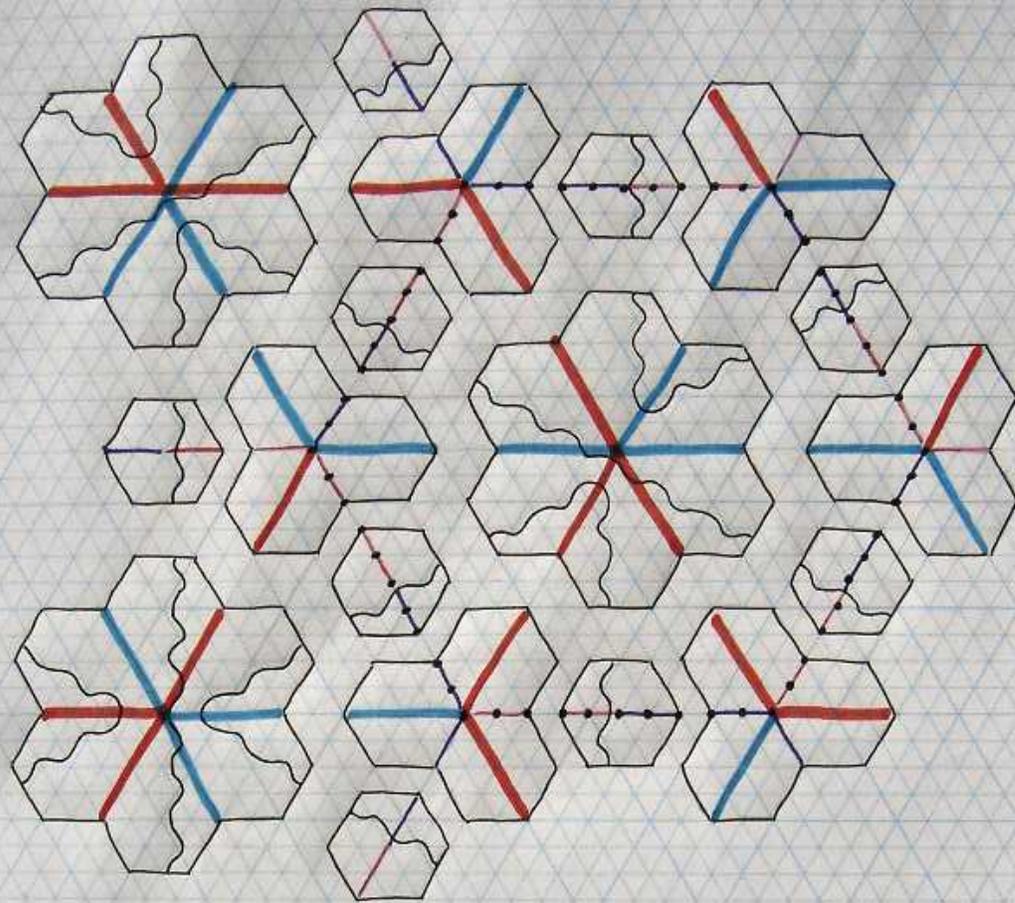


Fig. 7.2 Blue and red stripes can be used as a mnemonic for clockwise and anti-clockwise respectively. Recall that the tiling is made up of hexagonal cells - dotted line.

The 3 prototile set is functionally equivalent to a single tile.

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July, 2009

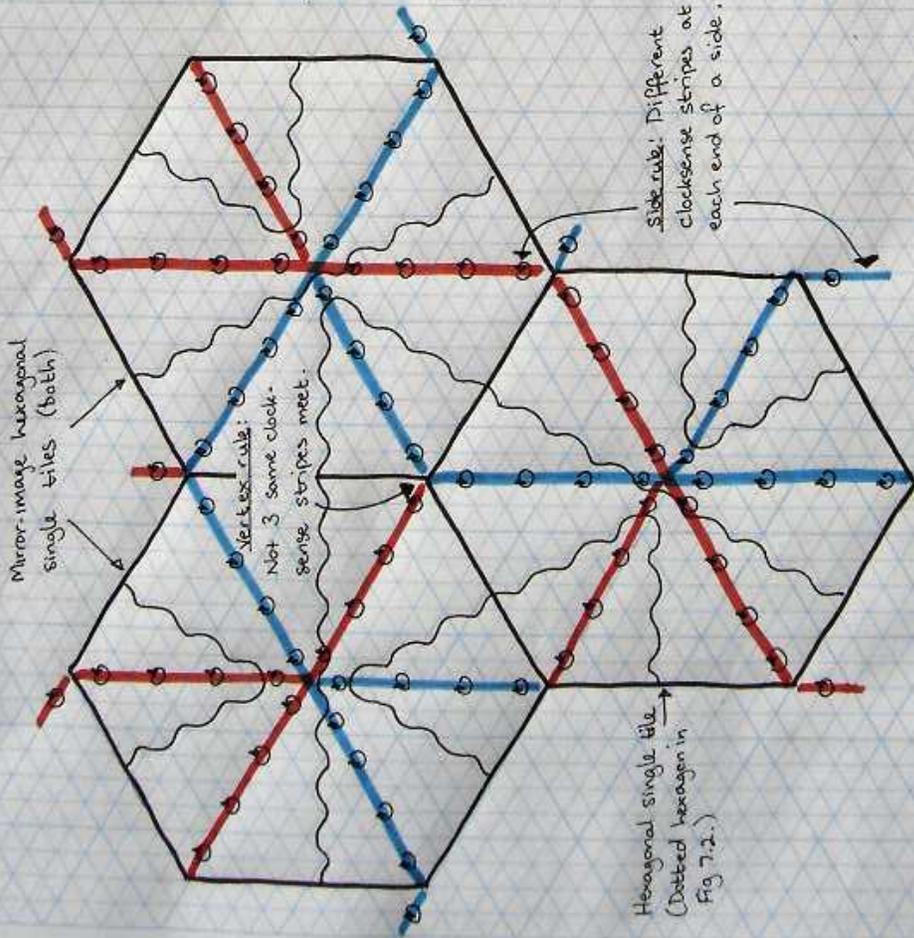


Fig 7.3 The glugon is replaced by the vertex rule and the C tile by the side rule. The black curved stripe is merely a stripe, not transmitted by lugs and notches. The clock-sense stripes can be thought of as arrays of little curved arrows. (The red and blue colours are merely mnemonic.)

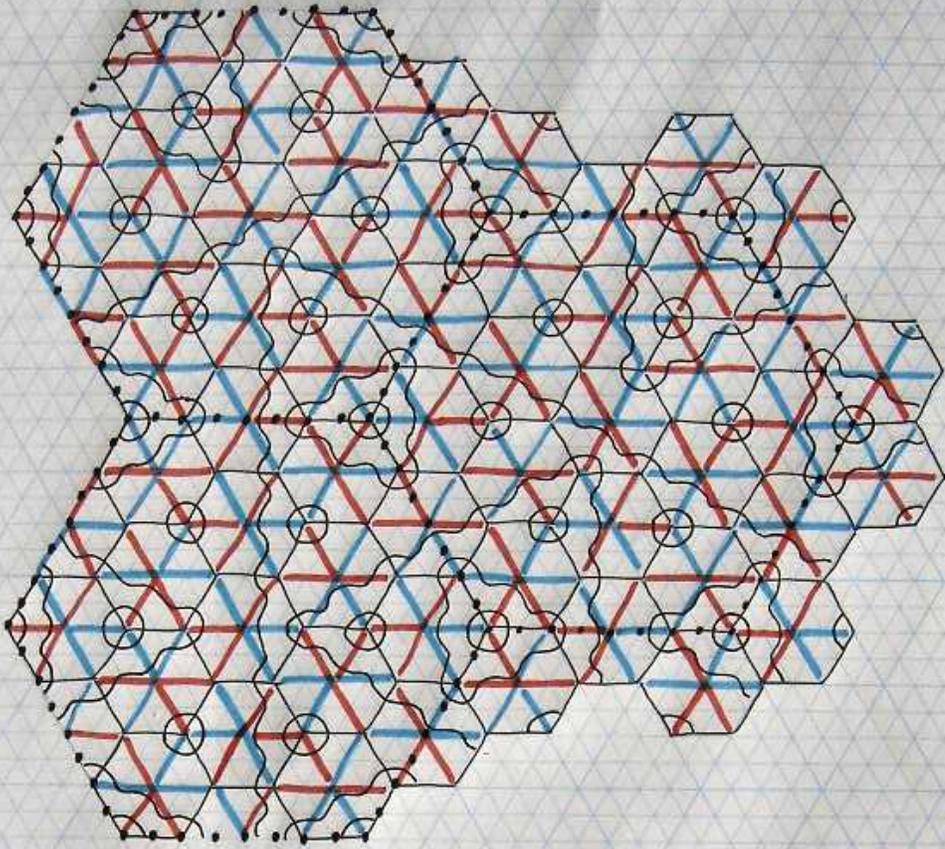


Fig 7.4 The individual hexagons, A to G, which make up the 3 tile set, as in Fig 7.1, can each assume the single tile character. Each of the A to G tiles is a single tile and each of \bar{A} to \bar{G} is the mirror-image single tile.

