Signed generating functions for odd inversions on descent classes [1]

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Abstract

We study the signed generating function for the number of odd inversions on descent classes of the symmetric group. We present operations on the descent class that leave the corresponding generating function unchanged, give sufficient conditions for it to be zero, and compute it explicitly for the alternating permutations and for a family of descent classes that includes all quotients.

1 Introduction

The signed (by number of inversions) enumeration of combinatorial statistics on the symmetric group is a well studied and classical topic (see, e.g., [1], [5], [6], [8], [13]). In [7] a new statistics on the symmetric group was introduced in relation with formed spaces. This statistic combines combinatorial and parity conditions and is now known as the number of odd inversions (see, e.g., [3], [4]) or odd length. In [7] it was conjectured that the signed generating function of this new statistic on any quotient of the symmetric group is given by a nice explicit product formula ([7, Conjecture C]). This conjecture was proved recently in [3].

Our purpose in this paper is to study the signed generating function of this new statistic on the descent classes of the symmetric group. In particular, we show that for a certain family of descent classes (which we call unmixed) this signed generating function again factors in a very explicit way (Theorem 4.1). Our result includes the main result of [3] on the quotients of the symmetric group as a special case. We also give sufficient conditions for the generating function to be zero, and we show that for the descent class of the alternating permutations it is always either zero or a monic power.

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The organization of the paper is as follows. In the next section we recall definitions and results that we use in the sequel. In §3 we study the effect that some operations that can be performed on a descent class have on the corresponding signed generating function. We also give sufficient conditions on a descent class for the corresponding signed generating function to be zero and we compute it explicitly for the descent class of the alternating permutations. In §4 we prove our main result. Namely we show that for any unmixed descent class the signed generating function for the number of odd inversions factors in a nice and explicit way. Unmixed descent classes include quotients, and in this case our result reduces to the main result of [3]. Finally, in §5, we present a conjecture about the generating function for the number of odd inversions on the symmetric group and the evidence that we have in its favor.

2 Preliminaries

For $m, n \in \mathbb{Z}$, $m \leq n$, we let $[m, n]$ denote the set $\{m, m+1, \ldots, n-1, n\}$ and for $n \in \mathbb{P}$ we let $[n] = [1, n]$. Given $J \subseteq [n-1]$ there are unique integers $a_1 < \ldots < a_s$ and $b_1 < \ldots < b_s$ such that $J = [a_1, b_1] \cup \ldots \cup [a_s, b_s]$ and $b_i + 1 < a_{i+1}$ for $i = 1, \ldots, s-1$. We call *connected components of* J the intervals $[a_1, b_1], \ldots, [a_s, b_s]$.

For $n \in \mathbb{N}$ we let $[n]_q := (1-q^n)/(1-q)$ (so $[0]_q = 0$), and $[n]_q! := \prod_{i=1}^{n} [i]_q$ (so $[0]_q! := 1$). For $n_1, \ldots, n_k \in \mathbb{N}$ such that $\sum_{i=1}^{k} n_i = n$ we let

$$\begin{bmatrix} n \\ n_1, \ldots, n_k \end{bmatrix}_q := \frac{[n]_q!}{[n_1]_q! \cdot \cdots \cdot [n_k]_q!}.$$

We refer to [2] for notation, terminology and basic facts about Coxeter groups.

The symmetric group S_n is the group of permutations of $[n]$. We let $S = \{s_1, \ldots, s_{n-1}\}$ denote the set of standard generators of S_n, where s_i denotes the i-th transposition $(i, i+1)$. It is well known that S_n, with respect to this set of generators, is a Coxeter group and that, for $\sigma \in S_n$, the Coxeter length $\ell(\sigma)$ and the descent set $D(\sigma)$ can be combinatorially described, respectively, as

$$\ell(\sigma) = |\{(i,j) \in [n]^2 \mid i < j, \sigma(i) > \sigma(j)\}|$$

and

$$D(\sigma) = |\{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}|. \tag{1}$$

Our main result concerns generating functions on descent classes of the symmetric group, which we now define.

Definition 2.1. For $I, J \subseteq S$, $I \subseteq S \setminus J$ we let

$$\mathcal{D}_I^J(S_n) := \{w \in S_n \mid I \subseteq D(w) \subseteq S \setminus J\} \tag{2}$$

$$S_n^J := \mathcal{D}_\emptyset^J. \tag{3}$$

Similarly, for subsets $X \subseteq S_n$, and $I, J \subseteq S$, $I \subseteq S \setminus J$ we denote $\mathcal{D}_I^J(X) := X \cap \mathcal{D}_I^J(S_n)$.

To state the main result of [3], which is also a special case of the main result of this paper, we need the following definitions. Let $n \in \mathbb{N}$. Set:

$$\begin{aligned} C_{n,+} &:= \{w \in S_n \mid i + w(i) \equiv 0 \pmod 2,\ i = 1, \dots, n\} \\ C_{n,-} &:= \{w \in S_n \mid i + w(i) \equiv 1 \pmod 2,\ i = 1, \dots, n\} \\ C_n &:= C_{n,+} \cup C_{n,-}. \end{aligned}$$

Note that

$$C_n = \{w \in S_n : i \equiv j \pmod 2 \Rightarrow w(i) \equiv w(j) \pmod 2,\ \text{for all } i, j \in [n]\}.$$

Elements in $C_{n,+}$ are called *even* chessboard elements, those in $C_{n,-}$ are called *odd* chessboard elements. In words, in a permutation which is an even chessboard element all the values agree in parity with their positions. In a permutation which is an odd chessboard element in every position there is an element of opposite parity.

For $n = 2m + 1$ clearly $C_{n,-} = \emptyset$ so $C_n = C_{n,+}$.

Note that the chessboard elements C_n form a subgroup of S_n and the even chessboard elements $C_{n,+}$ form a subgroup of C_n.

The odd length is defined as follows (see also [7] and [3]).

Definition 2.2. Let $n \in \mathbb{P}$ and $\sigma \in S_n$. The *odd length* of σ is

$$L(\sigma) := |\{(i,j) \in [n]^2 \mid i < j,\ \sigma(i) > \sigma(j),\ i \not\equiv j \pmod 2\}|. \tag{4}$$

The statistic L counts inversions between values in positions with opposite parity. In the next proposition we collect some properties satisfied by L.

Proposition 2.3. *Let $n \in \mathbb{P}$, let w_0 be the unique longest element of S_n. Then*

(i) $L(e) = 0$,

(ii) $L(s_i) = 1$, *for* $i = 1, \dots, n-1$,

(iii) $L(ww_0) = L(w_0w) = L(w_0) - L(w)$ *for all* $w \in S_n$,

(iv) w_0 *is the unique element on which* L *attains its maximum, and* $L(w_0) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$.

Proof. The only non trivial point is the last one. It follows from *(iii)* and the fact that the identity is the unique element on which L is zero. The last statement comes from the fact that, by definition, $L(w_0) = \sum_{i=1}^{n} \lceil \frac{i}{2} \rceil$. □

The following result, conjectured in [7] and proved in [3], shows that the distribution of the odd length signed by the Coxeter length factors nicely on all quotients of the symmetric groups.

Theorem 2.4. *Let $n \in \mathbb{P}$, $I \subseteq [n-1]$, and $I_1, \dots, I_s$ be the connected components of I. Then*

$$\sum_{\sigma \in \mathcal{D}^I_\emptyset(C_{n,+})} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \left[\begin{matrix} \widetilde{m} \\ \lfloor \frac{|I_1|+1}{2} \rfloor, \dots, \lfloor \frac{|I_s|+1}{2} \rfloor \end{matrix} \right]_{x^2} \prod_{k=\widetilde{m}+1}^{\lfloor \frac{n-1}{2} \rfloor} (1 - x^{2k})\,; \tag{5}$$

and

$$\sum_{\sigma\in\mathcal{D}^I_\emptyset(C_{n,-})}(-1)^{\ell(\sigma)}x^{L(\sigma)}=\begin{cases}0, & \text{if } \widetilde{m}=m\\ -x^m\displaystyle\sum_{\sigma\in\mathcal{D}^I_\emptyset(C_{n,+})}(-1)^{\ell(\sigma)}x^{L(\sigma)}, & \text{otherwise,}\end{cases}\tag{6}$$

if $n\equiv 0 \pmod 2$, *where* $\widetilde{m}:=\sum_{k=1}^{s}\left\lfloor\frac{|I_k|+1}{2}\right\rfloor$, *and* $m:=\lfloor\frac{n}{2}\rfloor$.

Our purpose in this work is to extend Theorem 2.4 to descent classes.

3 Shifting, compressing, and reversing

We derive in this section a number of preliminary results concerning operations that can be performed on the subsets defining the descent class, for which the signed generating function of the odd length remains the same, or changes in a controlled way. In particular we prove that the results about shifting and compressing that hold for quotients, hold more in general for descent classes. We also introduce a new technique, namely *reversing*, we give sufficient conditions on a descent class for the corresponding signed generating function to be zero, and we compute it explicitly for the descent class of the alternating permutations.

Recall that a permutation in the descent class $\mathcal{D}^I_J(S_n)$ is a permutation which is increasing in the positions corresponding to the indices $I\cup(I+1)$ and decreasing in $J\cup(J+1)$.

The proofs of the following two results are similar to those of [3, Lemma 3.1 and Proposition 3.3]. However, for the reader's convenience, and for completeness, we provide proofs here.

Lemma 3.1. *Let* $I,J\subseteq[n-1]$, $I\cap J=\emptyset$. *Then*

$$\sum_{\sigma\in\mathcal{D}^I_J(S_n)}(-1)^{\ell(\sigma)}x^{L(\sigma)}=\sum_{\sigma\in\mathcal{D}^I_J(C_n)}(-1)^{\ell(\sigma)}x^{L(\sigma)}.$$

Proof. Let $\sigma\in\mathcal{D}^I_J(S_n)\setminus\mathcal{D}^I_J(C_n)$. Then there exists $i\in[n-1]$ such that $\sigma^{-1}(i)\equiv\sigma^{-1}(i+1) \pmod 2$ (else either $\sigma^{-1}(i)\equiv i \pmod 2$ for all $i\in[n]$ or $\sigma^{-1}(i)\equiv i+1 \pmod 2$ for all $i\in[n]$ so $\sigma\in C_n$). Let i be minimal with this property and define $\sigma^*=s_i\sigma$. This is a well defined involution on $\mathcal{D}^I_J(S_n)\setminus\mathcal{D}^I_J(C_n)$ since $|\sigma^{-1}(i)-\sigma^{-1}(i+1)|\geq 2$. But $L(\sigma^*)=L(\sigma)$ and $\ell(\sigma^*)=\ell(\sigma)\pm 1$ so this implies the result. □

Proposition 3.2. *Let* $I,J\subseteq[n-1]$, $I\cap J=\emptyset$. *Let* $i\in\mathbb{P}$, $k\in\mathbb{N}$ *be such that* $[i,i+2k]$ *is a connected component of* $I\cup J$, $[i,i+2k]\subseteq I$, *and* $i+2k+2\notin I\cup J$.

Then

$$\sum_{\sigma\in\mathcal{D}^I_J(S_n)}(-1)^{\ell(\sigma)}x^{L(\sigma)}=\sum_{\sigma\in\mathcal{D}^{I\cup\tilde{I}}_J(S_n)}(-1)^{\ell(\sigma)}x^{L(\sigma)}=\sum_{\sigma\in\mathcal{D}^{\tilde{I}}_J(S_n)}(-1)^{\ell(\sigma)}x^{L(\sigma)}\tag{7}$$

where $\tilde{I}:=(I\setminus\{i\})\cup\{i+2k+1\}$.

Proof. Note first that, by our hypotheses, $(I \cup \tilde{I}) \cap J = \emptyset$. We have that

$$\sum_{\sigma \in \mathcal{D}_J^I(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\substack{\{\sigma \in \mathcal{D}_J^I(S_n) : \\ \sigma(i) > \sigma(i+2k+2)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{\substack{\{\sigma \in \mathcal{D}_J^I(S_n) : \ \sigma(i+2k+1) \\ < \ \sigma(i+2k+2)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} + \sum_{j=1}^{2k+1} \left(\sum_{\substack{\{\sigma \in \mathcal{D}_J^I(S_n) : \ \sigma(i+j-1) < \\ \sigma(i+2k+2) < \sigma(i+j)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} \right). \quad (8)$$

Let $r \in [k]$. Note that, by our hypotheses, $i-1 \notin J$ (else $i \in (J+1) \cap I$) and $i+2k+1 \notin J$ (since $i+2k+1 \in I+1$). Therefore the map $\sigma \mapsto \tilde{\sigma}$, where $\tilde{\sigma} := \sigma\ (i+2k+2\,, i+2r)$, is a bijection between $\{\sigma \in \mathcal{D}_J^I(S_n) : \ \sigma(i+2r) < \sigma(i+2k+2) < \sigma(i+2r+1)\}$ and $\{\sigma \in \mathcal{D}_J^I(S_n) : \ \sigma(i+2r-1) < \sigma(i+2k+2) < \sigma(i+2r)\}$. Furthermore, $\ell(\widetilde{\sigma}) = \ell(\sigma)+1$ and $L(\widetilde{\sigma}) = L(\sigma)$ so

$$\sum_{\substack{\{\sigma \in \mathcal{D}_J^I(S_n) : \sigma(i+2r) < \\ \sigma(i+2k+2) < \sigma(i+2r+1)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = - \sum_{\substack{\{\sigma \in \mathcal{D}_J^I(S_n) : \sigma(i+2r-1) < \\ \sigma(i+2k+2) < \sigma(i+2r)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}.$$

Similarly, the map $\sigma \mapsto \sigma\ (i+2k+2, i)$ shows that

$$\sum_{\{\sigma \in \mathcal{D}_J^I(S_n) : \sigma(i+2k+2) < \sigma(i)\}} (-1)^{\ell(\sigma)} x^{L(\sigma)} = - \sum_{\substack{\{\sigma \in \mathcal{D}_J^I(S_n) : \sigma(i) < \\ \sigma(i+2k+2) < \sigma(i+1)\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}.$$

Therefore, by (8),

$$\sum_{\sigma \in \mathcal{D}_J^I(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\{\sigma \in \mathcal{D}_J^I(S_n) : \sigma(i+2k+1) < \sigma(i+2k+2)\}} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

and the first equality in (7) follows.

The proof of the second equality is similar, and is therefore omitted. □

Note that the proof of the previous result actually yields that if $I, J \subseteq [n-1]$ are such that $I \cap J = \emptyset$, and if $i \in \mathbb{P}$, $k \in \mathbb{N}$ are such that $[i, i+2k+1]$ is a connected component of $I \cup J$ and $i+2k+1 \in J$, $[i, i+2k] \subseteq I$, then

$$\sum_{\sigma \in \mathcal{D}_J^I(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = 0.$$

This is a special case of a more general fact. Let $I, J \subseteq [n-1]$, $I \cap J = \emptyset$, and $i \in [n]$. We say that i is a *peak* of $\mathcal{D}_J^I(S_n)$ if $i \in (I+1) \setminus I$ or $i \in J \setminus (J+1)$. Similarly, i is a *valley* if $i \in I \setminus (I+1)$ or $i \in (J+1) \setminus J$.

Proposition 3.3. *Let $I, J \subseteq [n-1]$, $I \cap J = \emptyset$, and $i \in \mathbb{P}$, $k \in \mathbb{N}$ be such that $[i, i+2k+1]$ is a connected component of $I \cup J$ and $a, b \in [i, i+2k+2]$, a valley, b peak, implies $a \not\equiv b$* (mod 2)*. Then*

$$\sum_{\sigma \in \mathcal{D}_J^I(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = 0.$$

Proof. Let $\sigma \in \mathcal{D}_J^I(S_n)$. Let $\{a_1, \ldots, a_{2k+3}\}_< := \{\sigma(i), \sigma(i+1), \ldots, \sigma(i+2k+2)\}$. Let $a := \sigma^{-1}(a_1)$. Then a is a valley (for if $i < a < i+2k+2$ then $\sigma(a-1) > \sigma(a) < \sigma(a+1)$ so $a \in I \cap (J+1)$, while if $a = i$ then $\sigma(a) < \sigma(a+1)$ so $a \in I \setminus ((J+1) \cup (I+1))$, and if $a = i+2k+2$ then $\sigma(a-1) > \sigma(a)$ so $a \in (J+1) \setminus (J \cup I)$). Similarly, $\sigma^{-1}(a_{2k+3})$ is a peak. Therefore, by our hypotheses, $\sigma^{-1}(a_1) \not\equiv \sigma^{-1}(a_{2k+3}) \pmod 2$.

Let $j := \min\{r \in [2k+2] : \sigma^{-1}(a_r) \equiv \sigma^{-1}(a_{r+1}) \pmod 2\}$ (note that j certainly exists for if $\sigma^{-1}(a_1) \not\equiv \sigma^{-1}(a_2) \not\equiv \cdots \not\equiv \sigma^{-1}(a_{2k+3}) \pmod 2$ then $\sigma^{-1}(a_1) \equiv \sigma^{-1}(a_{2k+3}) \pmod 2$ which is a contradiction), and $\hat{\sigma} := (a_j, a_{j+1})\,\sigma$. Then $\hat{\sigma} \in \mathcal{D}_J^I(S_n)$, $\ell(\hat{\sigma}) = \ell(\sigma) \pm 1$, $L(\hat{\sigma}) = L(\sigma)$ and the map $\sigma \mapsto \hat{\sigma}$ is an involution. The result follows. □

Note that the converse of the previous result does not hold. For example, if $n = 8$, $I = \{1, 2, 4\}$, and $J = \{3, 5, 6\}$ then the signed generating function for $\mathcal{D}_J^I(S_8)$ is zero but $\mathcal{D}_J^I(S_8)$ has peaks $\{3, 5\}$ and valleys $\{1, 4, 7\}$. On the other hand, under the weaker hypothesis that there exist at least one peak and one valley with different parities the generating function is not, in general, zero. For example, if $n = 8$, $I = \{1, 2, 4\}$, and $K = \{3, 5, 6, 7\}$ then $\mathcal{D}_K^I(S_8)$ has peaks $\{3, 5\}$ and valleys $\{1, 4, 8\}$ but the corresponding generating function is $-x^6(1 + x^2 + x^4)$. It would be interesting to find necessary and sufficient conditions on I and J for the signed generating function on $\mathcal{D}_J^I(S_n)$ to be zero.

Proposition 3.3 implies that if $I \cup J$ has a "zig-zag" connected component K of even cardinality (i.e., if all even elements of K are in I and all odd ones are in J, or conversely) then the resulting signed generating function is zero. Thus, this is in particular true for the alternating permutations of a symmetric group of odd index. This makes it natural to investigate the corresponding generating function for all alternating permutations. For $n \in \mathbb{P}$ we let

$$E_n^- := \{\sigma \in S_n : \sigma(1) > \sigma(2) < \sigma(3) > \cdots\},$$

and

$$E_n^+ := \{\sigma \in S_n : \sigma(1) < \sigma(2) > \sigma(3) < \cdots\}.$$

We call the elements of E_n^- (resp. E_n^+) *alternating* (resp. *reverse alternating*) permutations (we refer the reader to, e.g., [10, §1.6] for further information about alternating permutations).

Proposition 3.4. *Let $n \in \mathbb{P}$. Then*

$$\sum_{\sigma \in E_n^-} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \begin{cases} 0, & \text{if } n \equiv 1 \pmod 2, \\ (-x)^{\frac{n}{2}}, & \text{if } n \equiv 0 \pmod 2, \end{cases} \tag{9}$$

and

$$\sum_{\sigma \in E_n^+} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \begin{cases} 0, & \text{if } n \equiv 1 \pmod 2, \\ x^{\frac{n}{2}(\frac{n}{2}-1)}, & \text{if } n \equiv 0 \pmod 2. \end{cases} \tag{10}$$

Proof. Note that $E_n^- = \mathcal{D}_I^J(S_n)$ where $I := \{i \in [n-1] : i \equiv 1 \pmod 2\}$ and $J := \{i \in [n-1] : i \equiv 0 \pmod 2\}$ so the first equation in (9) follows from Proposition 3.3. So assume that $n \equiv 0 \pmod 2$, say $n = 2m$ for some $m \in \mathbb{N}$. By Lemma 3.1 we have that

$$\sum_{\sigma \in E_n^-} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in \mathcal{D}_I^J(C_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)}.$$

We claim that $\mathcal{D}_I^J(C_{n,+}) = \emptyset$. Let $\sigma \in \mathcal{D}_I^J(C_{n,+})$. Let $i := \sigma^{-1}(1)$. Then $i \equiv 1 \pmod 2$ so $i \in I$ and hence $\sigma(i) > \sigma(i+1)$ which is a contradiction. Let now $\sigma \in \mathcal{D}_I^J(C_{n,-})$. We claim that then

$$\sigma = 2\,1\,4\,3\,6\,5 \cdots 2m\,2m-1.$$

We prove this claim by induction on $m \in \mathbb{P}$. If $m = 1$ the claim is clear. Let $m \geq 2$. Let $a := \sigma^{-1}(2m-1)$. Then $a \equiv 0 \pmod 2$ so $a = 2m$ (else $\sigma(a-1), \sigma(a+1) > \sigma(a) = 2m-1$) and hence $\sigma(2m-1) = 2m$. But $\sigma_{|[2m-2]} \in \mathcal{D}_{I\cap[n-3]}^{J\cap[n-3]}(C_{n-2,-})$ so the claim follows by induction. Since $\ell(2\,1\,4\,3 \cdots 2m\,2m-1) = m = L(2\,1\,4\,3 \cdots 2m\,2m-1)$ the second equation in (9) follows.

Since the map $\sigma \mapsto w_0\,\sigma$ is an involution between E_n^+ and E_n^-, the equations in (10) follow from those in (9) and Proposition 2.3. $\square$

We now return to our investigation of shifting, compressing, and reversing. The following is the "left" version of Proposition 3.2.

Proposition 3.5. *Let $I, J \subseteq [n-1]$, $I \cap J = \emptyset$. Let $i \in \mathbb{P}$, $k \in \mathbb{N}$ be such that $[i+1, i+2k+1]$ is a connected component of $I \cup J$, $[i+1, i+2k+1] \subseteq I$ and $i-1 \notin I \cup J$.*

Then

$$\sum_{\sigma \in \mathcal{D}_J^I(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in \mathcal{D}_J^{I \cup \bar{I}}(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in \mathcal{D}_J^{\bar{I}}(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

where $\bar{I} := (I \setminus \{i+2k+1\}) \cup \{i\}$.

Proof. From our hypotheses we have that $(I \cup \bar{I}) \cap J = \emptyset$, $[i, i+2k]$ is a connected component of $\bar{I} \cup J$, $[i, i+2k] \subseteq \bar{I}$, and $i+2k+2 \notin \bar{I} \cup J$, so the result follows from Proposition 3.2. $\square$

We now show that a connected component of even cardinality of the descents can be "transformed" (or "reversed") into a connected component of the ascents, by changing the generating function by a simple factor.

Lemma 3.6. *Let $I, J \subseteq [n-1]$, $I \cap J = \emptyset$, and i, $k \in \mathbb{P}$ be such that $K := [i, i+2k-1]$ is a connected component of $I \cup J$, $K \subseteq J$. Then*

$$\sum_{\sigma \in \mathcal{D}_J^I(C_{n,\pm})} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (-1)^k x^{k(k+1)} \sum_{\sigma \in \mathcal{D}_{J \setminus K}^{I \cup K}(C_{n,\pm})} (-1)^{\ell(\sigma)} x^{L(\sigma)}. \tag{11}$$

In particular,

$$\sum_{\sigma \in \mathcal{D}_J^I(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (-1)^k x^{k(k+1)} \sum_{\sigma \in \mathcal{D}_{J \setminus K}^{I \cup K}(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)}. \tag{12}$$

Proof. We have that

$$\sum_{\sigma \in \mathcal{D}_J^I(C_{n,+})} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\tau \in \mathcal{D}_{J \setminus K}^{I \cup K}(C_{n,+})} (-1)^{\ell(\bar{\tau})} x^{L(\bar{\tau})},$$

where $\bar{\tau} := [\tau(1), \ldots, \tau(i-1), \tau(i+2k), \ldots, \tau(i+1), \tau(i), \tau(i+2k+1), \ldots, \tau(n)]$. But

$$\ell(\bar{\tau}) = \ell(\tau) + (2k+1)k \quad \text{and, by Proposition 2.3} \quad L(\bar{\tau}) = L(\tau) + k(k+1);$$

thus

$$\sum_{\tau\in\mathcal{D}_{J\setminus K}^{I\cup K}(C_{n,+})} (-1)^{\ell(\bar\tau)}x^{L(\bar\tau)} = (-1)^k x^{k(k+1)} \sum_{\tau\in\mathcal{D}_{J\setminus K}^{I\cup K}(C_{n,+})} (-1)^{\ell(\tau)}x^{L(\tau)}$$

as desired. Similarly for $C_{n,-}$. $\square$

In a similar way, it is easy to determine the generating function on the descent class where all the descents are transformed into ascents, and conversely, as shown in the following result.

Proposition 3.7. *Let I, $J\subseteq[n-1]$, $I\cap J=\emptyset$. Then*

$$\sum_{\sigma\in\mathcal{D}_J^I(S_n)} (-1)^{\ell(\sigma)}x^{L(\sigma)} = (-1)^{\ell(w_0)}x^{L(w_0)} \sum_{\sigma\in\mathcal{D}_I^J(S_n)} (-1)^{\ell(\sigma)}\left(\frac{1}{x}\right)^{L(\sigma)}.$$

Proof. It is clear that the map $\sigma\mapsto w_0\sigma$ is a bijection from $\mathcal{D}_J^I(S_n)$ to $\mathcal{D}_I^J(S_n)$. Therefore, by Proposition 2.3 we have

$$\begin{aligned}\sum_{\sigma\in\mathcal{D}_J^I(S_n)} (-1)^{\ell(\sigma)}x^{L(\sigma)} &= \sum_{\tau\in\mathcal{D}_I^J(S_n)} (-1)^{\ell(w_0\tau)}x^{L(w_0\tau)}\\ &= (-1)^{\ell(w_0)}x^{L(w_0)}\sum_{\tau\in\mathcal{D}_I^J(S_n)} (-1)^{\ell(\tau)}\left(\frac{1}{x}\right)^{L(\tau)}. \quad\square\end{aligned}$$

Remark 3.8. The bijection $\sigma\mapsto w_0\sigma$ in the proof of Proposition 3.7 restricts to a bijection between chessboard elements of the descent classes. In particular, if n is even it is a bijection between $\mathcal{D}_J^I(C_{n,+})$ and $\mathcal{D}_I^J(C_{n,-})$.

The results about left and right shifting for connected components of the descents can be deduced from the analogous results for connected components of the ascents, as summarized in the following propositions.

Proposition 3.9. *Let I, $J\subseteq[n-1]$, $I\cap J=\emptyset$. Let $i\in\mathbb{P}$, $k\in\mathbb{N}$ be such that $[i,i+2k]$ is a connected component of $I\cup J$, $[i,i+2k]\subseteq J$, and $i+2k+2\notin I\cup J$. Then*

$$\sum_{\sigma\in\mathcal{D}_J^I(S_n)} (-1)^{\ell(\sigma)}x^{L(\sigma)} = \sum_{\sigma\in\mathcal{D}_{J\cup\tilde J}^I(S_n)} (-1)^{\ell(\sigma)}x^{L(\sigma)} = \sum_{\sigma\in\mathcal{D}_{\tilde J}^I(S_n)} (-1)^{\ell(\sigma)}x^{L(\sigma)},$$

where $\tilde J:=(J\setminus\{i\})\cup\{i+2k+1\}$.

Proof. By Proposition 3.2 and we have that

$$\sum_{\sigma\in\mathcal{D}_I^J(S_n)} (-1)^{\ell(\sigma)}x^{L(\sigma)} = \sum_{\sigma\in\mathcal{D}_I^{J\cup\tilde J}(S_n)} (-1)^{\ell(\sigma)}x^{L(\sigma)} = \sum_{\sigma\in\mathcal{D}_I^{\tilde J}(S_n)} (-1)^{\ell(\sigma)}x^{L(\sigma)}$$

so the result follows from Proposition 3.7. $\square$

In a similar way, using Proposition 3.5, we obtain the following.

Proposition 3.10. *Let I, $J \subseteq [n-1]$, $I \cap J = \emptyset$, and $i \in \mathbb{P}$, $k \in \mathbb{N}$ be such that $[i+1, i+2k+1]$ is a connected component of $I \cup J$, $[i+1, i+2k+1] \subseteq J$, and $i-1 \notin I \cup J$. Then*

$$\sum_{\sigma \in \mathcal{D}_J^I(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in \mathcal{D}_{J \cup \bar{J}}^I(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in \mathcal{D}_{\bar{J}}^I(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)},$$

where $\bar{J} := (J \setminus \{i+2k+1\}) \cup \{i\}$.

Computer calculations suggest that the operation of shifting can be performed under weaker hypotheses, namely even if the connected component to be shifted is not contained in I (as required in Proposition 3.2) and therefore not contained in J (in Proposition 3.9). More precisely, we conjecture the following.

Conjecture 3.11. *Let $I, J \subseteq [n-1]$, $I \cap J = \emptyset$. Let $i \in \mathbb{P}$, $k \in \mathbb{N}$ be such that $i+2k+2 \notin I \cup J$ and $[i, i+2k]$ is a connected component of $I \cup J$, say $[i, i+2k] = A \cup B$, where $A \subseteq I$ and $B \subseteq J$. Then*

$$\sum_{\sigma \in \mathcal{D}_J^I(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in \mathcal{D}_{\tilde{J}}^{\tilde{I}}(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

where $\tilde{I} := (I \setminus A) \cup (A+1)$ and $\tilde{J} := (J \setminus B) \cup (B+1)$.

4 Unmixed descent classes

In this section we prove the main result of this work. Namely we give an explicit closed product formula for the generating function of $(-1)^{\ell(\sigma)} x^{L(\sigma)}$ over any descent class $\mathcal{D}_J^I(S_n)$ for which the connected components of I and the connected components of J coincide with those of $I \cup J$. This result includes Theorem 2.4 as a special case.

Let $I, J \subseteq [n-1]$. We say that I and J are *unmixed* if

$$I \cap J = (I+1) \cap J = I \cap (J+1) = \emptyset. \tag{13}$$

Let $I, J \subseteq [n-1]$ be unmixed. Let $I_1, \ldots, I_s$ be the connected components of I and $J_1, \ldots, J_t$ be those of J. We say that (I, J) is *compressed* if $|I_1| \equiv \cdots \equiv |I_s| \equiv |J_1| \equiv \cdots \equiv |J_t| \equiv 1 \pmod 2$ and $|[n-1] \setminus (I \cup J)| = s + t - 1$. So, for example, $(\{1,7,8,9\}, \{3,4,5,11,12,13\})$ is compressed for $n = 14$ while $(\{1,3\}, \{7,8,9,11,12,13\})$ is not. Note that if $I, J \subseteq [n-1]$ as above are such that (I, J) is compressed then $n-1 = |I| + |J| + s + t - 1 \equiv -1 \pmod 2$ so n is even. Let $n \in \mathbb{P}$, $n \equiv 0 \pmod 2$. Say $n = 2m$. Let $I, J \subseteq [n-1]$ be such that $I \cap J = (I+1) \cap J = I \cap (J+1) = \emptyset$ and $I_1, \ldots, I_s, J_1, \ldots, J_t$ be the connected components of I and J, respectively. Then $I_1, \ldots, I_s, J_1, \ldots, J_t$ are the connected components of $I \cup J$. Therefore $\sum_{j=1}^{s} \left(\frac{|I_j|+1}{2}\right) + \sum_{k=1}^{t} \left(\frac{|J_k|+1}{2}\right) \leq m$ so (I, J) is compressed if and only if $m = \sum_{j=1}^{s} \left\lfloor \frac{|I_j|+1}{2} \right\rfloor + \sum_{k=1}^{t} \left\lfloor \frac{|J_k|+1}{2} \right\rfloor$.

We can now state our main result.

Theorem 4.1. *Let $I, J \subseteq [n-1]$ be unmixed. Let $I_1, \ldots, I_s$ be the connected components of I and $J_1, \ldots, J_t$ be the connected components of J. Then*

$$\sum_{\sigma \in \mathcal{D}_J^I(S_n)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \begin{cases} (-1)^{\|d\|} x^{\alpha(J)} \frac{x^{\|d\|}[\|b\|]_{x^2} + x^{\|b\|}[\|d\|]_{x^2}}{[M]_{x^2}} \begin{bmatrix} M \\ \mathbf{b}, \mathbf{d} \end{bmatrix}_{x^2}, & \text{if } n = 2M, \\ (-x)^{\|d\|} x^{\alpha(J)} \begin{bmatrix} M \\ \mathbf{b}, \mathbf{d} \end{bmatrix}_{x^2} \prod_{k=2M+2}^{n} (1 + (-1)^{k-1} x^{\lfloor \frac{k}{2} \rfloor}), & \text{otherwise,} \end{cases} \tag{14}$$

where $b_j := \left\lfloor \frac{|I_j|+1}{2} \right\rfloor$, *for* $j = 1, \ldots, s$, $d_k := \left\lfloor \frac{|J_k|+1}{2} \right\rfloor$, *for* $k = 1, \ldots, t$, $\|b\| := \sum_{i=1}^{s} b_i$, $\|d\| := \sum_{k=1}^{t} d_k$, $M := \|b\| + \|d\|$, $\mathbf{b} := b_1, \ldots, b_s$, $\mathbf{d} := d_1, \ldots, d_t$, *and* $\alpha(J) := \sum_{k=1}^{t} (d_k)^2$.

By Lemma 3.1 this result is a consequence of the following more precise one, which is what we actually prove.

Theorem 4.2. *Let $I, J \subseteq [n-1]$. Then in the hypotheses of Theorem 4.1, and keeping the same notation we have that*

$$\sum_{\sigma \in \mathcal{D}_J^I(C_{n,+})} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (-x)^{\|d\|} x^{\alpha(J)} \begin{bmatrix} M \\ \mathbf{b}, \mathbf{d} \end{bmatrix}_{x^2} \prod_{k=M+1}^{m} (1 - x^{2k}), \tag{15}$$

if n is odd, while

$$\sum_{\sigma \in \mathcal{D}_J^I(C_{n,+})} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \begin{cases} (-x)^{\|d\|} x^{\alpha(J)} \frac{[\|b\|]_{x^2}}{[m]_{x^2}} \begin{bmatrix} M \\ \mathbf{b}, \mathbf{d} \end{bmatrix}_{x^2}, & \text{if } m = M, \\ (-x)^{\|d\|} x^{\alpha(J)} \begin{bmatrix} M \\ \mathbf{b}, \mathbf{d} \end{bmatrix}_{x^2} \prod_{k=M+1}^{m-1} (1 - x^{2k}), & \text{otherwise,} \end{cases} \tag{16}$$

and

$$\sum_{\sigma \in \mathcal{D}_J^I(C_{n,-})} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \begin{cases} (-1)^{\|d\|} x^{\|b\| + \alpha(J)} \frac{[\|d\|]_{x^2}}{[m]_{x^2}} \begin{bmatrix} M \\ \mathbf{b}, \mathbf{d} \end{bmatrix}_{x^2}, & \text{if } m = M, \\ -(-x)^{\|d\|} x^{m + \alpha(J)} \begin{bmatrix} M \\ \mathbf{b}, \mathbf{d} \end{bmatrix}_{x^2} \prod_{k=M+1}^{m-1} (1 - x^{2k}), & \text{otherwise,} \end{cases} \tag{17}$$

if n is even, where $m := \lfloor \frac{n}{2} \rfloor$.

Proof. We let, for convenience, $\bar{b}_j := b_j + 1$, $\bar{d}_k := d_k + 1$, for $j \in [s]$ and $k \in [t]$, $\hat{\alpha}(J) := \alpha(J) + \|d\|$, and $\check{\alpha}(J) := \alpha(J) - \|d\|$.

We proceed by induction on $t \in \mathbb{N}$, the number of connected components of the descents. Let $t = 0$ (i.e., $J = \emptyset$). Then $(I, \emptyset)$ is compressed if and only if $\|b\| = \frac{n}{2}$ so Theorem 4.1 reduces to Theorem 2.4 in this case. So let $t \geq 1$.

Assume first that there exists $i \in [t]$ such that $|J_i| \equiv 0 \pmod 2$. Then by Lemma 3.6 and our induction hypothesis we have that

$$\sum_{\sigma \in \mathcal{D}_J^I(C_{n,+})} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (-1)^{\frac{|J_i|}{2}} x^{\frac{|J_i|(|J_i|+2)}{4}} \sum_{\sigma \in \mathcal{D}_{J \setminus J_i}^{I \cup J_i}(C_{n,+})} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

$$= (-1)^{d_i} x^{d_i \bar{d}_i} (-1)^{\|d\| - d_i} x^{\hat{\alpha}(J) - d_i \bar{d}_i} \begin{bmatrix} \|b\| + \|d\| \\ \mathbf{b}, \mathbf{d} \end{bmatrix}_{x^2} \prod_{k=\|b\|+\|d\|+1}^{\lfloor \frac{n-1}{2} \rfloor} (1 - x^{2k}),$$

so (15) and the second formula in (16) follow in this case.

In the same hypothesis, for the odd chessboard elements we similarly have that

$$\sum_{\sigma \in \mathcal{D}_J^I(C_{2m,-})} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (-1)^{\frac{|J_i|}{2}} x^{\frac{|J_i|(|J_i|+2)}{4}} \sum_{\sigma \in \mathcal{D}_{J \setminus J_i}^{I \cup J_i}(C_{2m,-})} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

$$= -(-1)^{d_i} x^{d_i \bar{d}_i} (-1)^{\|d\| - d_i} x^{m + \hat{\alpha}(J) - d_i \bar{d}_i} \begin{bmatrix} \|b\| + \|d\| \\ \mathbf{b}, \mathbf{d} \end{bmatrix}_{x^2} \prod_{k=\|b\|+\|d\|+1}^{m-1} (1 - x^{2k}),$$

so the second formula in (17) also follows.

We may therefore assume that $|J_1| \equiv |J_2| \equiv \cdots \equiv |J_t| \equiv 1 \pmod 2$.
Assume now that there exists $r \in [s]$ such that $|I_r| \equiv 0 \pmod 2$. Then by repeated application of Proposition 3.9 and 3.10, we have that

$$\sum_{\sigma \in \mathcal{D}_J^I(C_n, \pm)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in \mathcal{D}_{\widetilde{J}}^{\widetilde{I}}(C_n, \pm)} (-1)^{\ell(\sigma)} x^{L(\sigma)},$$

where $\widetilde{I}$ has connected components $\widetilde{I}_1 \cup \cdots \cup \widetilde{I}_s$, where $|\widetilde{I}_r| = |I_r| - 1$ and $|\widetilde{I}_k| = |I_k|$, $k \in [s] \setminus \{r\}$ and $\widetilde{J}$ has connected components $\widetilde{J}_1 \cup \cdots \cup \widetilde{J}_t$, where $|\widetilde{J}_1| = |J_1| + 1$ and $|\widetilde{J}_k| = |J_k|$, $k \in [2, t]$, and the connected components of $\widetilde{I} \cup \widetilde{J}$ are $\widetilde{I}_1, \ldots, \widetilde{I}_s, \widetilde{J}_1, \ldots, \widetilde{J}_t$. Thus, reasoning as in the previous case, and observing that $\left\lfloor \frac{|\widetilde{J}_1|+1}{2} \right\rfloor = \left\lfloor \frac{|J_1|+1}{2} \right\rfloor = d_1$ and $\left\lfloor \frac{|\widetilde{I}_r|+1}{2} \right\rfloor = \left\lfloor \frac{|I_r|+1}{2} \right\rfloor = b_r$, we conclude that

$$\sum_{\sigma \in \mathcal{D}_{\widetilde{J}}^{\widetilde{I}}(C_{n,+})} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (-1)^{\frac{|\widetilde{J}_1|}{2}} x^{\frac{|\widetilde{J}_1|(|\widetilde{J}_1|+2)}{4}} \sum_{\sigma \in \mathcal{D}_{\widetilde{J} \setminus \widetilde{J}_1}^{\widetilde{I} \cup \widetilde{J}_1}(C_{n+})} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

$$= (-1)^{d_1} x^{d_1 \bar{d}_1} (-1)^{\|d\| - d_1} x^{\hat{\alpha}(J) - d_1 \bar{d}_1} \begin{bmatrix} \|b\| + \|d\| \\ \mathbf{b}, \mathbf{d} \end{bmatrix}_{x^2} \prod_{k=\|b\|+\|d\|+1}^{\lfloor \frac{n-1}{2} \rfloor} (1 - x^{2k}),$$

and

$$\sum_{\sigma \in \mathcal{D}_{\widetilde{J}}^{\widetilde{I}}(C_{2m,-})} (-1)^{\ell(\sigma)} x^{L(\sigma)} = (-1)^{\frac{|\widetilde{J}_1|}{2}} x^{\frac{|\widetilde{J}_1|(|\widetilde{J}_1|+2)}{4}} \sum_{\sigma \in \mathcal{D}_{\widetilde{J} \setminus \widetilde{J}_1}^{\widetilde{I} \cup \widetilde{J}_1}(C_{2m-})} (-1)^{\ell(\sigma)} x^{L(\sigma)}$$

$$= -(-1)^{d_1} x^{d_1 \bar{d}_1} (-1)^{\|d\| - d_1} x^{m + \hat{\alpha}(J) - d_1 \bar{d}_1} \begin{bmatrix} \|b\| + \|d\| \\ \mathbf{b}, \mathbf{d} \end{bmatrix}_{x^2} \prod_{k=\|b\|+\|d\|+1}^{m-1} (1 - x^{2k}),$$

so the result again follows.

We may therefore assume that $|I_1| \equiv \cdots \equiv |I_s| \equiv |J_1| \equiv \cdots \equiv |J_t| \equiv 1 \pmod 2$.

Suppose first that $|[n-1] \setminus (I \cup J)| > s+t-1$. Therefore either $1 \notin I \cup J$ or $n-1 \notin I \cup J$ or there exists $i \in [n-1]$ such that $i,\, i+1 \notin I \cup J$. In any of these cases we can apply Propositions 3.9 and 3.10 to get

$$\sum_{\sigma \in \mathcal{D}^I_J(C_n,\pm)} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{\sigma \in \mathcal{D}^{\bar{I}}_{\bar{J}}(C_n,\pm)} (-1)^{\ell(\sigma)} x^{L(\sigma)},$$

where $\bar{I}$ has connected components $\bar{I}_1, \ldots, \bar{I}_s$ such that $|\bar{I}_j| = |I_j|$ for $j \in [s]$ and $\bar{J}$ has connected components $\bar{J}_1, \ldots, \bar{J}_t$ such that $|\bar{J}_1| = |J_1| + 1$ and $|\bar{J}_l| = |J_l|$ for $l \in [2,t]$. Then, reasoning as in the previous case, (15), and the second equations in (16) and (17) follow again by induction, since $\frac{|\bar{J}_1|}{2} = \left\lfloor \frac{|\bar{J}_1|+1}{2} \right\rfloor = \left\lfloor \frac{|J_1|+1}{2} \right\rfloor = d_1$.

Assume now that $|I_1| \equiv \cdots \equiv |I_s| \equiv |J_1| \equiv \cdots \equiv |J_t| \equiv 1 \pmod 2$ and $|[n-1] \setminus (I \cup J)| = s+t-1$, i.e., that (I,J) is compressed. Then $n \equiv 0 \pmod 2$, say $n = 2m$, and $m = \|b\| + \|d\|$.

For $i \in [s]$ let $a_i := \max I_i + 1$ and for $i \in [t]$ let $c_i := \min J_i$. Then $a_1 \equiv \cdots \equiv a_s \equiv 0 \pmod 2$ and $c_1 \equiv \cdots \equiv c_t \equiv 1 \pmod 2$. Therefore, if $\sigma \in \mathcal{D}^I_J(C_{2m,+})$, then $\sigma^{-1}(2m) \in \{a_1, \ldots, a_s\}$. Hence

$$\sum_{\sigma \in \mathcal{D}^I_J(C_{2m,+})} (-1)^{\ell(\sigma)} x^{L(\sigma)} = \sum_{j=1}^{s} \sum_{\substack{\{\sigma \in \mathcal{D}^I_J(C_{2m,+}): \\ \sigma^{-1}(2m)=a_j\}}} (-1)^{\ell(\sigma)} x^{L(\sigma)}.$$

Fix $j \in [s]$. Let $k := \max\{i \in [t] \,|\, c_i < a_j\}$ (where $k := 0$ if $\{i \in [t] \,|\, c_i < a_j\} = \emptyset$). Then the map $\tau \mapsto \bar{\tau}$, where $\bar{\tau}$ is obtained from τ by removing the maximum (which is in position a_j) and reversing the elements in each of the blocks of ascents and descents that are to the right of a_j, is a bijection between

$$\{\sigma \in \mathcal{D}^I_J(C_{2m,+}) \,:\, \sigma^{-1}(2m) = a_j\} \quad \text{and} \quad \mathcal{D}^{\varphi_j(I)}_{\varphi_j(J)}(C_{2m-1}),$$

where $\varphi_j(I) := I_1 \cup \cdots \cup I_{j-1} \cup (I_j \setminus \{a_j - 1\}) \cup (J_{k+1} - 1) \cup \cdots \cup (J_t - 1)$ and $\varphi_j(J) := J_1 \cup \cdots \cup J_k \cup (I_{j+1} - 1) \cup \cdots \cup (I_s - 1)$. Furthermore, we have that $\ell(\bar{\tau}) = \ell(\tau) + A$ and

$L(\bar{\tau}) = L(\tau) + B$, where, by Proposition 2.3

$$
\begin{aligned}
A &= \sum_{r=j+1}^{s} \binom{|I_r|+1}{2} - \sum_{h=k+1}^{t} \binom{|J_h|+1}{2} - (2m - a_j) \\
&= \sum_{r=j+1}^{s} b_r(2b_r - 1) - \sum_{h=k+1}^{t} d_h(2d_h - 1) - (2m - a_j) \\
&= \sum_{r=j+1}^{s} b_r(2b_r - 3) - \sum_{h=k+1}^{t} d_h(2d_h + 1) \\
B &= \sum_{r=j+1}^{s} \left(\frac{|I_r|+1}{2}\right)^2 - \sum_{h=k+1}^{t} \left(\frac{|J_h|+1}{2}\right)^2 - \frac{2m - a_j}{2} \\
&= \sum_{r=j+1}^{s} b_r^2 - \sum_{h=k+1}^{t} d_h^2 - \frac{2m - a_j}{2} \\
&= \sum_{r=j+1}^{s} b_r(b_r - 1) - \sum_{h=k+1}^{t} d_h(d_h + 1)
\end{aligned}
$$

since $2m - a_j = 2\left(\sum_{r=j+1}^{s} b_r + \sum_{h=k+1}^{t} d_h\right)$. Therefore we have, by our induction hypothesis (15), that

$$
\begin{aligned}
\sum_{\substack{\{\tau \in \mathcal{D}_J^I(C_{2m,+}): \\ \sigma^{-1}(2m) = a_j\}}} (-1)^{\ell(\tau)} x^{L(\tau)} &= (-1)^A x^{-B} \sum_{\bar{\tau} \in \mathcal{D}_{\varphi_j(J)}^{\varphi_j(I)}(C_{2m-1})} (-1)^{\ell(\bar{\tau})} x^{L(\bar{\tau})} \\
&= (-1)^{\|d\|} x^{\hat{\alpha}(\varphi_j(J)) - B} \begin{bmatrix} m-1 \\ b_1, \ldots, b_{j-1}, b_j - 1, b_{j+1}, \ldots, b_s, \mathbf{d} \end{bmatrix}_{x^2}.
\end{aligned} \tag{18}
$$

But

$$
\hat{\alpha}(\varphi_j(J)) = \sum_{r=1}^{k} d_r(d_r + 1) + \sum_{r=j+1}^{s} b_r(b_r + 1)
$$

so,

$$
\hat{\alpha}(\varphi_j(J)) - B = \hat{\alpha}(J) + 2 \sum_{r=j+1}^{s} b_r.
$$

Thus, the sum in (18) becomes

$$
(-1)^{\|d\|} x^{\hat{\alpha}(J)} x^{2\sum_{r=j+1}^{s} b_r} \begin{bmatrix} m-1 \\ b_1, \ldots, b_{j-1}, b_j - 1, b_{j+1}, \ldots, b_s, \mathbf{d} \end{bmatrix}_{x^2}.
$$

Therefore

$$
\begin{aligned}
\sum_{\sigma \in \mathcal{D}_J^I(C_{2m,+})} (-1)^{\ell(\sigma)} x^{L(\sigma)} &= (-1)^{\|d\|} x^{\hat{\alpha}(J)} \sum_{j=1}^{s} x^{2\sum_{r=j+1}^{s} b_r} \begin{bmatrix} m-1 \\ b_1, \ldots, b_{j-1}, b_j - 1, b_{j+1}, \ldots, b_s, \mathbf{d} \end{bmatrix}_{x^2} \\
&= (-1)^{\|d\|} x^{\hat{\alpha}(J)} \frac{[\|b\|]_{x^2}}{[m]_{x^2}} \begin{bmatrix} m \\ \mathbf{b}, \mathbf{d} \end{bmatrix}_{x^2}
\end{aligned}
$$

as desired.

In the same hypothesis, for the sum over odd chessboard elements we have, by Proposition 3.7, and the remark following it

$$\begin{aligned}
\sum_{\sigma \mathcal{D}_J^I(C_{2m,-})} (-1)^{\ell(\sigma)} x^{L(\sigma)} &= (-1)^{\ell(w_0)} x^{L(w_0)} \sum_{\tau \in \mathcal{D}_I^J(C_{2m,+})} (-1)^{\ell(\tau)} x^{-L(\tau)} \\
&= (-1)^{\binom{2m}{2}} x^{m^2} \sum_{\tau \in \mathcal{D}_I^J(C_{2m,+})} (-1)^{\ell(\tau)} x^{-L(\tau)} \\
&= (-1)^m x^{m^2} (-1)^{\|b\|} x^{-\sum_{j=1}^s b_j \bar{b}_j} \frac{[m-\|b\|]_{x^{-2}}}{[m]_{x^{-2}}} \left[\begin{matrix} m \\ \mathbf{b}, \mathbf{d} \end{matrix} \right]_{x^{-2}} \\
&= (-1)^{\|d\|} x^{m+\check{\alpha}(J)} \frac{[\|d\|]_{x^2}}{[m]_{x^2}} \left[\begin{matrix} m \\ \mathbf{b}, \mathbf{d} \end{matrix} \right]_{x^2}
\end{aligned}$$

and the result follows. This concludes the proof of the first equations in (16) and (17) and hence of the result. □

5 A conjecture

In this section we present a conjecture on the distribution of the number of odd inversions on the symmetric group, and the evidence that we have in its favor.

Given any statistic on S_n, it is natural to investigate the properties of the polynomial giving its distribution. For the odd length we denote this polynomial by $L_n(x) := \sum_{\sigma \in S_n} x^{L(\sigma)}$. Properties (*iii*) and (*iv*) in Proposition 2.3 imply that $L_n(x)$ is monic and symmetric for all $n \in \mathbb{N}$. For small values of n we have:

$$\begin{aligned}
L_3(x) &= 1 + 4x + x^2 \\
L_4(x) &= 1 + 8x + 6x^2 + 8x^3 + x^4 \\
L_5(x) &= 1 + 12x + 23x^2 + 48x^3 + 23x^4 + 12x^5 + x^6 \\
L_6(x) &= 1 + 16x + 59x^2 + 137x^3 + 147x^4 + 147x^5 + 137x^6 + 59x^7 + 16x^8 + x^9
\end{aligned}$$

With the exception of $n = 4$, for $n \leq 10$ the polynomials $L_n(x)$ are unimodal. We therefore conjecture the following.

Conjecture 5.1. *Let $n \geq 5$. The polynomial $L_n(x)$ is unimodal.*

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