# ON SOME EULER-MAHONIAN DISTRIBUTIONS

ANGELA CARNEVALE

ABSTRACT. We prove that the pair of statistics (des, maj) on multiset permutations is equidistributed with the pair (stc, inv) on certain quotients of the symmetric group. We define the analogue of the statistic stc on multiset permutations, whose joint distribution with the inversions equals that of (des, maj). We extend the definition of the statistic stc to hyperoctahedral and even hyperoctahedral groups. Such functions, together with the Coxeter length, are equidistributed, respectively, with (ndes,nmaj) and (ddes,dmaj).

## 1. INTRODUCTION

The first result about the enumeration of multiset permutations with respect to statistics now called *descent number* and *major index* is due to MacMahon. Let $\rho = (\rho_1, \ldots, \rho_m)$ be a composition of $N \in \mathbb{N}$. We denote by $S_\rho$ the set of all permutations of the multiset $\{1^{\rho_1}, \ldots, m^{\rho_m}\}$. The *descent set* $\mathrm{Des}(w)$ of $w = w_1 \cdots w_N \in S_\rho$ is $\mathrm{Des}(w) = \{i \in [N-1] \mid w_i > w_{i+1}\}$. The descent and major index statistics on $S_\rho$ are

$$\mathrm{des}(w) = |\mathrm{Des}(w)| \quad \text{and} \quad \mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i.$$

Then ([9, §462, Vol. 2, Ch. IV, Sect. IX])

$$(1.1) \qquad \sum_{k \geq 0} \left( \prod_{i=1}^{m} \binom{\rho_j + k}{k}_q \right) x^k = \frac{\sum_{w \in S_\rho} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)}}{\prod_{i=0}^{N}(1 - xq^i)} \in \mathbb{Z}[q][[x]],$$

where, for $n, k \in \mathbb{N}$ we put

$$\binom{n}{k}_p = \frac{[n]_p!}{[n-k]_p![k]_p!}, \qquad [n]_p! = \prod_{i=1}^{n}[i]_p, \qquad [n]_p = \sum_{i=0}^{n-1} p^i.$$

The well known result about the equidistribution, on multiset permutations, of the *inversion number* with the major index, goes also back to MacMahon; Foata and Schützenberger [6] proved that this equidistribution refines, in the case of the symmetric group, to inverse descent classes. A pair of statistics that is equidistributed with (des, maj) is called Euler-Mahonian. In [11] Skandera introduced an Eulerian statistic, which he called stc, on the symmetric group, and proved that the pair (stc, inv) is Euler-Mahonian.

In this note we prove that the joint distribution of (stc, inv) on certain quotients of the symmetric group is indeed the same as the distribution of (des, maj) on multiset permutations; we use this result to define a statistic mstc that is Eulerian on multiset permutations and that, together with inv is equidistributed with the pair (des, maj).

The Eulerian polynomial is (essentially) the descent polynomial on the symmetric group $S_n$. Frobenius proved (see [7]) that this polynomial has real, simple, negative roots, and that $-1$ features as a root if and only if $n$ is even. Simion proved later that the descent polynomials of permutations of any multiset are also real rooted, with simple, negative roots (see [10]). We use our first result of equidistribution to show that on the set of permutations of words in the alphabet $\{1^r, 2^r\}$, the polynomial of the joint distribution of des and maj admits, for odd

$r$ a unique *unitary* factor. This factorisation, together with the one of Carlitz's $q$-Eulerian polynomial (the polynomial of the joint distribution of des and maj on the symmetric group) that we show in [4], may be considered a refinement of Frobenius' result, and supports a conjecture we made in [4] and that we translate in Section 2 in terms of the joint distribution of (stc, inv) on quotients of the symmetric group.

Generalisations of MacMahon's result (1.1) to signed permutations were first obtained by Adin, Brenti and Roichman in [1] and to even-signed permutations by Biagioli in [2]. In the last section of this note we define Eulerian statistics nstc and dstc that, together with the length, are equidistributed, respectively, with the Euler-Mahonian pairs (ndes, nmaj) on the hyperoctahedral group and (ddes, dmaj) on the even hyperoctahedral group.

## 2. Stc on quotients of the symmetric group and multiset permutations

For $n, m \in \mathbb{N}$, $m \le n$ we denote with $[n] = \{1, \ldots, n\}$ and $[m, n] = \{m, m+1, \ldots, n\}$. For a permutation $\sigma \in S_n$ we use the one-line notation or the disjoint cycle notation.

The Coxeter length $\ell$ for $\sigma \in S_n$ coincides with the inversion number $\mathrm{inv}(\sigma) = |\{(i, j) \in [n] \times [n] \mid i < j, \sigma(i) > \sigma(j)\}|$. Also, for a (signed) permutation $\sigma \in S_n$ (respectively, $B_n$), we let $\mathcal{I}(\sigma) = \{(i, j) \in [n] \times [n] \mid \sigma(i) > \sigma(j)\}$.

It is well-known that the symmetric group $S_n$ is in bijection with the set of words $w = w_1 \cdots w_n \in E_n$ where

$$E_n = \{w_1 \cdots w_n \mid w_i \in [0, n-i], \text{ for } i = 1, \ldots, n-1\}.$$

One of such bijections is the Lehmer code, defined as follows.

For $\sigma \in S_n$, $\mathrm{code}(\sigma) = c_1 \cdots c_n \in E_n$ where $c_i = |\{j \in [i+1, n] \mid \sigma(i) > \sigma(j)\}|$. The sum of the $c_i$s gives, for each permutation, the inversion number. The statistic stc, that together with the length constitutes an Euler-Mahonian pair equidistributed with (des, maj), is defined as follows (cf. [11, Definition 3.1]): $\mathrm{stc}(\sigma) = \mathrm{st}(\mathrm{code}(\sigma))$, where for a word $w \in E_n$

$$\mathrm{st}(w) = \max\{r \in [n] \mid \exists \, 1 \le i_1, \cdots < i_r \le n \mid w_{i_1} \cdots w_{i_r} > (r-1)(r-2) \cdots 1\, 0\}$$

that is, the maximal $r$ for which there exists a subword of $w$ of length $r$ elementwise strictly greater than the *r-staircase word* $(r-1)(r-2) \cdots 1\, 0$.

For example let $\sigma = 452361 \in S_6$. Then $\mathrm{code}(\sigma) = 331110$, $\mathrm{inv}(\sigma) = \sum_i c_i = 9$, $\mathrm{stc}(\sigma) = \mathrm{st}(\mathrm{code}(\sigma)) = 3$. So defined, the statistic stc constitutes an Eulerian partner for the inversions on $S_n$.

**Theorem 2.1.** [11, Theorem 3.1] *Let $n \in \mathbb{N}$. Then*

$$\sum_{w \in S_n} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} = \sum_{w \in S_n} x^{\mathrm{stc}(w)} q^{\ell(w)}$$

Given a composition $\rho$ of $N$, the corresponding set of multiset permutations $S_\rho$ is naturally in bijection with certain quotients and inverse descent classes of $S_N$. In particular, for $\rho = (\rho_1, \ldots, \rho_m)$ a composition of $N$, for $i = 1 \ldots, m-1$ we let

$$(2.1) \qquad\qquad r_i = \sum_{k=1}^{i} \rho_k \text{ and } R = \{r_i \mid i \in [m]\} \subseteq [N-1].$$

We let $S_N^{R^c}$ and $IS_N^R$ denote, respectively, the quotient and the inverse descent class of the symmetric group

$$S_N^{R^c} = \{w \in S_N \mid \mathrm{Des}(w) \subseteq R\}, \qquad IS_N^R = \{w \in S_N \mid \mathrm{Des}(w^{-1}) \subseteq R\}.$$

A natural way to associate a permutation to a multiset permutation is the standardisation. Given $\rho$ a composition of $N$ and a word $w$ in the alphabet $\{1^{\rho_1}, \ldots, m^{\rho_m}\}$, $\mathrm{std}(w)$ is the

element of $S_N$ obtained by $w$ substituting, in the order of appearance in $w$ from left to right, the $\rho_1$ 1s with the sequence $1\, 2 \ldots \rho_1$, the $\rho_2$ 2s with the sequence $\rho_1 + 1 \ldots \rho_1 + \rho_2$ and so on. So for example if $\rho = (2, 3, 2)$ and $w = 1223132 \in S_\rho$, then $\mathrm{std}(w) = 1346275 \in S_7$.

The following result is due to Foata and Han.

**Proposition 2.2.** [5, Proprieté 2.2] *Let $n \in \mathbb{N}$, $J \subseteq [n-1]$. Then*

$$(2.2) \qquad \sum_{\substack{\{w \in S_n \mid \\ \mathrm{Des}(w) = J\}}} x^{\mathrm{des}(w^{-1})} q^{\mathrm{maj}(w^{-1})} = \sum_{\substack{\{w \in S_n \mid \\ \mathrm{Des}(w) = J\}}} x^{\mathrm{stc}(w)} q^{\ell(w)}.$$

**Proposition 2.3.** *Let $N \in \mathbb{N}$, $\rho$ a composition of $N$ and $R \subseteq [N-1]$ as in (2.1). The pair $(\mathrm{stc}, \ell)$ on $S_N^{R^c}$ is equidistributed with $(\mathrm{des}, \mathrm{maj})$ on $S_\rho$:*

$$(2.3) \qquad C_\rho(x, q) = \sum_{w \in S_\rho} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} = \sum_{w \in S_N^{R^c}} x^{\mathrm{stc}(w)} q^{\ell(w)}.$$

*Proof.* The standardisation std is a bijection between $S_\rho$ and $IS_N^R$, and preserves des and maj, so

$$\sum_{w \in S_\rho} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} = \sum_{w \in S_\rho} x^{\mathrm{des}(\mathrm{std}(w))} q^{\mathrm{maj}(\mathrm{std}(w))} = \sum_{w \in IS_N^R} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)}.$$

By Proposition 2.2 the last term is the desired distribution on $S_N^{R^c}$:

$$\sum_{w \in IS_N^R} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} = \sum_{w \in S_N^{R^c}} x^{\mathrm{stc}(w)} q^{\ell(w)}. \qquad \square$$

As an application, we prove a result about the bivariate factorisation of the polynomial $C_\rho(x, q)$, that in [4] is used to prove deduce analytic properties of some orbit Dirichlet series. We say that a bivariate polynomial $f(x, y) \in \mathbb{Z}[x, y]$ is unitary if there exist integers $\alpha, \beta \geq 0$ and $g \in \mathbb{Z}[t]$ so that $f(x, y) = g(x^\alpha y^\beta)$ and all the complex roots of $g$ lie on the unit circle (see also [4, Remark 2.9]).

**Proposition 2.4.** *Let $\rho = (r, r)$ where $r \equiv 1 \pmod 2$. Then*

$$(2.4) \qquad C_\rho(x, q) = (1 + xq^r)\widetilde{C}_\rho(x, q),$$

*where $\widetilde{C}_\rho(x, q)$ has no unitary factor.*

Before we prove Proposition 2.4, we give a nice description of the stc for permutations with at most one descent.

**Lemma 2.5.** *Let $\rho = (\rho_1, \rho_2)$, $N = \rho_1 + \rho_2$. Let $w \in S_N^{\{\rho_1\}^c}$. Then*

$$\mathrm{stc}(w) = |\{i \in [\rho_1] \mid w(i) > \rho_1\}|.$$

*Proof.* A permutation $w \in S_N^{\{\rho_1\}^c}$ has at most a descent at $\rho_1$, so its code is of the form $\mathrm{code}(w) = c_1 \cdots c_{\rho_1} 0 \cdots 0$, with $0 \leq c_1 \leq \ldots \leq c_{\rho_1}$. The first (possibly) non-zero element of the code is exactly the number of elements of the second block for which the image is in the first block. This number coincides with the length of the longest possible subword of the code which is elementwise greater than a staircase word. $\square$

*Proof of Proposition 2.4.* The polynomial $C_\rho(x, 1)$, descent polynomial of $S_\rho$, has all real, simple, negative roots (cf. [10, Corollary 2]). Thus a factorisation of the form (2.4) implies that $\widetilde{C}_\rho(x, q)$ has no unitary factor. To prove (2.4) we define an involution $\varphi$ on $S_N^{R^c}$ such that, for all $w \in S_N^{R^c}$, $|\ell(\varphi(w)) - \ell(w)| = r$ and $|\mathrm{stc}(\varphi(w)) - \mathrm{stc}(w)| = 1$.

For $w \in S_N^{R^c}$ we let

$$M_w = \{i \in [r] \mid w^{-1}(i) \leq r \text{ and } w^{-1}(i+r) > r \text{ or } w^{-1}(i) > r \text{ and } w^{-1}(i+r) \leq r\},$$

that is, the set of $i \in [r]$ for which $i$ and $i + r$ are not in the same ascending block. Since $r$ is odd, $M_w$ is non-empty for all $w \in S_\rho$. We then define $\varphi(w) = ((\iota, \iota + r)w)^{R^c}$, where $\iota = \min\{i \in M_w\}$ and, for $\sigma \in S_N$, $\sigma^{R^c}$ denotes the unique minimal coset representative in the quotient $S_N^{R^c}$. By Lemma 2.5 clearly $\mathrm{stc}(\varphi(w)) = \mathrm{stc}(w) \pm 1$. Suppose now that $w^{-1}(\iota) \leq r$ and $w^{-1}(\iota) > r$ (the other case is analogous). Then

$$\ell(\varphi(w)) = \ell(w) + |\{i \in [r] \mid w(i) > \iota\}| + |\{i \in [r+1, 2r] \mid w(i) < \iota + r\}| = \ell(w) + r - i + i. \quad \square$$

We reformulate [4, Conjecture B] in terms of the bivariate distribution of $(\mathrm{stc}, \ell)$ on quotients of the symmetric group.

**Conjecture A.** *Let $\rho$ be a composition of $N$ and $R \subseteq [N-1]$ constructed as in (2.1). Then $C_\rho(x, q) = \sum_{w \in S_N^{R^c}} x^{\mathrm{stc}(w)} q^{\ell(w)}$ has a unitary factor if and only if $\rho = (\rho_1, \ldots, \rho_m)$ where $\rho_1 = \ldots = \rho_m = r$ for some odd $r$ and even $m$. In this case*

$$\sum_{w \in S_N^{R^c}} x^{\mathrm{stc}(w)} q^{\ell(w)} = (1 + xq^{\frac{rm}{2}})\widetilde{C}_\rho(x, q)$$

*for some $\widetilde{C}_\rho(x, q) \in \mathbb{Z}[x, q]$ with no unitary factors.*

Proposition 2.3 suggests a natural extension of the definition of the statistic stc to multi-permutations, thus answering a question raised in [11].

For $w \in S_\rho$, $\mathrm{std}(w) \in IS_N^R$. So we have a bijection between multiset permutations $S_\rho$ and the quotient $S_N^{R^c}$

$$\mathrm{istd} : S_\rho \to S_N^{R^c}, \quad \mathrm{istd}(w) = (\mathrm{std}(w))^{-1}$$

which is inversion preserving: $\mathrm{inv}(w) = \mathrm{inv}(\mathrm{istd}(w))$.

*Definition* 2.6. Let $\rho$ be a composition of $N$. For a multiset permutation $w \in S_\rho$ the *multistc* is

$$\mathrm{mstc}(w) = \mathrm{stc}(\mathrm{istd}(w)).$$

The pair $(\mathrm{mstc}, \mathrm{inv})$ is equidistributed with $(\mathrm{des}, \mathrm{maj})$ on $S_\rho$, as

$$\sum_{w \in S_\rho} x^{\mathrm{mstc}(w)} q^{\mathrm{inv}(w)} = \sum_{w \in S_N^{R^c}} x^{\mathrm{stc}(w)} q^{\mathrm{inv}(w)} = \sum_{w \in S_\rho} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)},$$

which together with (1.1) proves the following theorem.

**Theorem 2.7.** *Let $\rho$ be a composition of $N \in \mathbb{N}$. Then*

$$\sum_{k \geq 0} \left( \prod_{i=1}^{m} \binom{\rho_j + k}{k}_q \right) x^k = \frac{\sum_{w \in S_\rho} x^{\mathrm{mstc}(w)} q^{\mathrm{inv}(w)}}{\prod_{i=0}^{N}(1 - xq^i)} \in \mathbb{Z}[q][[x]].$$

## 3. Signed and even-signed permutations

MacMahon's result (1.1) for the symmetric group (i.e. for $\rho_1 = \ldots \rho_m = 1$) is often present in the literature as Carlitz's identity, satisfied by Carlitz's $q$-Eulerian polynomial $A_n(x, q) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)}$.

Such result was extended, for suitable statistics, to the groups of signed and even-signed permutations. The major indices so defined are in both cases equidistributed with the Coxeter length $\ell$. In this section we define type $B$ and type $D$ analogues of the statistic stc, that together with the length satisfy these generalised Carlitz's identities.

3.1. **Eulerian companion for the length on $B_n$.** Let $n \in \mathbb{N}$. The hyperoctahedral group $B_n$ is the group of permutations $\sigma = \sigma_1 \cdots \sigma_n$ of $\{\pm 1, \ldots, \pm n\}$ for which $|\sigma| = |\sigma_1| \ldots |\sigma_n| \in S_n$. For $\sigma \in B_n$, the negative set and negative statistic are

$$\text{Neg}(\sigma) = \{i \in [n] \mid \sigma(i) < 0\} \quad \text{neg}(\sigma) = |\text{Neg}(\sigma)|.$$

The Coxeter length $\ell$ for $\sigma$ in $B_n$ has the following combinatorial interpretation (see, for instance [3]):

$$\ell(\sigma) = \text{inv}(\sigma) + \text{neg}(\sigma) + \text{nsp}(\sigma),$$

where inv is the usual inversion number and $\text{nsp}(\sigma) = |\{(i,j) \in [n] \times [n] \mid i < j, \ \sigma(i) + \sigma(j) < 0\}|$ is the number of *negative sum pairs*.

In [1] an Euler-Mahonian pair of the *negative* type was defined as follows. The negative descent and negative major index are, respectively,

$$(3.1) \qquad \text{ndes}(\sigma) = \text{des}(\sigma) + \text{neg}(\sigma), \qquad \text{nmaj}(\sigma) = \text{maj}(\sigma) - \sum_{i \in \text{Neg}(\sigma)} \sigma(i).$$

The pair $(\text{ndes}, \text{nmaj})$ satisfies the following generalised Carlitz's identity.

**Theorem 3.1.** [1, Theorem 3.2] *Let $n \in \mathbb{N}$. Then*

$$(3.2) \qquad \sum_{r \geq 0} [r+1]_q^n x^r = \frac{\displaystyle\sum_{\sigma \in B_n} x^{\text{ndes}(\sigma)} q^{\text{nmaj}(\sigma)}}{(1-x) \displaystyle\prod_{i=1}^{n} (1 - x^2 q^{2i})} \quad \text{in } \mathbb{Z}[q][[x]].$$

Motivated by (3.1) and the well-known fact that the length in type $B$ may be also written as

$$(3.3) \qquad \ell(\sigma) = \text{inv}(\sigma) - \sum_{i \in \text{Neg}(\sigma)} \sigma(i),$$

we define the analogue of the statistic stc for signed permutations as follows.

*Definition* 3.2. Let $\sigma \in B_n$. Then

$$\text{nstc}(\sigma) = \text{stc}(\sigma) + \text{neg}(\sigma).$$

**Theorem 3.3.** *Let $n \in \mathbb{N}$. Then*

$$\sum_{\sigma \in B_n} x^{\text{nstc}(\sigma)} q^{\ell(\sigma)} = \sum_{\sigma \in B_n} x^{\text{ndes}(\sigma)} q^{\text{nmaj}(\sigma)}.$$

*Proof.* We use essentially the same argument as in the proof of [8, Theorem 3]. There, the following decomposition of $B_n$ is used. Every permutation $\tau \in S_n$ is associated with $2^n$ elements of $B_n$, via the choice of the $n$ signs. More precisely, given a signed permutation $\sigma \in B_n$ one can consider the ordinary permutation in which the elements are in the same relative positions as in $\sigma$. We write $\pi(\sigma) = \tau$. Then

$$B_n = \bigcup_{\tau \in S_n} B(\tau),$$

where $B(\tau) = \{\sigma \in B_n \mid \pi(\sigma) = \tau\}$. So every $\sigma \in B_n$ is uniquely identified by the permutation $\tau = \pi(\sigma)$ and the choice of signs $J(\sigma) = \{\sigma(j) \mid j \in \text{Neg}(\sigma)\}$.

Clearly, for $\sigma \in B_n$ we have $\mathfrak{I}(\sigma) = \mathfrak{I}(\pi(\sigma))$, and thus $\text{stc}(\sigma) = \text{stc}(\pi(\sigma))$. So, for $\tau = \pi(\sigma)$

$$x^{\text{nstc}(\sigma)} q^{\ell(\sigma)} = x^{\text{stc}(\tau)} q^{\text{inv}(\tau)} \prod_{j \in J(\sigma)} x q^j.$$

The claim follows, as

$$\sum_{\sigma \in B_n} x^{\text{nstc}(\sigma)} q^{\ell(\sigma)} = \sum_{\sigma \in B(\tau)} \sum_{\tau \in S_n} x^{\text{stc}(\tau)} q^{\text{inv}(\tau)} \sum_{J \subseteq [n]} \prod_{j \in J} x q^j = A_n(x, q) \prod_{i=1}^{n} (1 + x q^i). \qquad \square$$

**Corollary 3.4.** *Let* $n \in \mathbb{N}$. *Then*

$$(3.4) \qquad \sum_{r \geq 0} [r + 1]_q^n x^r = \frac{\displaystyle\sum_{\sigma \in B_n} x^{\text{nstc}(\sigma)} q^{\ell(\sigma)}}{(1 - x) \displaystyle\prod_{i=1}^{n} (1 - x^2 q^{2i})} \quad \text{in } \mathbb{Z}[q][[x]].$$

3.2. **Eulerian companion for the length on** $D_n$**.** The even hyperoctahedral group $D_n$ is the subgroup of $B_n$ of signed permutations for which the negative statistic is even:

$$D_n = \{\sigma \in B_n \mid \text{neg}(\sigma) \equiv 0 \pmod 2\}.$$

Also for $\sigma$ in $D_n$ the Coxeter length can be computed in terms of statistics:

$$(3.5) \qquad\qquad\qquad \ell(\sigma) = \text{inv}(\sigma) + \text{nsp}(\sigma).$$

The problem of finding an analogue, on the group $D_n$ of even signed permutations, was solved in [2], where type $D$ statistics des and maj were defined, as follows. For $\sigma \in D_n$

$$(3.6) \qquad \text{ddes}(\sigma) = \text{des}(\sigma) + |\text{DNeg}(\sigma)|, \qquad \text{dmaj}(\sigma) = \text{maj}(\sigma) - \sum_{i \in \text{DNeg}(\sigma)} (\sigma(i) + 1),$$

where $\text{DNeg}(\sigma) = \{i \in [n] \mid \sigma(i) < -1\}$. The following holds.

**Theorem 3.5.** [2, Theorem 3.4] *Let* $n \in \mathbb{N}$. *Then*

$$(3.7) \qquad \sum_{r \geq 0} [r + 1]_q^n x^r = \frac{\displaystyle\sum_{\sigma \in D_n} x^{\text{ddes}(\sigma)} q^{\text{dmaj}(\sigma)}}{(1 - x)(1 - x q^n) \displaystyle\prod_{i=1}^{n-1} (1 - x^2 q^{2i})} \quad \text{in } \mathbb{Z}[q][[x]].$$

*Definition* 3.6. Let $\sigma \in D_n$. We set

$$\text{dstc}(\sigma) = \text{stc}(\sigma) + |\text{DNeg}(\sigma)| = \text{stc}(\sigma) + \text{neg}(\sigma) + \varepsilon(\sigma),$$

where

$$\varepsilon(\sigma) = \begin{cases} -1 & \text{if } \sigma^{-1}(1) < 0, \\ 0 & \text{otherwise .} \end{cases}$$

We now show that the statistic just defined constitutes an Eulerian partner for the length on $D_n$, that is, the following holds.

**Theorem 3.7.** *Let* $n \in \mathbb{N}$. *Then*

$$\sum_{\sigma \in D_n} x^{\text{dstc}(\sigma)} q^{\ell(\sigma)} = \sum_{\sigma \in D_n} x^{\text{ddes}(\sigma)} q^{\text{dmaj}(\sigma)}.$$

*Proof.* We use, as in [2] the following decomposition of $D_n$. Let

$$(3.8) \qquad\qquad T_n = \{\alpha \in D_n \mid \text{des}(\alpha) = 0\} = \{\alpha \in D_n \mid \mathfrak{I}(\alpha) = \varnothing\}$$

then $D_n$ can be rewritten as the following disjoint union:

$$(3.9) \qquad\qquad\qquad D_n = \bigcup_{\tau \in S_n} \{\alpha\tau \mid \alpha \in T_n\}.$$

For $\alpha \in T_n$ and $\tau \in S_n$ the following holds:

$$\ell(\alpha\tau) = \ell(\alpha) + \ell(\tau) = \text{nsp}(\alpha) + \text{inv}(\tau), \quad \text{nsp}(\alpha\tau) = \text{nsp}(\alpha), \quad \text{dstc}(\alpha\tau) = \text{stc}(\tau) + \text{neg}(\alpha) + \varepsilon(\sigma).$$

The last equation follows from the second equality in (3.8). Thus

$$\sum_{\sigma \in D_n} x^{\mathrm{dstc}(\sigma)} q^{\ell(\sigma)} = \sum_{\alpha \in T_n} \sum_{\tau \in S_n} x^{\mathrm{stc}(\tau) + \mathrm{neg}(\alpha) + \varepsilon(\alpha)} q^{\ell(\alpha) + \ell(\tau)}$$

$$= \sum_{\alpha \in T_n} x^{\mathrm{neg}(\alpha) + \varepsilon(\alpha)} q^{\mathrm{nsp}(\alpha)} \sum_{\tau \in S_n} x^{\mathrm{stc}(\tau)} q^{\mathrm{inv}(\tau)}$$

$$= \prod_{i=1}^{n-1} (1 + x q^i) A_n(x, q)$$

for the last equality see [2, Lemma 3.3]. The result follows, as

$$\sum_{\sigma \in D_n} x^{\mathrm{ddes}(\sigma)} q^{\mathrm{dmaj}(\sigma)} = \prod_{i=1}^{n-1} (1 + x q^i) A_n(x, q). \qquad \square$$

**Corollary 3.8.** *Let* $n \in \mathbb{N}$. *Then*

$$(3.10) \qquad \sum_{r \geq 0} [r + 1]_q^n x^r = \frac{\displaystyle\sum_{\sigma \in D_n} x^{\mathrm{dstc}(\sigma)} q^{\ell(\sigma)}}{(1 - x)(1 - x q^n) \displaystyle\prod_{i=1}^{n-1} (1 - x^2 q^{2i})} \quad \text{in } \mathbb{Z}[q][[x]].$$

## REFERENCES

1. R. M. Adin, F. Brenti, and Y. Roichman, *Descent numbers and major indices for the hyperoctahedral group*, Adv. in Appl. Math. **27** (2001), no. 2-3, 210–224, Special issue in honor of Dominique Foata's 65th birthday (Philadelphia, PA, 2000).
2. R. Biagioli, *Major and descent statistics for the even-signed permutation group*, Adv. in Appl. Math. **31** (2003), no. 1, 163–179.
3. A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005.
4. A. Carnevale and C. Voll, *Orbit Dirichlet series and multiset permutations*, arXiv:1607.01292.
5. D. Foata and G. Han, *Une nouvelle transformation pour les statistiques Euler-mahoniennes ensemblistes*, Mosc. Math. J. **4** (2004), no. 1, 131–152, 311.
6. D. Foata and M.-P. Schützenberger, *Major index and inversion number of permutations*, Math. Nachr. **83** (1978), 143–159.
7. G. Frobenius, *Über die Bernoullischen Zahlen und die Eulerschen Polynome*, Sitzungsber. Preuss. Akad. Wiss. (1910), 809 – 847.
8. L. M. Lai and T. K. Petersen, *Euler-Mahonian distributions of type* $B_n$, Discrete Math. **311** (2011), no. 8-9, 645–650.
9. P. A. MacMahon, *Combinatory analysis. Vol. I, II (bound in one volume)*, Dover Phoenix Editions, Dover Publications, Inc., Mineola, NY, 2004, Reprint of *An introduction to combinatory analysis* (1920) and *Combinatory analysis. Vol. I, II* (1915, 1916).
10. R. Simion, *A multi-indexed Sturm sequence of polynomials and unimodality of certain combinatorial sequences*, J. Combin. Theory Ser. A **36** (1984), no. 1, 15–22.
11. M. Skandera, *An Eulerian partner for inversions*, Sém. Lothar. Combin. **46** (2001/02).

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, D-33501 BIELEFELD, GERMANY
*E-mail address*: acarneva1@math.uni-bielefeld.de