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Proper actions and proper invariant metrics

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Abstract

We show that if a locally compact group G acts properly on a locally compact σ-compact space X, then there is a family of G-invariant proper continuous finite-valued pseudometrics which induces the topology of X. If X is, furthermore, metrizable, then G acts properly on X if and only if there exists a G-invariant proper compatible metric on X.

1. *Introduction*

We establish a close connection between proper group actions and groups of isometries. There is an old result in this direction, proved in 1928 by van Dantzig and van der Waerden; see [**4**]. It says that, for a locally compact connected metric space (X, d), its group $G = \mathrm{Iso}(X, d)$ of isometries is locally compact and acts properly. That the action is proper is no longer true in general, if X is not connected, although G is sometimes still locally compact; see [**13**]. Concerning properness of the action, Gao and Kechris [**5**] proved the following result. If (X, d) is a proper metric space, then G (is locally compact and) acts properly on X. Recall that a metric d on a space X is called proper if every ball has compact closure.

There is the following converse result. If a locally compact group G acts properly on a locally compact σ-compact metrizable space X, then there is a compatible G-invariant metric d on X (see [**11**]). In this paper, we prove that under these hypotheses there is actually a compatible G-invariant *proper* metric on X. We call a metric on a topological space compatible if it induces its topology. Note that a proper metric space is σ-compact. For the records, here is one version of our main result, namely the one for metrizable spaces (see also Theorem 4.2).

Theorem 1.1. *Suppose the* (*locally compact*) *topological group G acts properly on the metrizable locally compact σ-compact topological space X. Then there is a G-invariant proper compatible metric on X.*

These results raise the question of whether they generalize to the non-metrizable case. We give a complete answer as follows. Recall that a pseudometric on X is a function d on $X \times X$, which has all the properties of a metric, except that its value may be ∞ and that $d(x, y) = 0$ may not imply that $x = y$. For a precise definition see Definition 2.1. A locally compact space is σ-compact if and only if has a proper finite-valued continuous pseudometric, as is easily seen; see, for example, the proof of Corollary 5.3. It then actually has a family of such pseudometrics which induces the topology of X. The corresponding statement for the equivariant situation is the following version of the main result of our paper, namely for not necessarily metrizable spaces (see also Theorem 4.1).

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Theorem 1.2. *Let G be a (locally compact) topological group that acts properly on a locally compact σ-compact Hausdorff space X. Then there is a family of G-invariant proper finite-valued continuous pseudometrics on X which induces the topology of X.*

Let us point out that the existence of a compatible family of G-invariant finite-valued continuous pseudometrics, not necessarily proper ones, under these hypotheses was already proved in [**1**], actually under weaker hypotheses.

The connection of Theorems 1.1 and 1.2 is given by the following result. We are in the case of Theorem 1.1 if and only if there is a countable family as in Theorem 1.2. For a precise statement, see Corollary 4.4.

Note that continuity of the pseudometrics follows from the other properties; see Remark 5.5.

This theorem may be considered as the converse of the following theorem; see Theorem 3.1.

Theorem 1.3. *Let X be a topological space and let $\mathcal{D}$ be a family of proper continuous finite-valued pseudometrics on X which induces the topology of X. Let G be the group of all bijective maps $X \to X$ leaving every $d \in \mathcal{D}$ invariant. Endow G with the compact-open topology. Then G is a locally compact topological group and acts properly on X.*

The main result of our paper has been proved already for the special case of a smooth manifold. Namely Kankaanrinta proved in [**8**] that if a Lie group G acts properly and smoothly on a smooth manifold M, then M admits a complete G-invariant Riemannian metric. A consequence of our main result for the metrizable case is the following result of Struble [**14**]. Every second countable locally compact group has a left-invariant compatible proper metric which generates its topology; see Corollary 9.4. The authors thank Kechris for pointing out that this result, which we first saw in the paper [**6**] of Haagerup and Przybyszewska, is already contained in the earlier paper [**14**] by Struble. Proper G-invariant metrics have been used in several fields of mathematics; see [**7**, **9**]. For more information about related work, open questions and miscellaneous remarks, see the final section of this paper.

2. *Preliminaries*

2.1. *Pseudometrics*

Definition 2.1. A pseudometric d on a set X is a function $d : X \times X \to [0, +\infty]$ which fulfils for $x, y, z \in X$ the following properties:

(i) $d(x, x) = 0$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) + d(y, z) \geqslant d(x, z)$.

Thus, loosely speaking, a pseudometric is a metric except that its values may be $+\infty$, and $d(x, y) = 0$ does not imply $x = y$. A family $\mathcal{D}$ of pseudometrics on X induces a topology on X, for which finite intersections of balls $B_d(x, r) := \{y \in X; d(x, y) < r\}$ with $x \in X$, $d \in \mathcal{D}$ and $r \in [0, \infty)$ form a basis. This topology is the coarsest topology for which every $d \in \mathcal{D}$ is a continuous function on $X \times X$. The topology of a topological space X is induced by a family of pseudometrics if and only if X is completely regular; see [**2**, Chapter X, § 1.4 Theorem 1 and § 1.5 Theorem 2]. A topological space X is called *metrizable* if its topology is induced by an appropriately chosen metric d on X. Such a metric d on X is then called a *compatible metric*.

From now on we call a locally compact Hausdorff space simply a 'space', for short. Recall that a space is called σ-compact if it can be written as a countable union of compact subsets. A σ-compact space is metrizable if and only if it is second countable, that is, its topology has a countable base; see [**2**, Chapter IX, § 2.9 Corollary].

A pseudometric d on a space X will be called *proper* if every ball of finite radius has compact closure. A space X together with a compatible proper metric d will be called a *proper metric space*. It is also called a *Heine–Borel space* by some authors and a *finitely compact space* by others. Important examples of proper metric spaces are the Euclidean spaces and the space $\mathbb{Q}_p$ of rational p-adics with their usual metrics.

The topology of a space can be induced by a family of pseudometrics, since a space (understood: locally compact Hausdorff) is completely regular. The topology of a σ-compact space can be induced by a family of proper finite-valued pseudometrics (see Corollary 5.3). One of our main results, Theorem 1.2, spells out for which actions there is a family of invariant proper finite-valued pseudometrics inducing the topology, namely the proper actions. And Theorem 1.3 says that these are essentially the only ones for which such a family exists.

Now let $(X, \mathcal{D})$ be a space X together with a family $\mathcal{D}$ of pseudometrics inducing its topology. A case of particular importance is when $\mathcal{D}$ consists of just one metric, which by assumption induces the topology of X. Let $G = \mathrm{Iso}(X, \mathcal{D})$ be the group of isometries of $(X, \mathcal{D})$, that is, the group of all bijections $X \to X$ leaving every $d \in \mathcal{D}$ invariant. Endow G with the topology of pointwise convergence. Then G will be a topological group [**2**, Chapter X, § 3.5 Corollary]. On G there is also the topology of uniform convergence on compact subsets, which is the same as the compact-open topology. In our case, these topologies coincide with the topology of pointwise convergence, and the natural action of G on X is continuous [**2**, Chapter X, § 2.4 Theorem 1 and § 3.4 Corollary 1]. We will prove soon that if at least one of the pseudometrics d in $\mathcal{D}$ is proper, then G is locally compact. In this case, the natural action of G on X is even proper. We shall discuss this notion now.

DEFINITION 2.2. A continuous map $f : X \to Y$ between spaces is called *proper* if one of the following two equivalent conditions holds:
(i) $f^{-1}(K)$ is compact for every compact subset K of Y;
(ii) f is a closed map and the inverse image of every singleton is compact.

Let G be a topological group. Suppose a continuous action of G on a space X is given.

DEFINITION 2.3 AND PROPOSITION. The following conditions are equivalent.
(i) The map $G \times X \to X \times X$, $(g, x) \mapsto (gx, x)$, is proper.
(ii) For every pair A and B of compact subsets of X the transporter

$$G_{AB} := \{g \in G;\ gA \cap B \neq \emptyset\},$$

from A to B is compact.
(iii) Whenever we have two nets $(g_i)_{i\in I}$ in G and $(x_i)_{i\in I}$ in X, for which both $(x_i)_{i\in I}$ and $(g_i x_i)_{i\in I}$ converge, then the net $(g_i)_{i\in I}$ has a convergent subnet.
The action of G on X is called proper if one of these conditions holds.

For a proof; see [**2**, Chapter I, § 10.2 Theorem 1 and Chapter III, § 4.4 Proposition 7]. For more information on proper group actions, see the forthcoming book 'Proper transformation groups' by the first author and Strantzalos.

Note that if the action of G on X is proper, then G is locally compact, by (ii). And furthermore, if X is σ-compact, then G is also σ-compact, by (ii).

It is useful to rephrase the definition of properness in terms of limit sets. Let $(x_i)_{i\in I}$ be a net in the, not necessarily locally compact, topological space X. We say that the net $(x_i)_{i\in I}$ diverges and write $x_i \underset{i\in I}{\to} \infty$, if the net $(x_i)_{i\in I}$ has no convergent subnet. If X is locally compact, then a net $(x_i)_{i\in I}$ in X diverges if and only if it converges to the additional point ∞ of the one-point (also called Alexandrov-) compactification of X.

Again let the topological group G act on the space X. For $x \in X$ the *limit set* $L(x)$ is defined by

$$\begin{aligned} L(x) := \{y;\ &\text{there exists a divergent net } (g_i)_{i\in I} \text{ in } G \\ &\text{such that } (g_i x)_{i\in I} \text{ converges to } y\} \end{aligned}$$

and the *extended limit set* $J(x)$ is defined by

$$\begin{aligned} J(x) := \{y;\ &\text{there exists a divergent net } (g_i)_{i\in I} \text{ in } G \\ &\text{and a net } (x_i)_{i\in I} \text{ in } X \text{ converging to } x, \\ &\text{such that } (g_i x_i)_{i\in I} \text{ converges to } y\}. \end{aligned}$$

Thus, the action of G on X is proper if and only if the following condition holds:

(iv) $J(x) = \emptyset$ *for every* $x \in X$,

since condition (iv) is equivalent to condition (iii). Furthermore, if $\mathcal{D}$ is a family of pseudometrics inducing the topology of X and every $g \in G$ leaves every $d \in \mathcal{D}$ invariant, then it is easy to see that the following condition holds.

(v) $L(x) = \emptyset$ *implies* $J(x) = \emptyset$.

3. *The group of isometries of a proper metric space*

Again let X be a locally compact Hausdorff space, let $\mathcal{D}$ be a family of pseudometrics inducing the topology of X and let G be the group of isometries of $(X, \mathcal{D})$ with its natural topology, as above.

Theorem 3.1. *If at least one of the pseudometrics in* $\mathcal{D}$ *is proper, then* G *is locally compact and the natural action of* G *on* X *is proper.*

The special case that $\mathcal{D}$ consists of just one metric is due to Gao and Kechris [**5**], as follows.

Theorem 3.2. *If* (X, d) *is a proper metric space, then its group* G *of isometries is locally compact and its natural action of* G *on* X *is proper.*

Proof of Theorem 3.1. It suffices to show that the natural action of G on X is proper. To prove this, we will show that the limit set $L(x)$ is empty for every $x \in X$. Thus, let $(g_i)_{i\in I}$ be a net in G for which $(g_i x)_{i\in I}$ converges to a point, say y, in X. We have to show that the net $(g_i)_{i\in I}$ has a convergent subnet. We may assume that $g_i x$ is contained in the relatively compact ball $B_d(y, r)$ for every $i \in I$, where d is a proper pseudometric in $\mathcal{D}$ and $r > 0$. We use the Arzela–Ascoli theorem. Let $z \in X$. The points $g_i z$, $i \in I$, are contained in the ball $B_d(z, R)$, where $R = r + d(x, z)$. Thus, the set $\{g_i z; i \in I\}$ is relatively compact for every $z \in X$. The family of maps $\{g_i; i \in I\}$ is uniformly equicontinuous, being a subset of the uniformly equicontinuous family G of maps from X to X. It follows from the Arzela–Ascoli theorem that the net $(g_i)_{i\in I}$ has a subnet $(g_j)_{j\in J}$ that converges uniformly on compact subsets to a map g. Clearly, g leaves every $d \in \mathcal{D}$ invariant. To see that g is actually invertible look at the net

$(g_j^{-1})_{j\in J}$. We have $g_j^{-1}y \in B_d(x,r)$ and hence $g_j^{-1}z \in B_d(z,R')$ where $R' = r + d(x,z)$. Then the net $(g_j^{-1})_{j\in J}$ has a subnet which converges uniformly on compact subsets to a map f. It then follows that f and g are inverse of each other. □

Remark 3.3. The sets $K(E) := \{x \in X;\ Ex \text{ is relatively compact}\}$, where $E \subset \mathrm{Iso}(X,d)$ played a crucial role in [**12**, **13**], where it is proved that they are open–closed subsets of X. In the case of a proper metric space (X,d), the set $K(E)$ is either the empty set or the whole space X as shown in the proof of Theorem 3.1. Using Bourbaki [**2**, Chapter X, Exercise 13, p. 323], we may also show that sets $K(E)$ are open–closed subsets of X, but we must be careful! Even in the legendary 'Topologie Générale' of Bourbaki there is at least one mistake! Precisely in the aforementioned Exercise 13 of Chapter X, p. 323, part (d), it is said that if E is a uniformly equicontinuous family of homeomorphisms of a locally compact uniform space X, then $K(E)$ is a closed subset of X. This is not true if E is not a subset of a uniformly equicontinuous *group* of homeomorphisms of X as we can easily see by the following counterexample; see [**12**].

Counterexample 3.4. Let

$$X = \bigcup_{k=1}^{\infty} \left\{(x,y);\, x = \frac{1}{k},\, y \geqslant 0\right\} \cup \{(x,y);\ x = 0,\ y > 0\}$$

be endowed with the Euclidean metric. Consider the family $E = \{f_n\}$ of selfmaps of X defined by $f_n(x,y) = (x, y/n)$. The family E consists of uniformly equicontinuous homeomorphisms of X and $K(E) = \bigcup_{k=1}^{\infty}\{(x,y);\ x = 1/k,\ y \geqslant 0\}$ as can be easily checked. Hence, the set $K(E)$ is not closed in X.

4. *Proper invariant metrics and pseudometrics, outline of the proof*

The main results of our paper are the following converses of Theorems 3.1 and 3.2. Again, X is a space, that is, a locally compact Hausdorff space, and G is a Hausdorff topological group. Suppose that we are given a continuous action of G on X.

Theorem 4.1. *Suppose that X is σ-compact. If the action of G on X is proper, then there is a family $\mathcal{D}$ of proper finite-valued G-invariant pseudometrics on X, which induces the topology of X.*

Theorem 4.2. *Suppose that X is σ-compact. If the action of G on X is proper and X is metrizable, then there is a compatible G-invariant proper metric d on X.*

Remark 4.3. If the action is proper, then it is easy to see that the kernel of the action $K := \{g \in G;\ gx = x \text{ for every } x \in X\}$ is compact and the action map induces an isomorphism of topological groups of G/K onto a closed subgroup of $\mathrm{Iso}(X,\mathcal{D})$ or $\mathrm{Iso}(X,d)$, respectively. We thus have a complete correspondence between proper actions and isometry groups of proper metrics or pseudometrics.

Corollary 4.4. *Suppose that X is σ-compact and G acts properly on X. Then the following properties of X are equivalent.*

(a) *The space X is metrizable.*
(b) *There is a compatible G-invariant proper metric on X.*

(c) *There is a countable family of finite-valued pseudometrics on X, which induces the topology of X.*

(d) *There is a countable family of proper finite-valued G-invariant pseudometrics on X, which induces the topology of X.*

Proof. (a) $\Rightarrow$ (b) by Theorem 4.2; (b) $\Rightarrow$ (d) and (d) $\Rightarrow$ (c) are trivial and (c) $\Rightarrow$ (a) is a well-known theorem of topology [**2**, Chapter IX, § 2.4 Corollary 1] whose proof is similar to the argument in the last paragraph of the proof of Lemma 8.7(a). □

The proofs of Theorems 4.1 and 4.2 take up most of the remainder of the paper. Let us briefly describe the plan of the proof. We describe the plan for the case of a family of pseudometrics, the proof for the metrizable case simplifies at some points.

(1) We first construct a family $\mathcal{D}$ of pseudometrics on X, with values in [0,1] which induces the topology of X; see Section 5.

(2) Next we show how to make every $d \in \mathcal{D}$ G-invariant; see Section 6.

(3) Then we make every $d \in \mathcal{D}$ orbitwise proper; see Section 7.

(4) These steps are fairly routine. We then present our main tool, namely the 'measuring stick construction'. Imagine a family of measuring sticks given by distances of closely neighbouring points. We then define a pseudometric $d(x, y)$ on X, for x, y in X, as the infimum of all measurements along sequences of points $x = x_0, \dots, x_n = y$ such that the distance of any two adjacent points is given by measuring sticks. For a precise definition, actually several equivalent ones, see Section 8. It turns out that we then get for an appropriate family of measuring sticks a proper pseudometric. The disadvantage of this construction is that there may be points that cannot be connected by sequences as above. Equivalently, there may be points x, y with $d(x, y) = \infty$.

(5) We then use our 'bridge construction'; see Section 9. Think of pairs of points with $d(x, y) < \infty$ as lying on the same island. Thus, what we call an island is an equivalence class of the equivalence relation defined as $x \sim y$ if and only if $d(x, y) < \infty$. We connect (some of) these islands by bridges and attribute (high) weights to these bridges. We then define a new pseudometric in a similar way to above using the already defined pseudometric on the islands and the weights of bridges. We thus obtain a proper pseudometric with finite values and actually a whole family of such, which induces the topology of X. All these constructions are done in a G-invariant way, so that the resulting pseudometrics are G-invariant.

5. *A compatible metric and proper pseudometrics*

Again, by a space we mean a locally compact Hausdorff space. Recall the following basic metrization result; see [**2**, Chapter IX, § 2.9 Corollary].

THEOREM 5.1. *For a space X the following properties are equivalent*:

(a) *X is second countable, that is, its topology has a countable base*;

(b) *the one-point compactification $\bar{X}$ of X is metrizable*;

(c) *X is metrizable and σ-compact.*

If a space is metrizable, we may assume that the metric d inducing the topology has values in [0,1]. We just have to replace d by d_1 with $d_1(x, y) := d(x, y)/(1 + d(x, y))$.

For the general case of a not necessarily metrizable σ-compact space, and for later use, we need the following easy lemma, whose proof is left to the reader.

LEMMA 5.2. *A space X is σ-compact if and only if there is a proper continuous function $f : X \to [0, \infty)$.*

COROLLARY 5.3. *On every σ-compact space X there is a family $\mathcal{D}$ of proper finite-valued pseudometrics inducing the topology of X.*

Proof. Let $\mathcal{D}_0$ be the family of pseudometrics on X of the form

$$d_f(x, y) := |f(x) - f(y)|,$$

for $x, y \in X$, where $f : X \to \mathbb{R}$ is a continuous function. Then $\mathcal{D}_0$ induces the topology of X. Here we do not use that X is σ-compact. But in the next step we do. If X is σ-compact, then let $\mathcal{D}$ be the family $\mathcal{D} := \{d + d_f;\ d \in \mathcal{D}_0\}$, where $f : X \to \mathbb{R}$ is proper and continuous. Then $\mathcal{D}$ induces the topology of X and consists of proper finite-valued pseudometrics. □

The same trick yields the following corollary.

COROLLARY 5.4. *The following properties of a space X are equivalent*:
(a) *X has a compatible proper metric*;
(b) *X is metrizable and σ-compact*;
(c) *X is metrizable and separable*;
(d) *X is second countable.*

REMARK 5.5. Note that if a pseudometric d belongs to a family of pseudometrics inducing the topology of X, then d is continuous, since then $B_d(x, r)$ is a neighbourhood of x for every $x \in X$ and every $r > 0$, and hence the function $y \mapsto d(x, y)$ is continuous at x for every $x \in X$, which easily implies that d is continuous by the triangle inequality.

6. *Making the metrics or pseudometrics G-invariant*

Now suppose that X is a space, G is a Hausdorff topological group, and a proper continuous action of G on X is given.

STEP 2. *If X is σ-compact, then there is a family of G-invariant continuous finite-valued pseudometrics inducing the topology of X. Furthermore, if X is metrizable, then there is a compatible G-invariant metric on X.*

We present two proofs.

The first one is due to Koszul [**11**] and uses the concept of a fundamental set, a concept we shall need again, later on. The second one is taken from the second author's PhD thesis [**12**]. It uses the notion of an equicontinuous action on the one-point compactification of X. Unfortunately, in the process we lose the property that our (pseudo-)metrics are proper. Note that [**1**] contains a more general result for Palais proper actions on completely regular spaces.

DEFINITION 6.1. A subset F of X is called a *fundamental set* for the action of G on X if the following two conditions hold:
(a) $GF = X$;
(b) G_{KF} has compact closure for every compact subset K of X.

Concerning (b), recall the definition of the transporter $G_{AB} = \{g \in G;\ g\,A \cap B \neq \emptyset\}$ from A to B. Note that only proper actions can have a fundamental set, since (a) implies that

$$G_{AB} \subset G_{BF}^{-1} \cdot G_{AF}$$

and hence G_{AB} is relatively compact if A and B are compact, by (b), and then G_{AB} is actually compact, by continuity of the action. We have the following converse; see [**11**].

Proposition 6.2. *If X is σ-compact, then there is an open fundamental set for every proper action.*

Step 2 (*First proof*). Let F be an open fundamental set for the action of G on X. Let d be a continuous finite-valued pseudometric on X. Let d' be the supremum of all pseudometrics on X with the property that $d' \mid F \times F \leqslant d$ and $d' \mid (X \smallsetminus F) \times (X \smallsetminus F) = 0$. Explicitly, let r be the function on X with $r_d(x) = d(x, X \smallsetminus F) := \inf\{d(x,y)\ ;\ y \in X \smallsetminus F\}$. Then

$$d'(x,y) = \min\{d(x,y),\ r_d(x) + r_d(y)\}.$$

Note that, for every $x \in F$, there is a neighbourhood of x where d and d' coincide. The function d' is a finite-valued continuous pseudometric and the function $G \to \mathbb{R}$, $g \mapsto d'(gx, gy)$ is continuous and has compact support, namely, contained in $G_{\{x,y\},F}$. Define

$$d''(x,y) = \int_G d'(gx, gy)\, dg,$$

where dg is a right invariant Haar measure on G. Then d'' is a G-invariant pseudometric on X. The pseudometric d'' is actually a metric if d is a metric. Furthermore, d'' is continuous for every $d \in \mathcal{D}$, by a uniform equicontinuity argument for functions on compact spaces. Thus, the family $\mathcal{D}'' = \{d'';\ d \in \mathcal{D}\}$ induces a weaker topology than $\mathcal{D}$. The two topologies are actually equal since, for every neighbourhood V of $x \in X$, there are a compact neighbourhood V_1 of x in X and a compact neighbourhood U_1 of e in G such that $U_1 V_1 \subset V$ and $U_1(X \smallsetminus V) \subset X \smallsetminus V_1$, and hence

$$d''(x,y) \geqslant d'(x, X \smallsetminus V_1) \cdot \int_{U_1} dg,$$

for every $y \in X \smallsetminus V$, which implies our claim for $x \in F$ and hence for every x by G-invariance of the two topologies. □

Step 2 (*Second proof*). This proof is based on the notion of an equicontinuous group action. Consider the one-point compactification $\bar{X} = X \cup \{\infty\}$. The action of G on X extends to an action of G on $\bar{X}$ by defining $g(\infty) = \infty$ for every $g \in G$. The extended action is continuous. Let $\mathcal{D}$ be a family of pseudometrics on $\bar{X}$ which induces the topology of $\bar{X}$. Without further assumptions on X we can take the family $\{d_f; f : \bar{X} \to [0,1] \text{ continuous}\}$; see the proof of Corollary 5.3. If $\bar{X}$ is metrizable, we can take $\mathcal{D}$ to consist of just one element. This is the case if and only if X is metrizable and σ-compact; see Theorem 5.1. In any case, define, for $d \in \mathcal{D}$ and $x, y \in X$,

$$d'(x,y) := \sup_{g \in G} d(gx, gy),$$

and set $\mathcal{D}' = \{d';\ d \in \mathcal{D}\}$. We claim that $\mathcal{D}'$ induces the topology of X. Obviously, the topology induced by $\mathcal{D}'$ is finer than the topology of X, since $d' \geqslant d$ and $\mathcal{D}$ induces the topology of X.

Concerning the converse, consider the following property. The action of G on X is called *pointwise equicontinuous with respect* to $\mathcal{D}$ if, for every $x \in X$, $d \in \mathcal{D}$ and $\epsilon > 0$, there is a neighbourhood U of x such that, for $y \in U$, we have $d(gx, gy) < \epsilon$ for every $g \in G$. Clearly, if this holds then the topology defined by $\mathcal{D}'$ is weaker than the topology of X and our claim is proved. It thus remains to show the following lemma.

LEMMA 6.3. *Let X be a space and let G be a topological group acting properly on X. Let $\mathcal{D}$ be a family of pseudometrics on $\bar{X}$ inducing the topology of $\bar{X}$. Then G acts pointwise equicontinuously on X with respect to $\mathcal{D}$.*

Proof. Arguing by contradiction, assume that there are $d \in \mathcal{D}$, $x \in X$, $\epsilon > 0$ and a net $(x_i)_{i\in I}$ in X converging to x and a net $(g_i)_{i\in I}$ in G such that $d(g_i x, g_i x_i) \geqslant \epsilon$ for every $i \in I$. It follows that $g_i \to \infty$, since otherwise the net $(g_i)_{i\in I}$ has a convergent subnet, say $(g_j)_{j\in J}$, converging to $g \in G$. Then $g_j x \underset{j\in J}{\to} gx$ and $g_j x_j \underset{j\in J}{\to} gx$, contradicting $d(g_i x, g_i x_i) \geqslant \epsilon$ for every $i \in I$. It follows next that $g_i x_i \underset{i\in I}{\to} \infty$, since otherwise there would be a subnet $(g_j x_j)_{j\in J}$ converging to a point of X, which implies that there would be a convergent subnet of $(g_j)_{j\in J}$, by properness of the action. Thus, $g_i x_i \underset{i\in I}{\to} \infty$ and $g_i \underset{i\in I}{\to} \infty$, which implies $g_i x \underset{i\in I}{\to} \infty$, again by properness of the action. But then $d(g_i x, g_i x_i) \underset{i\in I}{\to} 0$, since d is continuous on $\bar{X}$. This contradicts our assumption and completes the proof. □

REMARK 6.4. The second proof shows Step 2 for the metrizable case only under the additional assumption that $\bar{X}$ is metrizable, that is, that X is metrizable and σ-compact. This is enough for our main results though, because there all spaces are σ-compact.

REMARK 6.5. The pseudometrics we obtain by these proofs are not proper, in general. This is clear for the second proof. For the first proof, even if we start from a proper (pseudo-) metric d, in the case where the orbit space $G\backslash X$ is compact, so F is relatively compact, we obtain that d'' has an upper bound.

REMARK 6.6. One could rephrase the notion of pointwise equicontinuity in terms of the unique uniformity on the compact space $\bar{X}$. We choose here to use the language of pseudometrics since proper (pseudo-) metrics are our final goal.

7. *Orbitwise proper metrics and pseudometrics*

If G acts on X, then we denote by $\pi : X \to G\backslash X$ the natural map to the orbit space. We call a pseudometric d on X *orbitwise proper* if $\pi(B_d(x, r))$ has compact closure for every $x \in X$ and $0 < r < \infty$. Again, we assume the notation and hypotheses of the last section.

STEP 3. *If X is σ-compact, then there is a family of G-invariant orbitwise proper finite-valued pseudometrics on X inducing the topology of X. If X is furthermore metrizable there is a G-invariant orbitwise proper compatible metric on X.*

Proof. If X is a space with a proper action, then the orbit space $G\backslash X$ is Hausdorff as well; see [2]. Clearly, $G\backslash X$ is locally compact. Furthermore, if X is σ-compact, so is $G\backslash X$. So there is a proper continuous function $f : G\backslash X \to [0, \infty)$; see Lemma 5.2. The pseudometric $d' := d_{f\circ\pi}$

on X, defined by

$$d'(x,y) = |f\pi(x) - f\pi(y)|,$$

for $x, y \in X$, is orbitwise proper, continuous and G-invariant. Hence, if $\mathcal{D}$ is a family of finite-valued G-invariant pseudometrics on X inducing the topology of X, then so is $\mathcal{D}' = \{d + d';\ d \in \mathcal{D}\}$ and, furthermore, every pseudometric of this family is orbitwise proper. □

8. *The measuring stick construction*

We first present our measuring stick construction in three equivalent ways. We then give a sufficient condition under which the resulting pseudometric is proper. This will be applied to our situation and yields Step 4 of our proof.

8.1. Let X be a set, d be a pseudometric on X and $\mathcal{U}$ be a covering of X. We then define a new pseudometric $d' = d'(d,\mathcal{U})$ on X depending on d and $\mathcal{U}$ as follows: d' is the supremum of all pseudometrics d'' on X with the property that $d''|U \times U \leqslant d|U \times U$ for every $U \in \mathcal{U}$.

8.2. We think of pairs (x, y) of points lying in one $U \in \mathcal{U}$ as *measuring sticks* or *sticks*, for short. A sequence $x = x_0, x_1, \dots, x_n = y$ of points in X, such that any two consecutive points form a stick, will be called a stick path from x to y of length n and d-length $\sum_{i=1}^{n} d(x_{i-1}, x_i)$. We claim that $d'(x, y)$ is the infimum of d-lengths of all stick paths from x to y, since, on the one hand, defining d' in this way clearly gives a pseudometric on X and $d'|U \times U \leqslant d|U \times U$, and, on the other hand, for every pseudometric d'' with the two properties above we have that $d''(x, y)$ is at most equal to the d-length of any stick path from x to y, because for every stick path $x = x_0, x_1, \dots, x_n = y$, we have

$$d''(x,y) \leqslant \sum_{i=1}^{n} d''(x_{i-1}, x_i) \leqslant \sum_{i=1}^{n} d(x_{i-1}, x_i)$$

We thus obtain the following properties of $d' = d'(d,\mathcal{U})$:

(a) $d' \geqslant d$;

(b) $d'|U \times U = d|U \times U$;

(c) if d is finite-valued on every $U \in \mathcal{U}$, then $d(x, y) < \infty$ if and only if there is a stick path from x to y.

8.3. An alternative way to describe this construction is the following. Let $\Gamma_{\mathcal{U}}$ be the following graph. The vertices of $\Gamma_{\mathcal{U}}$ are the points of X and the edges of $\Gamma_{\mathcal{U}}$ are the sticks, that is, the pairs (x, y) contained in one $U \in \mathcal{U}$. So the graph $\Gamma_{\mathcal{U}}$ is closely related to the nerve of the covering $\mathcal{U}$. To every edge (x, y) of $\Gamma_{\mathcal{U}}$ we can associate the weight $d(x, y)$. Then, for points x, y in X, the pseudometric $d'(x, y)$ is the graph distance of the corresponding vertices of this weighted graph.

Let us now return to the case that we are interested in. Thus, let X be a σ-compact space with a proper action of a locally compact topological group G. Let F be an open fundamental set for G in X. We consider the covering $\mathcal{U}$ by the translates of F, so $\mathcal{U} = \{gF;\ g \in G\}$. We apply the measuring stick construction for an appropriate pseudometric d and show that the resulting pseudometric d' is proper, but may be infinite-valued. We do this first for the case where the orbit space $G\backslash X$ is compact and then for the general case. We shall need an auxiliary result about Lebesgue numbers of our covering; see Lemma 8.2. The problem of infinite values of d' will be dealt with in the next section. The method will be the 'bridge construction'.

We start with a well-known result, for which we include a proof for the convenience of the reader.

Lemma 8.1. *If the orbit space $G\backslash X$ is compact, then every fundamental set is relatively compact. Conversely, if $G\backslash X$ is compact, then every relatively compact subset F of X with the property that $GF = X$ is a fundamental set for G in X.*

Proof. The second claim is clear, since property (b) of a fundamental set follows immediately from the hypothesis that the action of G on X is proper; see Proposition and Definition 2.3(ii). To prove the first claim choose a compact neighbourhood U_x for every point $x \in X$. A finite number of the $\pi(U_x)$, with $x \in X$, cover $G\backslash X$, where π is the natural map $\pi : X \to G\backslash X$, which is known to be an open map. Let us say $G\backslash X = \pi(U_{x_1}) \cup \ldots \cup \pi(U_{x_n})$; so $X = GU_{x_1} \cup \ldots \cup GU_{x_n}$. Hence, $A \subset G_{U_{x_1},A}U_{x_1} \cup \ldots \cup G_{U_{x_n},A}U_{x_n}$ for every subset A of X. For $A = F$ the subsets $G_{U_{x_i},F}$ of G are relatively compact, by property (b) of a fundamental set; see Definition 6.1. Hence, F is relatively compact. □

A family $\mathcal{D}$ of pseudometrics is called *saturated* if d_1, $d_2 \in \mathcal{D}$ implies $\sup(d_1, d_2) \in \mathcal{D}$.

Lemma 8.2. *Let $\mathcal{D}$ be a saturated family of G-invariant pseudometrics inducing the topology of X. Suppose that the orbit space $G\backslash X$ is compact. Then there is a pseudometric $d \in \mathcal{D}$ and a positive number ϵ such that, for every $x \in X$, the ball $B_d(x, \epsilon)$ is contained in one translate of F.*

A number ϵ with this property is called a *Lebesgue number* for the covering $\{gF; g \in G\}$ with respect to d.

Proof. By G-invariance, it suffices to show this for points $x \in F$. Since $\bar{F}$ is compact, it is covered by a finite number of gF, say $\bar{F} \subset g_1F \cup \ldots \cup g_nF$. Recall that F is supposed to be open. The set of balls $B_d(x, r)$, $d \in \mathcal{D}$, $x \in X$, $r > 0$, form a base of the topology of X, not only their finite intersections, since $\mathcal{D}$ is saturated. Thus, there is for every $x \in \bar{F}$ a pseudometric $d_x \in \mathcal{D}$ and a radius r_x such that $B_{d_x}(x, r_x)$ is contained in one translate of F, since F is open. A finite number of balls $B_{d_x}(x, r_x/2)$ cover $\bar{F}$, say those for $x = x_1, \ldots, x_n$. Thus, for every $y \in \bar{F}$ there is an x_i, $i = 1, \ldots, n$, such that $y \in B_{d_{x_i}}(x_i, r_{x_i}/2)$ and hence $B_{d_{x_i}}(y, r_{x_i}/2) \subset B_{d_{x_i}}(x_i, r_{x_i})$ is contained in one translate of F. Hence, our claim holds for $d = \sup(d_{x_1}, \ldots, d_{x_n}) \in \mathcal{D}$ and $\epsilon = \inf(r_{x_1}, \ldots, r_{x_n})$. □

Now let again $\mathcal{U} = \{gF;\ g \in G\}$ and for a G-invariant pseudometric d on X let $d' = d'(d, \mathcal{U})$ be the pseudometric obtained by the measuring stick construction.

Proposition 8.3. *Suppose the orbit space $G\backslash X$ is compact. Let d be a continuous G-invariant pseudometric on X, for which there is a Lebesgue number for $\mathcal{U}$. Then d' is a proper pseudometric, that is, $B_{d'}(x, R)$ is relatively compact for every $x \in X$ and every $R < \infty$.*

Proof. We may assume that $x \in F$, by G-invariance. Then $y \in B_{d'}(x, R)$ if and only if there is a stick path $x = x_0, x_1, \ldots, x_n = y$ with d-length $\sum_{i=1}^{n} d(x_{i-1}, x_i) < R$. We may assume that no three consecutive points x_{i-1}, x_i, x_{i+1} of our stick path are contained in one translate of F, because otherwise we can leave out x_i from our stick path and obtain a stick path of not greater than d-length. Let ϵ be the Lebesgue number for $\mathcal{U}$ with respect to d. It follows that $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) \geqslant \epsilon$ for every $i = 1, \ldots, n-1$, because otherwise x_{i-1}, x_i, x_{i+1} are contained in one translate of F. We thus obtain the following upper bound for the length n of our stick path:

$$n < \frac{2R}{\varepsilon} + 1.$$

Thus, let $N \in \mathbb{N} \cup \{0\}$ and let B_N be the set of points $y \in X$ for which there is a stick path of length N starting at a point $x \in F$ and ending at y. We have to show that B_N is relatively compact for every $N \in \mathbb{N} \cup \{0\}$. For $N = 0$ we have $B_N = F$. If $y \in B_{N+1}$ there is a point $y' \in B_N$ such that (y', y) is a stick, say $\{y', y\} \subset g\,F$. Then $y' \in B_N \cap g\,F$ and hence $g \in G_{F,B_N} = G_{B_N^{-1},F}$. This subset of G is relatively compact by induction and property (b) of a fundamental set. Thus, $y \in g\,F \subset G_{F,B_N}F$, hence $B_{N+1} \subset G_{F,B_N}F$ and so B_{N+1} is relatively compact. □

This yields Step 4 of our proof for the case that the orbit space is compact. For the general case we need *one* pseudometric d for which there is a Lebesgue number for every subset of X of the form $\pi^{-1}(K)$, where K is a compact subset of $G\backslash X$. Here we have to suppose that the orbit space is σ-compact.

Before we proceed to do this, we need to figure out where d' is finite. Let F and $\mathcal{U}$ be as above. We do not suppose that the orbit space is compact. Let d be a G-invariant pseudometric on X for which $d|F \times F$ has finite values. Let the symbol '$\sim$' denote the smallest G-invariant equivalence relation on X for which F is contained in one equivalence class. Recall that $G_{FF} = \{g \in G;\ gF \cap F \neq \emptyset\}$. Let G_0 be the subgroup of G generated by G_{FF}.

Lemma 8.4. *Let x and y be points of X. The following properties of the pair (x, y) are equivalent.*

(a) $d'(x, y) < \infty$.
(b) *There is a stick path from x to y.*
(c) $x \sim y$.
(d) *The vertices x and y of the graph $\Gamma_{\mathcal{U}}$ belong to the same connected component of $\Gamma_{\mathcal{U}}$.*
(e) *If $x \in g\,F$ and $y \in h\,F$, then $g^{-1}h \in G_0$.*

The equivalence classes will be called *islands* from now on.

Proof. (a)⇔(b) was noted above, and (b)⇔(d) and (b)⇔(c) follow immediately from the definitions.

(b)⇒(e). Let $x \in g\,F$ and $y \in h\,F$ and let (x, y) be a stick, say $\{x, y\} \subset k\,F$ for some $k \in G$. Then $g^{-1}k \in G_{FF}$ and $h^{-1}k \in G_{FF}$, hence $g^{-1}h \in G_0$. The claim (b)⇔(e) follows now by induction on the length of the stick path.

(e)⇒(c). Let Y be an equivalence class of $\sim$. Thus, if one point of a translate $g\,F$ of F is contained in Y, then $g\,F$ is contained in Y. By the same argument applied to $g\,k\,F$ with $k \in G_{FF}$, it then follows that $g\,G_{FF}F \subset Y$, hence $g \cdot G_{FF} \cdot G_{FF}\,F \subset Y$, and so on. So $g\,G_0\,F \subset Y$ if $g\,F \cap Y \neq \emptyset$, which proves our claim. □

Corollary 8.5. *The map $g\,G_0 \mapsto g\,G_0\,F$ establishes a bijection between the set G/G_0 of left cosets of G_0 in G and the set of islands in X.*

Corollary 8.6. *If $G\backslash X$ is σ-compact, the so are $\bar{F}$, $G_{\bar{F},\bar{F}}$, G_0 and every island.*

Proof. If K is a compact subset of $G\backslash X$, then so is $F_K := \bar{F} \cap \pi^{-1}(K) = \overline{\pi^{-1}(K) \cap F}$, by Lemma 8.1, and hence also G_{F_K,F_K}, since the action of G on X is proper and continuous. It follows that if $G\backslash X$ is σ-compact, then so are $\bar{F}$, $G_{\bar{F},\bar{F}}$, the subgroup G_1 of G generated by $G_{\bar{F},\bar{F}}$ and $G_1\bar{F}$. It thus remains to be shown that $G_0 = G_1$ and $G_0\bar{F} = G_0F$. But clearly $G_{\bar{F}F} = G_{FF}$ since F is open, hence $G_{\bar{F}\bar{F}} \subset G_{\bar{F}F}^{-1} \cdot G_{\bar{F}F}$, by the formula following Definition 6.1, and thus $G_{\bar{F},\bar{F}} \subset G_0$ and hence $G_1 = G_0$. Furthermore, $\bar{F} \subset G_{\bar{F}F}^{-1}F$, by Definition 6.1(a), and hence $G_0\bar{F} = G_0F$. □

We come back to the Lebesgue number and show properness of d' for the case where the orbit space is σ-compact. This accomplishes Step 4 of our plan in Section 4. Note that at this point we do not need that X be σ-compact, only that the orbit space be σ-compact.

LEMMA 8.7. *Suppose that the orbit space $G\backslash X$ is σ-compact.*

(a) *Then there is a continuous orbitwise proper G-invariant pseudometric d on X with the following properties: d is finite-valued on every island and, for every compact subset K of $G\backslash X$, there is a Lebesgue number for the covering $\mathcal{U}|\pi^{-1}(K)$ of the G-space $\pi^{-1}(K)$ with respect to the restriction of d to $\pi^{-1}(K)$.*

(b) *If d is as in* (a), *then d' is proper, which means that the ball $B_{d'}(x,R)$ has compact closure for every $x \in X$ and every $0 < R < \infty$.*

Proof. (a) Let K_n, $n \in \mathbb{N}$, be a sequence of compact subsets of $G\backslash X$ such that $\bigcup_{n=1}^{\infty} K_n = G\backslash X$ and $K_n \subset \overset{\circ}{K}_{n+1}$ for every $n \in \mathbb{N}$. Put $X_n = \pi^{-1}(K_n)$. Then X_n is a closed G-invariant subset of X on which G acts properly with compact orbit space K_n. The set $F_n := F \cap X_n$ is an open fundamental set for G in X_n, hence relatively compact in X_n and in X. So there is a continuous orbitwise proper G-invariant finite-valued pseudometric d_n on X such that there is a Lebesgue number for the covering $\{gF_n; g \in G\}$ of X_n with respect to the pseudometric d_n restricted to X_n. Note that d_n is defined and finite-valued on all of X. To see the existence of such a d_n, we apply Lemma 8.2 to the family $d|X_n \times X_n$, where d runs through a saturated family of finite-valued G-invariant pseudometrics on X inducing the topology of X, which we may assume to be orbitwise proper, by Step 3 in Section 7.

Let Y be the island G_0F containing F. We use here the notation of Lemma 8.4 and its corollaries. Since Y is σ-compact, there is a family L_n, $n \in \mathbb{N}$, of compact subsets of Y such that $\bigcup_{n=1}^{\infty} L_n = Y$ and $L_n \subset \overset{\circ}{L}_{n+1}$. We may assume that $d_n|L_n \times L_n$ has values at most 1, by rescaling. Now define

$$d(x,y) = \begin{cases} \Sigma \frac{1}{2^n} d_n(x,y) & \text{if} x \sim y, \\ \infty & \text{otherwise.} \end{cases}$$

Then d is G-invariant continuous orbitwise proper pseudometric on X, which is finite-valued on $Y \times Y$ and hence on every island. There is a Lebesgue number for the covering $\{g\,F_n;\, g \in G\}$ of X_n with respect to d, since there is one for d_n and $d \geqslant (1/2^n)d_n$. Here we think of d and d_n as restricted to $X_n \times X_n$. This implies our claim under (a).

(b) Islands are of the form $g\,G_0\,F$, hence open, since F is supposed to be open. It follows that they are also closed. Again, let $Y = G_0F$ be the island containing F. Let $B_{d'}(x,R)$, $x \in X$, $0 < R < \infty$, be a ball for the pseudometric d' and let B be its closure. We have to show that B is compact. We know that $K := \pi(B)$ is compact, since d is orbitwise proper and hence so is d', since $d' \geqslant d$ by Subsection 8.2(a). We may assume that $x \in F$ and hence $B_{d'}(x,R) \subset Y$, and thus $B \subset Y$.

The subgroup G_0 of G is open since it is generated by the open subset G_{FF}. It follows that G_0 is a closed subgroup of G. Then the action of G_0 on Y is proper, since the restricted action of G_0 on X is proper and Y is a closed G_0-invariant subset of X. And F is an open fundamental set for G_0 in Y. Let $Z = Y \cap \pi^{-1}(K)$. This is a closed G_0-invariant subset of Y, and $F_Z := Z \cap F = F \cap \pi^{-1}(K)$ is an open fundamental set for G_0 in Z. The orbit space $G_0 \backslash Z$ is compact; it can be identified with K. So we can apply Proposition 8.3 to the G_0-space Z, the pseudometric $d|Z \times Z$ and the covering $\mathcal{U}_Z := \{gF_Z;\, g \in G_0\}$ to obtain that the resulting stick path pseudometric $d'' := d'(d|Z \times Z, \mathcal{U}_Z)$ is proper. It remains to see that $B_{d''}(x, R) = B_{d'}(x, R)$. Clearly $d''(x, y) < R$ implies $d'(x, y) < R$, by looking at the stick paths for $\mathcal{U}_Z$. Conversely, if $d'(x, y) < R$, then there is a stick path $x = x_0, x_1, \dots, x_n = y$ for $\mathcal{U}$ with $\Sigma d(x_{i-1}, x_i) < R$. Then all the x_i are in $B_{d'}(x, R) \subset Y$ and $\pi(x_i) \in K$, hence $x_i \in Z$ and every pair x_{i-1}, x_i is contained in some translate gF of F. But then $g \in G_0$, by Lemma 8.4(e), and so $\{g^{-1}x_{i-1}, g^{-1}x_i\}$ is contained in F and in Z, hence in F_Z. Thus, our stick path is also a stick path for $\mathcal{U}_Z$ in Z, and thus $d''(x, y) < R$. □

9. *Bridges*

Again, let X be a σ-compact space and let the locally compact group G act properly on X. Note that then G is σ-compact as well, since if X is the union of countably many compact subsets K_n, then G is the union of the countably many sets G_{K_n,K_n} that are compact since the action of G on X is both proper and continuous. Let us again fix an open fundamental set F for G in X. Then, using the notation of the last section, G_0 is an open subgroup of G and hence G/G_0 is a countable discrete space. We can thus choose a finite or infinite sequence of elements g_n, $n = 0, 1, \dots$, such that G is the union of the disjoint cosets g_nG_0. We may assume that g_0 is the identity element. Let S be the set of indices, so that $S = \mathbb{N} \cup \{0\}$ or $S = \{0, 1, \dots, N\}$ for some $N \in \mathbb{N} \cup \{0\}$. Thus, $G = \bigcup_{n \in S} g_nG_0$ and hence X is the union of the disjoint subsets g_nG_0F, $n \in S$, by Corollary 8.5. Recall that the sets of the form $g\,G_0F$ are called islands. Consequently, we define a *bridge* to be a 2-point subset of X of the form $\{gx, gg_nx\}$ with $g \in G$, $n \in S$, $n \neq 0$, and $x \in F$. Note that gx and gg_nx are always on different islands since $n \neq 0$. But the representation of a bridge in the form above may not be unique. Now suppose that a G-invariant pseudometric d on X is given. We then define the *bridge path pseudometric* d_B on X as the supremum of all pseudometrics d'' with the following two properties.

9.1. (a) For every island Y in X we have $d''|Y \times Y \leqslant d|Y \times Y$.
(b) $d''(gx, gg_nx) \leqslant n$ for $g \in G$, $n \in S$ and $x \in F$.

There is an alternative description of d_B in terms of paths. Let us define the length of a bridge $\{y, z\}$ as the smallest number $n \in S$ such that $\{y, z\} = \{gx, gg_nx\}$ for some $g \in G$ and $x \in F$. Thus, the length of a bridge is always an integer greater or equal than 1. Let us call a sequence of points $x = x_0, x_1, \dots, x_n = y$ a *bridge path* of length n from x to y if any two consecutive points either lie on a common island or form a bridge, that is, for every $i = 1, \dots, n$ either there is an island Y such that $\{x_{i-1}, x_i\} \subset Y$ or $\{x_{i-1}, x_i\}$ is a bridge. Define the d-length of such a bridge path as $\sum_{i=1}^{n} d_i$ where $d_i = d(x_{i-1}, x_i)$ if $\{x_{i-1}, x_i\}$ is on one island or, if $\{x_{i-1}, x_i\}$ is a bridge, then let d_i be the length of this bridge.

9.2. The length $d_B(x, y)$ is the infimum of d-lengths of all bridge paths from x to y.

Proof. The pseudometric d'' defined by the statement of 9.2 has the properties 9.1(a) and (b). Conversely, if d'' is a pseudometric with the properties 9.1(a) and (b), then $d''(x, y)$ is at most equal to the d-length of any bridge path from x to y; cf. the similar proof in Subsection 8.2. □

PROPOSITION 9.3 Properties of d_B.
(a) *The pseudometric d_B is G-invariant.*
(b) *The pseudometric d_B is finite-valued if $d|Y \times Y$ is finite-valued for one (equivalently every) island Y.*
(c) *If x is a point of the island Y, then the balls $B_d(x,r) \cap Y$ and $B_{d_B}(x,r)$ coincide for $r < 1$.*
(d) *If d is continuous, then so is d_B.*
(e) *Suppose that d is continuous, proper and, for every island Y, has finite values on $Y \times Y$. Then d_B is continuous, proper and finite-valued (everywhere).*

Proof. Property(a) follows from our construction.

Property(b) follows from the fact that d_B is G-invariant and every island can be reached from F by a bridge.

Property(c) follows from Subsection 9.2 and the fact that every bridge has length at least 1.

By property(d), a pseudometric is continuous if it is continuous near the diagonal, by the triangle inequality. So (d) follows from (c).

Property(e) is the main point of these properties. It remains to be shown that d_B is proper if d is proper, continuous and on every island finite-valued. Thus, let $x \in X$ and $0 < R < \infty$. We have to show that $B_{d_B}(x,R)$ has compact closure. For a point $y \in X$ we have $d_B(x,y) < R$ if there is a bridge path $x = x_0, \ldots, x_n = y$ with d-length $\Sigma d_i < R$. We may assume that the three consecutive points x_{i-1}, x_i and x_{i+1} of our bridge path are not on a common island, since otherwise we could leave out x_i without increasing the d-length of our path, by the triangle inequality for d. So our path has at least $(n-1)/2$ bridges, all of length at least 1. We thus have an upper bound for the length n of our bridge path, namely, $n \leqslant 2R + 1$. Furthermore, every bridge in our path has length at most R and every step $d_i = d(x_{i-1}, x_i)$ on one island has length at most R. It thus suffices to prove the following two claims.

(i) If K is a compact subset of X, then $B_d(K,R) = \{y \in X\,;\, d(x,y) < R\}$ has compact closure.

(ii) If K is a compact subset of X, then the set $B(K,R) := \{z \in X;$ there is a bridge $\{y,z\}$ from a point $y \in K$ to z of length at most $R\}$ has compact closure.

Proof of (i). The set K is contained in a finite union of islands, since K is compact and the islands are open and disjoint and form a cover of X. It thus suffices to prove our claim for the case where K is contained in one island, say Y. Let x be a point of K. Then the function $y \mapsto d(x,y)$ is continuous and finite-valued on Y, hence has a finite maximum on K, so $K \subset B_d(x,r)$ for some $0 < r < \infty$. Then $B_d(K,R) \subset B_d(x, r+R)$, which has compact closure by hypothesis. This shows our claim.

Proof of (ii). The bridges $\{y,z\}$ starting from a point of K and having length at most R are of the form $\{gx, gg_nx\}$ with $x \in F$ and $n \leqslant R$, and either $gx \in K$ or $gg_nx \in K$. Hence, $g \in G_{FK}$ or $g \in G_{g_nF,K} = G_{FK} \cdot g_n^{-1}$ and thus the endpoint z of our bridge is of the form $z = gg_nx \in G_{FK}g_nK$ in the first case or of the form $z = gx \in G_{FK}g_n^{-1}K$ in the second case; thus, every endpoint z of such a bridge is contained in the relatively compact set $\bigcup_{n \leqslant R} G_{FK}g_n^{\pm 1}K$, as required. □

9.3. We are now ready to finish the proof of our main Theorems 1.1 and 1.2. Let X be a σ-compact Hausdorff space and suppose that the locally compact topological group G acts properly on X. We have shown that then there is a family of continuous G-invariant pseudometrics on X inducing the topology of X (see Step 2 in Section 6), which we may furthermore assume to be finite-valued and orbitwise proper, by Step 3 in Section 7. Then the stick construction of Section 8 gave us a pseudometric, which is continuous, proper and on every island finite-valued, namely the pseudometric d' of Lemma 8.7. Continuity of d' follows from property (b) of Subsection 8.2 and finiteness on islands from Lemma 8.4. If we use this pseudometric in the bridge construction of Section 9, then the resulting pseudometric d_B is continuous, finite-valued and proper. If now $\mathcal{D}$ is a family of G-invariant pseudometrics inducing the topology of X (we know that such a family exists, by Step 2 in Section 6) then the family $\{d + d_B;\, d \in \mathcal{D}\}$ has all the properties we want in Theorem 1.2 (Theorem 4.1). Furthermore, if X is metrizable, then there is a compatible G-invariant metric d on X, by Step 2 in Section 6. Again, there is a pseudometric d_B which is continuous, proper, finite-valued and G-invariant. Then the metric $d + d_B$ has all these properties, too, and is furthermore a compatible metric. This proves Theorem 1.1 (Theorem 4.2).

Let us point out the following corollary by Struble [**14**].

Corollary 9.4. *Every second countable locally compact group has a left-invariant compatible proper metric.*

Proof. The underlying space of such a group G is metrizable and σ-compact, by Corollary 5.4. The action of G on itself by left translations is obviously proper, so there is a compatible left-invariant proper metric on G, by Theorem 1.1. □

As a special case we obtain the following old result of Busemann [**3**].

Corollary 9.5. *The group of isometries of a proper metric space admits a compatible left-invariant proper metric.*

Proof. The group G of isometries of a proper metric space is locally compact and Hausdorff (see Theorem 3.2) and second countable (see [**2**, Chapter X, § 3.3 Corollary]), which implies our claim by the previous corollary. □

10. *Concluding remarks*

In this section, we discuss applications and related work, mention open questions and make other remarks.

10.1. In the non-equivariant context, that is, if we consider just the topological space X without any group action, it is well known that a σ-compact locally compact metrizable space has a compatible proper metric; see Corollary 5.4. More precisely, in [**15**] it is proved that if d is a complete metric on such a space X, then there is a proper metric on X which is locally identical with d, that is, for every point $x \in X$ there is a neighbourhood of x where the two metrics coincide. Note that in our construction the metric is not changed locally in Steps 4 and 5 of Section 4. Thus, in the situation of Theorem 1.1, if d is a compatible G-invariant metric on X which is orbitwise proper, then there is a G-invariant compatible proper metric on X which is locally identical with d. One may thus ask the following question. Suppose, in the situation of Theorem 1.1, we are given a G-invariant complete compatible metric on X. Is there a G-invariant proper (compatible) metric on X which is locally identical with d?

10.2. Given an isometric action of a group G on a σ-compact locally compact metric space X with metric d, it is not true in general that there is a compatible proper metric d_p for which the action of G is isometric. For an example, let $X = \{(x, y) \in \mathbb{R}^2; x = 0 \text{ or } x = 1\}$ endowed with the metric $d = \min\{d_E, 1\}$, where d_E is the Euclidean metric of $\mathbb{R}^2$ restricted to X. Let G be the group of isometries of (X, d). There is no compatible proper metric d_p on X for which G acts isometrically, for the following reason. The group H of isometries of (X, d_p), endowed with the compact open topology, acts properly, hence the isotropy group $H_{(0,0)}$ of the point $(0, 0)$ is compact and hence has compact orbits. On the other hand, let $G_{(0,0)}$ be the isotropy group of the point $(0, 0)$ in G. The orbit $G_{(0,0)}(1, 0)$ of $(1, 0)$ is $\{1\} \times \mathbb{R}$ and is not relatively compact in X. So G is not contained in H. The point of the example is that the action of G is not proper, no matter which topology we put on G.

10.3. Let us consider the following question. Under which conditions is it true that, given a compatible metric d on a locally compact σ-compact space X, there is a compatible proper metric d_p with the same group of isometries? A sufficient condition was given by Janos [**7**], namely, if (X, d) is a connected uniformly locally compact metric space.

10.4. Note that if we have a closed subgroup G of the group of isometries of a proper metric space (X, d), then it is not true in general that there is a metric d_1 on X for which G is the precise group of isometries. For example, the space $X = \mathbb{R}$ of real numbers with the Euclidean metric has the group $G = \mathbb{R}$ as a closed subgroup of its group of isometries. But, for every G-invariant metric d_1 on X, we have $d_1(x, 0) = d_1(0, -x)$, hence the group of isometries of d_1 contains the reflections of $\mathbb{R}$ and is thus strictly larger than $\mathbb{R}$.

10.5. Given a proper action of a locally compact topological group G on a locally compact metrizable space X, one can ask if there is a G-invariant metric. This is known to be equivalent to $G\backslash X$ being paracompact; see [**1**, **11**] and the forthcoming book 'Proper transformation groups' by the first author and Strantzalos. The answer is positive in many cases; see [**1**] and the forthcoming book 'Proper transformation groups' by the first author and Strantzalos. If X is no longer locally compact, the answer is known to be negative if the action is Bourbaki-proper but again unknown in general for Palais-proper actions [see the forthcoming book 'Proper transformation groups' by the first author and Strantzalos].

10.6. Our Theorem 1.1 has potential applications for the Novikov conjecture; namely, let G be a locally compact second countable group and let μ be a Haar measure on G. Then, using a proper left-invariant compatible metric on G, Haagerup and Przybyszewska have proved in [**6**] that there is a proper affine isometric action of G on some separable strictly convex reflexive Banach space. Kasparov and Yu have recently proved that the Novikov conjecture holds for every discrete countable group that has a uniform embedding into a uniformly convex Banach space; see [**10**].

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