LOCALIZATION–COMPLETION STRIKES AGAIN: RELATIVE K_1 IS NILPOTENT BY ABELIAN

A. BAK, R. HAZRAT, AND N. VAVILOV

Abstract. Let G and E stand for one of the following pairs of groups:

• Either G is the general quadratic group $U(2n, R, \Lambda)$, $n \geq 3$, and E its elementary subgroup $\mathrm{EU}(2n, R, \Lambda)$, for an almost commutative form ring (R, Λ),

• or G is the Chevalley group $G(\Phi, R)$ of type Φ, and E its elementary subgroup $E(\Phi, R)$, where Φ is a reduced irreducible root system of rank ≥ 2 and R is commutative.

Using Bak's localization-completion method in [7], it was shown in [18] and [19] that G/E is nilpotent by abelian, when R has finite Bass–Serre dimension. In this note, we combine localization-completion with a version of Stein's relativization [34], which is applicable to our situation [11], and carry over the results in [18] and [19] to the relative case. In other words, we prove that not only absolute K_1 functors, but also the relative K_1 functors, are nilpotent by abelian.

1. Introduction

In [7], the first author developed a powerful localization-completion method which allowed him to prove that $\mathrm{SK}_1(n, R, I)$ is nilpotent, and, more generally, $K_1(n, R, I)$ is nilpotent-by-abelian, whenever the Bass–Serre dimension $\delta(R)$ of the ground ring R is finite. In [19], [20] one can find a slightly less technical description of localization-completion and its detailed comparison with other localization methods.

In [18] and [19] the second and the third authors addressed extensions of these results to unitary groups over form rings and to Chevalley groups over commutative rings. However, in [18] and [19] we succeeded only in establishing analogues of the results of [7] in the *absolute* case.

In the present paper we make the final step and prove *relative* versions of the above results. More precisely, the main results of the present work may be summarized as constructions of descending G-central series in

- congruence subgroups of unitary groups,
- congruence subgroups of Chevalley groups.

The terms of these central series are indexed by the Bass–Serre dimension on the codomains of the ground ring. In the case of finite-dimensional rings this leads to the following

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theorems, which are immediate corollaries of the more powerful Theorems 3 and 4 which we prove in Sections 4 and 5, respectively.

Theorem 1. *Let (R, Λ) be a form ring which is module finite over a commutative ring A of finite Bass–Serre dimension $\delta(A)$, and let (I, Γ) be a form ideal of (R, Λ). Then for any $n \geq 3$ the quotient $U(2n, I, \Gamma)/E(2n, I, \Gamma)$ is nilpotent-by-abelian of nilpotent class at most $\delta(R) + 1$.*

Theorem 2. *Let Φ be a reduced irreducible root system of rank ≥ 2. Let R be a commutative ring of finite Bass–Serre dimension $\delta(R)$, and let $I \trianglelefteq R$ be an ideal of R. Then for any Chevalley group $G(\Phi, R)$ of type Φ over R the quotient $G(\Phi, R, I)/E(\Phi, R, I)$ is nilpotent-by-abelian of nilpotent class at most $\delta(R) + 1$.*

The interrelation between the absolute and the relative case of a problem varies according to the kind of problem one has.

• In some problems, such as normality of the elementary subgroup, the relative case immediately follows from the absolute one via the procedure of relativization.

• In some other problems, such as the classification of subgroups normalized by a relative elementary subgroup, the relative case is noticeably harder than the absolute one, and does not directly follow (see §6).

In this scale of events our paper is somewhere in the middle.

On the one hand, it is classically known, that relative K_1-functors may be non-trivial even when the absolute ones are. The first such examples occur already for totally imaginary Hasse domains, as discovered by Bass–Milnor–Serre [13] and Matsumoto [27]. Thus, our Theorems 1 and 2 do not immediately follow from the *results* of [18] and [19], pertaining to the absolute case.

On the other hand, looking inside the proofs, it is easy to discover, that the nilpotent filtration in the absolute case can be successfully used to beget a corresponding nilpotent filtrations in the relative case. Thus, our proof is a blend of localization-completion with a version of Stein's relativization.

In the case of commutative rings, needed to establish our result for Chevalley groups, these tools are well-known. They are less familiar for the case of almost commutative *form rings*. We need here to prove the corresponding results for unitary groups.

The rest of the paper is organized as follows. In Sections 2 and 3 we recall some necessary machinery from [18] and [11]. After that our main results are established in Sections 4 and 5. Finally, in Section 6 we state and briefly discuss some closely related problems.

Our general background references for unitary groups are [6], [15], [11], [18], [30], where one can find many further references. Unfortunately, there are no books on Chevalley groups over rings. The basic definitions we need, and many additional references can be found in [1], [27], [33], [35], [46], [48], and we do not try to reproduce them here.

2. Localizations and completions of form rings

Let X be a topological space. The *dimension* of X is the length n of the longest chain $X_0 \subsetneqq X_1 \subsetneqq \cdots \subsetneqq X_n$ of nonempty closed *irreducible* subsets X_i of X, see [12], §III. Define $\delta(X)$ to be the smallest nonnegative integer d such that X is a finite union of irreducible *Noetherian* subspaces of dimension $\leq d$. If there is no such d, then by definition $\delta(X) = \infty$.

Now, let R be a commutative ring. Let $\mathrm{Spec}(R)$ denote the spectrum of R, considered as a topological space with respect to the Zariski topology, and let $\mathrm{Max}(R)$ denote maximal spectrum of R, i.e., the subspace of $\mathrm{Spec}(R)$ of all maximal ideals. The *Bass–Serre dimension* $\delta(R)$ of the ring R is defined as $\delta(\mathrm{Max}(R))$. It is a finer version of the usual Jacobson dimension $\dim(\mathrm{Max}(R))$. This dimension will be used in Section 5. Below, we define Bass–Serre dimension for form rings.

Let R_A denote a pair consisting of an associative ring R with identity and a commutative ring $A \leq \mathrm{center}(R)$. Thus R is an algebra over A. A *morphism* $R_A \to R'_{A'}$ of algebras is a ring homomorphism $f : R \to R'$ such that $f(A) \leq A'$. Recall that if (R, Λ) and (R', Λ') are form rings relative to symmetries λ and λ', respectively, then a morphism of form rings $\eta : (R, \Lambda) \to (R', \Lambda')$ is a homomorphism of rings such that

$$\eta(\Lambda) \subseteq \Lambda', \qquad \eta(\lambda) = \lambda', \qquad \eta(\bar{a}) = \overline{\eta(a)},$$

for all $a \in A$.

A *form algebra over a commutative ring* A is a form ring (R_A, Λ) where the involution leaves A invariant. A *morphism* $(R_A, \Lambda) \to (R'_{A'}, \Lambda')$ of form algebras is a morphism of form rings which defines an algebra morphism $R_A \to R'_{A'}$. A form algebra (R_A, Λ) is called *module finite*, if R is module finite over A. If (R_A, Λ) is a form algebra, let A_0 denote the subring of A generated by all $a\bar{a}$ such that $a \in A$. Define the *Bass–Serre dimension of* (R_A, Λ) by

$$\delta(R_A, \Lambda) = \begin{cases} \delta(A_0), & \text{if } (R_A, \Lambda) \text{ is module finite,} \\ \infty, & \text{otherwise.} \end{cases}$$

Let S be an involution invariant subset of R. Then

$$RSR = \left\{ \sum a_i s_i a'_i \mid a_i, a'_i \in R, s_i \in S \right\}$$

is the involution invariant ideal in R generated by S and is denoted by $\langle S \rangle$.

We need to consider morphisms to localizations and completions form rings. We define them here. Let M be an R-module, over a commutative ring R. For $s \in R$, we denote by $\langle s \rangle$ the multiplicative set $\{1, s, s^2, \cdots\}$ generated by s. Further, let M_s denote the module of $\langle s \rangle$-fractions of M and let

$$\widehat{M}_{(s)} = \varprojlim_{i \geq 0} M/Ms^i$$

denote the *completion* of M at s.

In our setting, we will need to use *finite* completions, as in [7], rather than ordinary completions. This is because we want the localizations-completions squares defined later to be pullback squares. Finite completions are defined as follows. For an R-module M, we define its *finite completion* at s as

$$\widetilde{M}_{(s)} = \varinjlim_J (\widehat{M_j})_{(s)},$$

where

- $\{R_j \mid j \in J\}$ is any directed system of subrings $R_j \leq R$ such that each R_j is finitely generated as a $\mathbb{Z}$-algebra, contains s, and $\varinjlim_J R_j = R$,
- $\{M_j \mid j \in J\}$ is any directed system of abelian subgroups $M_j \leq M$ such that each M_j is a finitely generated R_j-module and $\varinjlim_J M_j = M$.

Clearly, $\widetilde{M}_{(s)} = \widehat{M}_{(s)}$ if M is Noetherian and R is finitely generated as a $\mathbb{Z}$-algebra (see [8]).

Let (R, Λ) be a form algebra and let $s \in A_0$. Define the *finite completion of* (R, Λ) *at* s by

$$\widetilde{(R,\Lambda)}_{(s)} = \widetilde{(R,\Lambda)}_{\widetilde{A}_{0,(s)}} = \Big((\widetilde{R}_{(s)})_{\widetilde{A}_{0,(s)}}, (\widetilde{\Lambda}_{(s)})_{\widetilde{A}_{0,(s)}}\Big).$$

It is useful to recall, that the ordinary *completion of* (R, Λ) *at* s is defined as follows:

$$\widehat{(R,\Lambda)}_{(s)} = (\widehat{R}_{(s)}, \widehat{\Lambda}_{(s)}).$$

3. Relativization with two parameters

The usual version of relativization with one parameter was proposed by Michael Stein [34], and has been widely used since then, notably by John Milnor [28], Andrei Suslin and Vyacheslav Kopeiko [39], Leonid Vaserstein [43], the first author [7], Alexei Stepanov and the third author [36]. This form of relativization suffices for the application to Chevalley groups we have in mind. However, relative subgroups in unitary groups are associated with form ideals, and not just ideals, so a slightly fancier relativization with *two parameters* is needed here. It was developed in [11], [18], [30] and below we briefly recall the basic idea.

To treat the relative groups corresponding to the form ideals we have to recall some notation related to Stein's relativization [34]. First, let I be any ideal of a ring R. The reason, why relative notions do not immediately follow from the absolute ones, is that usually the canonical projection $R \longrightarrow R/I$ is not split, or, in other words, does not have a section. There are two common ways to embed I as an ideal in another ring, for which the canonical projection $S \longrightarrow S/I$ has a section.

- We can define the *double* $R \times_I R$ of a ring R with respect to an ideal $I \leq R$ by the Cartesian square

$$\begin{tikzcd} R \times_I R \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & R \arrow[d, "\pi"] \\ R \arrow[r, "\pi"'] & R/I \end{tikzcd}$$

where $\pi_1(a,b) = a$ and $\pi_2(a,b) = b$. In other words $R \times_I R$ consists of all the pairs $(a,b) \in R \times R$ such that $a \equiv b \pmod{I}$. Clearly $\mathrm{Ker}\ \pi_1 = (0,I)$ and $\mathrm{Ker}\ \pi_2 = (I,0)$. The diagonal embedding $\delta : R \to R \times_I R$ given by $\delta(a) = (a,a)$ splits both π_1 and π_2.

• On the other hand, for a pair (R,I) one can define *semidirect product* $R \ltimes I$ of R and I as the set of pairs (a,c), such that $a \in R$, $c \in I$, with addition defined component-wise and multiplication given by the following formula: $(a,c)(b,d) = (ab, ad + cb + cd)$.

The following is well-known (and obvious).

Lemma 1. *The ring $R \times_I R$ is isomorphic to the semidirect product $R \ltimes I$ of $\delta(R) \cong R$ and $\mathrm{Ker}\,\pi_1 \cong I$.*

In the sequel we identify I with $\mathrm{Ker}\,\pi_1$. Of course if we write (as we do in the sequel) the Cartesian square above in terms of semidirect products, rather than doubles, we must define π_1, π_2 by $\pi_1(a,c) = a$, $\pi_2(a,c) = a + c$. Usually the relative questions for the ideal I in R can be reduced to absolute questions for the ring $R \ltimes I$.

The main result of [34] asserts that $G(\Phi,R,I)$ and $E(\Phi,R,I)$ can be identified with $G(\Phi, R \ltimes I, 0 \ltimes I)$ and $E(\Phi, R \ltimes I, 0 \ltimes I)$, respectively, and that one has the following equality, which reduces problems about relative groups to ones about absolute groups: For any root system Φ and any pair $I \trianglelefteq R$ one has

$$E(\Phi, R \ltimes I) \cap G(\Phi, R \ltimes I, 0 \ltimes I) = E(\Phi, R \ltimes I, 0 \ltimes I).$$

We need to add some equivariant detail to this statement. The ring homomorphism $\pi_2 : R \ltimes I \to R, (r,x) \mapsto r + x$, induces an isomorphism

$$\pi_2 : G(\Phi, R \ltimes I, 0 \ltimes I) \xrightarrow{\cong} G(\Phi, R, I)$$

of groups. Let π_2^{-1} denote the inverse isomorphism. The ring homomorphism $\pi_1 : R \ltimes I \to R, (r,x) \mapsto r$, induces an isomorphism $\pi_1 : G(\Phi, R \ltimes 0) \xrightarrow{\cong} G(\Phi, R)$ of groups. Let π_1^{-1} denote the inverse isomorphism. Let $G(\Phi,R) \curvearrowright G(\Phi,R,I)$ denote the action of $G(\Phi,R)$ on $G(\Phi,R,I)$ by conjugation and let $G(\Phi, R \ltimes 0) \curvearrowright G(\Phi, R \ltimes I, 0 \ltimes I)$ denote the action of $G(\Phi, R \ltimes 0)$ on $G(\Phi, R \ltimes I, 0 \ltimes I)$ by conjugation.

Lemma 2. *The pair (π_1^{-1}, π_2^{-1}) defines an isomorphism*

$$G(\Phi,R) \curvearrowright G(\Phi,R,I) \xrightarrow{\cong} G(\Phi, R \ltimes 0) \curvearrowright G(\Phi, R \ltimes I, 0 \ltimes I)$$

of group actions, taking the subaction $G(\Phi,R) \curvearrowright E(\Phi,R)$ onto the subaction

$$G(\Phi, R \ltimes 0) \curvearrowright E(\Phi, R \ltimes I, 0 \ltimes I).$$

Furthermore,

$$E(\Phi, R \ltimes I) \cap G(\Phi, R \ltimes I, 0 \ltimes I) = E(\Phi, R \ltimes I, 0 \ltimes I).$$

In [11], [30] it was shown that the same results hold with $G(\Phi,R)$ replaced by $U(2n,R,\Lambda)$, and now we briefly recall this construction.

Let (R, Λ) be a form ring. Let I be a (two-sided) ideal in R invariant with respect to the involution, i.e. such that $\overline{I} = I$. Set $\Gamma_{\max} = I \cap \Lambda$ and

$$\Gamma_{\min} = \left\{\xi - \lambda\overline{\xi} \mid \xi \in I\right\} + \left\{\xi\alpha\overline{\xi} \mid \xi \in I,\ \alpha \in \Lambda\right\}.$$

By definition $\Gamma_{\min}$ and $\Gamma_{\max}$ depend both on the *absolute* form parameter Λ *and* an ideal I in R. The form parameter Λ is fixed and will not be accounted in the notation. Sometimes it is necessary to stress the dependence of $\Gamma_{\min}$ and $\Gamma_{\max}$ on I. In such cases we write $\Gamma_{\min}(I)$ and $\Gamma_{\max}(I)$.

A *relative form parameter* Γ in (R, Λ) *of level* I is an additive subgroup of I such that

- $\Gamma_{\min} \subseteq \Gamma \subseteq \Gamma_{\max}$,
- $\alpha\Gamma\overline{\alpha} \subseteq \Gamma$ for all $\alpha \in R$.

Now, let (I, Γ) be a form ideal in a form ring (R, Λ). Then we can define the double of Λ along Γ in exactly the same way, as above,

$$\Lambda \times_\Gamma \Lambda = \{(a, c) \in \Lambda \times \Lambda \mid a - c \in \Gamma\}.$$

It is easy to see that $\Lambda \times_\Gamma \Lambda$ is a form parameter in $R \times_I R$ and that $(R \times_I R, \Lambda \times_\Gamma \Lambda)$ is identified with $(R \ltimes I, \Lambda \ltimes \Gamma)$ under the map $(r, s) \mapsto (r, s - r)$. In fact, the following result is Lemma 5.2.15 of [15].

Lemma 3. $(R \ltimes I, \Lambda \ltimes \Gamma)$ *is a form ring with respect to the component-wise involution and* $\lambda = (\lambda, 0)$.

Another form ring which can be associated with this form ideal is the factor ring $(R/I, \Lambda/\Gamma_{\max})$ (see [15], Lemma 5.2.12). Then we have a *commutative*, but not in general pullback, square of form rings:

$$\begin{tikzcd}
(R \ltimes I, \Lambda \ltimes \Gamma) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & (R, \Lambda) \arrow[d, "\pi"] \\
(R, \Lambda) \arrow[r, "\pi"'] & (R/I, \Lambda/\Gamma_{\max})
\end{tikzcd}$$

which is analogous to the Cartesian square above. This commutative square is actually *Cartesian* when $\Gamma = \Gamma_{\max}$. Since the functor $U(2n, R, \Lambda)$ from form rings to groups commutes with limits the commutative/Cartesian squares of form rings above lead to commutative/Cartesian squares of groups $U(2n, R, \Lambda)$.

Let $\pi : R \ltimes I \to R$ be defined by $(a, c) \mapsto a + c$. This map induces a homomorphism

$$\pi : U(2n, R \ltimes I, \Lambda \ltimes \Gamma) \to U(2n, R, \Lambda),$$

that is both surjective and split, and the most convenient way to define the congruence subgroup $U(2n, I, \Gamma)$ is to identify it with the kernel $U(2n, 0 \ltimes I, 0 \ltimes \Gamma)$ of π.

Similarly, restricting this to the the elementary subgroups, we get

$$1 \to E(2n, I, \Gamma) \to E(2n, R \ltimes I, \Lambda \ltimes \Gamma) \to E(2n, R, \Lambda) \to 1,$$

where, as above, we identify $E(2n, I, \Gamma)$ and $E(2n, 0 \ltimes I, 0 \ltimes \Gamma)$.

The following result is Lemma 5.4 in [10].

Lemma 4. *Let (I,Γ) be a form ideal of a form ring (R,Λ). Then*

$$E(2n, A \ltimes I, \Lambda \ltimes \Gamma) \cap U(2n, I, \Gamma) = E(2n, I, \Gamma).$$

Our proof heavily relies on the following result, established in [11], which will be applied whenever necessary, without any specific reference.

Lemma 5. *Let $n \geq 3$ and (I,Γ) be a form ideal of an almost commutative form ring (R,Λ). Then the elementary subgroup $E(2n,I,\Gamma)$ is normal in $U(2n,R,\Lambda)$.*

In the process of the proof of Lemma 6 below the second author in [18] gave another proof of this result, and in fact much stronger results.

4. Nilpotent filtration of relative unitary groups

Recall from [18] the following piece of notation.

Definition 1. Let (R,Λ) be a module finite form ring. Let $s \in A_0$. Define

$$G(s^{-1},R) = \ker\big(U(2n,R,\Lambda) \longrightarrow U(2n,R_s,\Lambda_s)/E(2n,R_s,\Lambda_s)\big),$$
$$G(\widehat{s},R) = \ker\big(U(2n,R,\Lambda) \longrightarrow U(2n,\widetilde{(R,\Lambda)}_{(s)})/E(2n,\widetilde{(R,\Lambda)}_{(s)})\big).$$

Our Theorem 3 below heavily depends on the following result, which may be considered one of the main results of [18], Theorem 4.6.

Lemma 6. *Let $n \geq 3$ and $s \in A_0$. Then*

$$[G(s^{-1},R), G(\hat{s},R)] \subseteq E(2n,R,\Lambda).$$

Next, we define the terms of our nilpotent filtration.

Definition 2. Let (R,Λ) be a module finite form ring, and (I,Γ) a form ideal. If $\mu : R \to R'$ is a homomorphism of rings with involution, let Λ' denote the form parameter of R' generated by $\mu(\Lambda)$, I' the involution invariant ideal of R' generated by $\mu(I)$ and Γ' the relative form parameter in the form ring (R',Λ') of level I' generated by $\mu(\Gamma)$. Define

$$S^dU(2n,I,\Gamma) = \bigcap_{\substack{R\to R'\\ \delta(R')\leq d}} \operatorname{Ker}\Big(U(2n,I,\Gamma) \longrightarrow U(2n,I',\Gamma')/E(2n,I',\Gamma')\Big).$$

Replacing (I,Γ) with (R,Λ) and $d=0$ we get

$$S^0U(2n,R,\Lambda) = \bigcap_{\substack{R\to R'\\ \delta(R')=0}} \operatorname{Ker}\Big(U(2n,R,\Lambda) \longrightarrow U(2n,R',\Lambda')/E(2n,R',\Lambda')\Big).$$

We are ready to prove the main Theorem of this section. As we promised in the introduction, the proof uses Bak's localization-completion and Stein's relativization.

Theorem 3. *Let (R,Λ) be a module finite form ring, (I,Γ) a form ideal and $n \geq 3$. Then the sequence*

$$S^0U(2n,I,\Gamma) \geq S^1U(2n,I,\Gamma) \geq S^2U(2n,I,\Gamma) \geq \cdots$$

is a descending $S^0U(2n,R,\Lambda)$-central series and $S^dU(2n,I,\Gamma) = E(2n,I,\Gamma)$, whenever $d \geq \delta(A_0) = \delta(R)$. Moreover, each $S^dU(2n,I,\Gamma),(d \geq 0)$ is normal in $U(2n,R,\Lambda)$ and the action via conjugation of $U(2n,R,\Lambda)$ on $U(2n,I,\Gamma)/S^0U(2n,I,\Gamma)$ is trivial. In particular, $K_1(2n,I,\Gamma) := U(2n,I,\Gamma)/EU(2n,I,\Gamma)$ is nilpotent by abelian.

Proof. Clearly $E(2n,I,\Gamma) \subseteq S^dU(2n,I,\Gamma)$ for any d. If $\delta(R) \leq d$, then the identity map $R \to R$ is included in the definition of $S^dU(2n,I,\Gamma)$, and thus this group coincides with $E(2n,I,\Gamma)$.

We proceed by induction on $\delta(R)$. If $\delta(R) = 0$ then $S^0U(2n,R,\Lambda) = E(2n,R,\Lambda)$ and $S^0U(2n,I,\Gamma) = E(2n,I,\Gamma)$. Since $E(2n,I,\Gamma)$ is a normal subgroup of $E(2n,R,\Lambda)$ the theorem holds for zero-dimensional rings.

Since

$$U(2n,I,\Gamma)/S^{d+1}U(2n,I,\Gamma) \to \prod_{\substack{R \to R' \\ \delta(R') \leq d+1}} U(2n,I',\Gamma')/E(2n,I',\Gamma')$$

is a monomorphism, it is enough to prove the theorem for rings of dimension $d+1$.

Thus, we have to show that if $\sigma \in S^0U(2n,R,\Lambda)$ and $\rho \in S^dU(2n,I,\Gamma)$, then $[\sigma,\rho] \in E(2n,I,\Gamma)$. For this we use the localization-completion method introduced in [7].

Let $X_1 \cup \ldots \cup X_r$ be a decomposition of $\mathrm{Max}(A_0)$ into irreducible Noetherian subspaces of dimension $\leq \delta(R)$. For any $1 \leq i \leq r$, let $M_i \in X_i$. Take the multiplicative set $S = A_0 \setminus (M_1 \cup \cdots \cup M_r)$. Since $S^{-1}A_0$ is a semi-local ring, $\delta(\varinjlim A_{0(s)}) = \delta(S^{-1}A_0) = 0$, where the limit is taken over all $s \in S$. Thus $\delta(S^{-1}R, S^{-1}\Lambda) = \delta(S^{-1}A_0) = 0$. Therefore, one can find an $s \in A_0$ such that $\sigma \in G(s^{-1},R)$.

On the other hand, by Lemma 4.17 in [7] for any $s \in S$, we have

$$\delta(\widetilde{R_A,\Lambda})_{(s)} = \delta(\widetilde{A}_{0(s)}) < \delta(A_0) = \delta(R)$$

and the value of $\rho \in U(2n,I,\Gamma)$ in $U(2n,(\widetilde{R_A,\Lambda})_{(s)})$ lies in $E(2n,(\widetilde{I,\Gamma})_{(s)})$.

Since during the course of the entire proof so far, only a finite number of elements from I play a role, one can replace, if necessary, I by a smaller ideal which is finitely generated over R and hence finitely generated over A. Doing this, $R \ltimes I$ becomes module finite over A. We shall now use relativization technique to reduce to the absolute case.

By analog of Lemma 2 for $U(2n,-)$, we have an isomorphism

$$(\pi_1^{-1},\pi_2^{-1}) : U(2n,R,\Lambda) \curvearrowright U(2n,I,\Gamma) \longrightarrow U(2n,R\ltimes 0,\Lambda\ltimes 0) \curvearrowright U(2n,0\ltimes I,0\ltimes\Gamma)$$

of group actions. Let $\sigma' = \pi_1^{-1}(\sigma)$ and $\rho' = \pi_2^{-1}(\rho)$. Clearly, $\pi_2^{-1}[\sigma,\rho] = [\sigma',\rho']$. By Lemma 6, $[\sigma',\rho'] \in E(2n,R\ltimes I,\Lambda\ltimes\Gamma)$. Thus

$$[\sigma',\rho'] \in E(2n,R\ltimes I,\Lambda\ltimes\Gamma) \cap U(2n,0\ltimes I,0\ltimes\Gamma).$$

But $E(2n, R \ltimes I, \Lambda \ltimes \Gamma) \cap U(2n, 0 \ltimes I, 0 \ltimes \Gamma) = E(2n, 0 \ltimes I, 0 \ltimes \Gamma)$ by Lemma 4. Thus $[\sigma, \rho] \in E(2n, I, \Gamma)$, again, by Lemma 4.

Since each relative elementary group $E(2n, I', \Gamma')$ is normal in $U(2n, R', \Lambda')$, it follows that $S^d U(2n, I, \Gamma)$ is an intersection of normal subgroups of $U(2n, R, \Lambda)$ and hence normal. Since

$$U(2n, I, \Gamma)/S^0 U(2n, I, \Gamma) \longrightarrow \prod_{\substack{R \to R' \\ \delta(R')=0}} U(2n, I', \Gamma')/E(2n, I', \Gamma')$$

is a monomorphism and the action via conjugation of $U(2n, R', \Lambda')$ on

$$U(2n, I', \Gamma')/E(2n, I', \Gamma')$$

is trivial (see [4]), it follows that the action via conjugation of $U(2n, R, \Lambda)$ on

$$U(2n, I, \Gamma)/S^0 U(2n, I, \Gamma)$$

is trivial.

The remaining assertions in the theorem are clear. □

5. Nilpotent filtration of relative Chevalley groups

Let Φ be a reduced irreducible root system and R a commutative ring. We consider the corresponding simply connected Chevalley group $G = G(\Phi, R)$ and its elementary subgroup $E(\Phi, R)$.

When $\mathrm{rk}(\Phi) \geq 2$ it was proved by Suslin and Kopeiko [38], [39], [21] for the classical cases and by Taddei [41] for the exceptional cases, that $E(\Phi, R)$ is normal in $G(\Phi, R)$.

Let I be an ideal of R. The *principal congruence subgroup of level I* is defined as the kernel of the reduction homomorphism $G(\Phi, R) \to G(\Phi, R/I)$ and is denoted by $G(\Phi, R, I)$.

The normal subgroup of the elementary group $E(\Phi, R)$ generated by all the elementary root unipotent elements of level I, i.e., elements conjugate to $x_\alpha(\xi)$ for some $\alpha \in \Phi$ and some $\xi \in I$ is denoted by $E(\Phi, R, I)$.

If $\theta : R \to R'$ is a ring homomorphism and I an ideal of R, let I' denote the ideal of R' generated by $\theta(I)$.

Similarly to Section 4, we need to consider the canonical morphisms to the localization ring R_s and the finite completion ring $\widetilde{R}_s = \varinjlim (\widehat{R}_i)_s$, where the limit is taken over all finitely generated subrings R_i of R, as follows,

$$\begin{array}{ccc} G(\Phi, R) & \xrightarrow{F_s} & G(\Phi, R_s) \\ \Big\downarrow{\scriptstyle \widetilde{F}_s} & & \\ G(\Phi, \widetilde{R}_s) & & \end{array}$$

Definition 3. Let R be a commutative ring and $s \in R$. Define,

$$G(s^{-1}, R) = \ker\big(G(\Phi, R) \longrightarrow G(\Phi, R_s)/E(\Phi, R_s)\big),$$
$$G(\hat{s}, R) = \ker\big(G(\Phi, R) \longrightarrow G(\Phi, \widetilde{R}_{(s)})/E(\Phi, \widetilde{R}_{(s)})\big).$$

The following inclusion is one of the main results of [19], Theorem 6.1. It relies on all previous calculations of that paper, and plays a crucial role in the proof of Theorem 4 below.

Lemma 7. *Let* $\mathrm{rk}(\Phi) \geq 2$. *Then for every commutative ring* R *and every* $s \in R$ *one has*

$$[G(s^{-1}, R), G(\hat{s}, R)] \subseteq E(\Phi, R).$$

Next, we define the terms of filtration

Definition 4. Let R be a commutative ring and I an ideal of R. Define

$$S^d G(\Phi, R, I) = \bigcap_{\substack{R \to R' \\ \delta(R') \leq d}} \ker \big(G(\Phi, R, I) \longrightarrow G(\Phi, R', I')/E(\Phi, R', I') \big).$$

where I' is defined as above.

Replacing I with R and $d = 0$ we get

$$S^0 G(\Phi, R) = \bigcap_{\substack{R \to R' \\ \delta(R') = 0}} \ker \big(G(\Phi, R) \longrightarrow G(\Phi, R')/E(\Phi, R') \big).$$

Now we are all set to state the second main result of the present paper.

Theorem 4. *Let* R *be a commutative ring,* Φ *an irreducible root system of rank* ≥ 2 *and* $G(\Phi, R)$ *the Chevalley group of* Φ *with coefficients in* R. *Let* I *be an ideal of* R *and* $G(\Phi, R, I)$ *the congruence subgroup of level* I. *Then the sequence*

$$S^0 G(\Phi, R, I) \geq S^1 G(\Phi, R, I) \geq S^2 G(\Phi, R, I) \geq \cdots$$

is a descending $S^0 G(\Phi, R)$*-central series and* $S^d G(\Phi, R, I) = E(\Phi, R, I)$ *whenever* $d \geq \delta(R)$. *Moreover, each* $S^d G(\Phi, R, I), (d \geq 0)$ *is normal in* $G(\Phi, R)$ *and the action via conjugation of* $G(\Phi, R)$ *on* $G(\Phi, R, I)/S^0 G(\Phi, R, I)$ *is trivial. In particular* $K_1 G(\Phi, R, I) := G(\Phi, R, I)/E(\Phi, R, I)$ *is nilpotent by abelian.*

Proof. The proof is similar to that of Theorem 3. One only needs to replace the functor $U(2n, -, -)$ by $G(\Phi, -)$ and refer to the corresponding results of [41] and [19], instead of [11] and [18].

We proceed by induction on $\delta(R)$. The theorem holds for zero dimensional rings. For rings of dimension $d+1$ it suffices to show, that for any $\sigma \in S^0 G(\Phi, R)$ and $\rho \in S^d G(\Phi, R, I)$, one has $[\sigma, \rho] \in S^{d+1} G(\Phi, R, I)$.

But $S^{d+1} G(\Phi, R, I) = E(\Phi, R, I)$, because the identity map $R \to R$ is among those taken to define $S^{d+1} G$. Using the localization-completion method, in exactly the same way as in the proof of Theorem 3, one can find an element $s \in R$ such that $F_s(x) \in E(\Phi, R_s)$ and $\widetilde{F}_s(y) \in E(\Phi, \widetilde{R}_{(s)}, \widetilde{I}_{(s)})$. One reduces now to the case that I is finitely generated over R, exactly as in the proof of Theorem 3.

Now using Lemma 7 for the ring $R \ltimes I$ we have that $[\pi_1^{-1}\sigma, \pi_2^{-1}\rho] \in E(\Phi, R \ltimes I)$. But by Lemma 2, (π_1^{-1}, π_2^{-1}) is an isomorphism of group actions with the property that $[\pi_1^{-1}\sigma, \pi_2^{-1}\rho] \in E(\Phi, R \ltimes I)$ implies $[\sigma, \rho] \in E(\Phi, R, I)$.

Since each $E(\Phi, R', I')$ is normal in $G(\Phi, R')$, it follows that $S^d G(\Phi, R, I)$ is the intersection of normal subgroups of $G(\Phi, R)$ and hence normal. Since

$$G(\Phi, R, I)/S^0 G(\Phi, R, I) \longrightarrow \prod_{\substack{R \to R' \\ \delta(R')=0}} G(\Phi, R', I')/E(\Phi, R', I')$$

is a monomorphism and the action via conjugation of $G(\Phi, R')$ on

$$G(\Phi, R', I')/E(\Phi, R', I')$$

is trivial by (see [33]), it follows that the action via conjugation of $G(\Phi, R)$ on

$$G(\Phi, R, I)/S^0 G(\Phi, R, I)$$

is trivial.

The remaining assertions in the theorem are clear. □

6. Where next?

In this section we state and briefly discuss some further relativization problems, closely related to our Theorems 1 and 2. In fact, currently there is substantial progress on all of these problems, including the (difficult!) Problems 5 and 6.

Unfortunately, our results in this paper are not as definitive for Chevalley groups, as they are for unitary groups. In fact, relative groups in Chevalley groups are parameterized by admissible pairs (A, B), introduced by Abe and Abe–Suzuki [1]–[3], and Stein [33]. Let A be an ideal of R. Denote by A_2 the ideal, generated by 2ξ and ξ^2 for all $\xi \in A$. The first component A of an admissible pair is an ideal of R, parametrising short roots. When $\Phi \neq C_l$ the second component B, $A_2 \leq B \leq A$, is also an ideal, parameterizing long roots. In the exceptional case $\Phi = C_l$ the second component B is an additive subgroup stable under multiplication by ξ^2, $\xi \in R$ i.e., a form parameter. A similar notion can be introduced for the type G_2, as well, but in this case one should replace 2 by 3 *everywhere* in the above definition.

Now the relative elementary subgroup is defined as follows:

$$E(\Phi, R, A, B) = \langle x_\alpha(\xi), \alpha \in \Phi_s, \xi \in A;\ x_\beta(\zeta), \beta \in \Phi_l, \zeta \in B \rangle^{E(\Phi,R)}.$$

By the very definition the relative elementary subgroup $E(\Phi, R, A, B)$, is normal in the absolute elementary group $E(\Phi, R)$. However, its normality in the Chevalley group $G(\Phi, R)$ itself cannot be found in the existing literature, see the discussion in [47], [49].

On the other hand, in [1]–[3] and [43] the full congruence subgroup $C(\Phi, R, A, B)$ is *defined* as the following transporter

$$C(\Phi, R, A, B) = \{g \in G(\Phi, R) \mid [g, E(\Phi, R)] \leq E(\Phi, R, A, B)\}.$$

In analogy with the one parameter case, one should have defined it as the following transporter

$$C(\Phi, R, A, B) = \{g \in G(\Phi, R) \mid [g, G(\Phi, R)] \leq G(\Phi, R, A, B)\},$$

and then *prove* the previous commutator relation, known as the second standard commutator formula.

Problem 1. Establish the standard commutator formulae

$$[G(\Phi,R),E(\Phi,R,A,B)]=[E(\Phi,R),C(\Phi,R,A,B)]=E(\Phi,R,A,B)$$

for relative subgroups in Chevalley groups, parameterized by admissible pairs

Of course, it could be established by localization arguments, as in [40], [10], but it would be much more interesting to develop an analogue of [34], [11], that directly reduces normality of such an elementary subgroup to the absolute case.

Problem 2. Develop version of Stein's relativization with two parameters for relative subgroups in Chevalley groups, parameterized by admissible pairs.

Initially we planned to include these results in the present paper. However, the difficulty here is that in the pair $(R\times_A R, R\times_B R)$ the second term is not an ideal of the first one. This requires defining congruence subgroups modulo additive subgroups, which are not normal in the Chevalley group, and we do not know how to do it, without considering representations of G. For classical groups this is done in [11], [30], and for G_2 it can be easily done by hand. But for F_4 this requires a thorough look at the 27-dimensional representation.

Another step forward is to characterize the subnormal subgroups of classical-like groups. This turns out to be directly related to subgroups normalized by relative elementary subgroups.

Problem 3. Classify subgroups of unitary groups/Chevalley groups normalized by relative elementary subgroups.

The development of this line of research starts as follows. In [5] (the original manuscript of which goes back to 1967), the first named author studied the subgroups of $\mathrm{GL}(n,R)$ normalized by $E(n,R,I)$, for a ring R of finite stable rank and obtained a sandwich classification for such subgroups. His motivation for this was to positively answer a question credited to Borel. Consider the general linear group $\mathrm{GL}(n,K)$ where K is a global field. If $n\geq 3$ and H is a noncentral subgroup of $\mathrm{GL}(n,K)$, normalized by an arithmetic subgroup of $\mathrm{GL}(n,K)$, then does H contain an arithmetic subgroup of $\mathrm{GL}(n,K)$? The first author observed that the answer to this would follow if one could establish a sandwich condition similar to the absolute case for subgroups of the special linear group $\mathrm{SL}(n,R)$ normalized by relative elementary groups where R is the ring of integers in K.

Problem 3 is completely solved only for the case of the general linear group $\mathrm{GL}(n,R)$, as follows:

Theorem 5. *Let R be a commutative ring, $n\geq 3$ and H a subgroup of $\mathrm{GL}(n,R)$ normalized by $E(n,R,J)$ for an ideal J. Then there exist an ideal I and an integer m such that*

$$E(n,R,I)\leq H\leq C(n,R,I:J^m).$$

Recall that for two ideals I and J of the commutative ring R, $(I : J) = \{r \in R \mid rJ \subseteq I\})$. An important technical aspect of this problem, which received attention over many years, consists in finding the *smallest* possible m such that these inclusions hold for all such subgroups H. For example, in the case of $\mathrm{GL}_{n\geq 3}$, m took the following consecutive values in the period 1973–1989: $m = 7$ [50], $m = 6$ [42], $m = 5$ [45], $m = 4$ [44] and has not been improved since then.

Theorems of the above nature are a key in classifying the subnormal subgroups of $GL(n, R)$ (see proof of Theorem 1 in [44]). Namely, if

$$H = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_d = \mathrm{GL}(n, R)$$

is a subnormal subgroup of $\mathrm{GL}(n, R)$, then thanks to the above Theorem, there is an ideal J of R such that

$$E(n, R, J^4) \leq H \leq C(n, R, J).$$

In [5] the first author conjectured that his Sandwich Classification Theorem holds as well in the setting of general quadratic groups over rings with stable rank condition (in [5] Conjecture 1.3). Indeed, in light of recent developments in the theory, one can formulate the following conjecture:

Conjecture 1. *Let (R, Λ) be a form ring with R module finite, and let (J, Γ_J) be a form ideal. Let H be a subgroup of $G(2n, R, \Lambda)$, which is normalized by $E(2n, J, \Gamma_J)$. Then there is a form ideal (I, Γ_I) and a positive integer m such that*

$$E(2n, I, \Gamma_I) \leq H \leq G(2n, I : J^m, (I : J^m) \cap \Lambda).$$

This conjecture for a commutative ring R satisfying the stable rank condition, was settled positively by Habdank [17]. Recently Zhang [51] proved the conjecture in the stable case with only the commutativity assumption on the ring (and obtained a much finer range than is predicated by the conjecture), and consequently a description of subnormal subgroups of quadratic groups in this setting followed. His refinement was to replace $(I : J^m) \cap \Lambda$ by a certain smaller relative form parameter $\Gamma_{(I:J^m)}$ and it is conjectured that the conclusion of the conjecture above holds also for this smaller relative form parameter.

For the case of $\mathrm{GL}(n, R)$ there are several very interesting notes by Alec Mason [23]–[26], which study relative commutator subgroups such as

$$[E(n, R, A), E(n, R, B)], \qquad [\mathrm{GL}(n, R, A), \mathrm{GL}(n, R, B)],$$

etc, for two ideals $A, B \trianglelefteq R$. In particular, he gives counter-examples, which show that these commutators are not what you expect them to be even in the case of Dedekind rings.

For rings satisfying appropriate stability conditions Mason establishes the following standard commutator formula with two parameters, which simultaneously generalize both usual standard commutator formulae with one parameter,

$$[E(n, R, A), \mathrm{GL}(n, R, B)] = [E(n, R, A), E(n, R, B)].$$

In [37] we prove that this formula in fact holds for all commutative rings, the proof being another variation of the decomposition of unipotents [36]. It is only natural to ask, whether similar result holds for other groups.

Problem 4. Prove the standard commutator formula with two ideals (form ideals, admissible pairs) for unitary groups/Chevalley groups.

In view of the above, such a generalization would in general involve up to *four* parameters!

In [19] we mentioned, that nilpotency of K_1 can be considered as a *very* strong form of normality of the elementary subgroup. Thus, whenever we can establish normality, there is the hope of being able to prove nilpotency as well. After the publication of [18] and [19], two new contexts emerged, of remarkable generality, when the elementary subgroup has been shown to be normal.

One of them is the work of Viktor Petrov on odd unitary groups, [29], [30]. Another one is a very recent paper of Viktor Petrov and Anastasia Stavrova [31], where normality of the elementary subgroup is established for the group of points of an isotropic reductive group over an arbitrary commutative ring, under the assumption that all localizations have only components of semi-simple ranks ≥ 2 (see [31] for the precise statements).

Problem 5. Establish nilpotency of the K_1-functor modeled on odd unitary groups.

Problem 6. Establish nilpotency of the K_1 functor modeled on isotropic reductive groups, all of whose localizations have only components of semi-simple ranks ≥ 2.

These problems, especially the last one, are considerably harder than the rest, since K_1-functors modeled on non-split simple groups can be non-trivial even in the field case. In fact, triviality of such K_1-functors constituted the celebrated Kneser–Tits conjecture. The first counter-examples to the positive solution of the Kneser–Tits conjecture were constructed by Platonov and Yanchevski. Even today, after decades of sustained efforts, it remains open for some forms of exceptional groups. One can find an account of this theory in the marvelous book by Platonov and Rapinchuk [32] (see also [14]). Nevertheless, we are positive, that the results of [31] suffice to generalize most of the calculations of [19] to this case.

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Department of Mathematics, University of Bielefeld, P. O. Box 100131, 33501 Bielefeld, Germany
E-mail address: `bak@mathematik.uni-bielefeld.de`

Department of Pure Mathematics, Queen's University, Belfast BT7 1NN, U.K.
E-mail address: `r.hazrat@qub.ac.uk`

Department of Mathematics and Mechanics, St. Petersburg State University, St. Petersburg 198504, Russia
E-mail address: `nikolai-vavilov@yandex.ru`