Sensitivity analysis of impact and lift–off events in constrained mechanical systems

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Abstract We consider mechanical systems with holonomic constraints given by equalities and unilateral constraints that arise in contact problems. Impact or lift–off events lead to jump discontinuities in the velocities and δ–functions in the Lagrange multipliers, and we find piecewise smooth solutions of the equations of motion. In the special case of one isolated event we study the sensitivity of the solution w.r.t. small perturbations in the equations and in the constraints. We derive error bounds for the position and velocity coordinates and illustrate the results by a numerical example: the collision of a hydrostatic skeleton with an obstacle.

1 Introduction

A perturbation analysis for the equations of motion of constrained mechanical systems plays an important role in the derivation of efficient numerical methods. Various mathematical formulations can be compared w.r.t. the sensitivity of the analytical solution subject to small perturbations in the equations. Furthermore, the influence of roundoff errors can be studied by a perturbation analysis of various discretization schemes.

We consider mathematical models that consist of a set of discrete mass–points. The forces that lead to the motion of the system might be given by internal forces exerted by massless springs that connect two mass–points, or external forces like gravitation or fluid forces. In most practical applications we find constraints in the system that are given by equalities, i.e. there are relations between the mass–points that have to be fulfilled during the motion. Common examples are rigid connections between single mass–points, but also global constraints that involve the whole set of mass–points are possible such as conservation of volume which will be studied in our numerical example, see Sect. 4. The constraints are incorporated into the equations of motion by a Lagrange multiplier approach (see e.g. [8]) and we end up with a system of differential–algebraic equations (DAEs).

In addition, the motion of the system might be constrained by an obstacle, e.g. in crash simulations. Each mass–point in the system is either distant from the obstacle (passive) or in contact with the obstacle (active) and we find a unilateral constraint

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for the relative distance. Furthermore, in a non–adhesive contact situation there is a contact force acting in the direction of the outward normal of the obstacle. It has to vanish as soon as the mass–point lifts off and this leads to a unilateral constraint for the contact force. These two unilateral constraints of “no penetration” and “no adhesion” are connected by a complementarity condition that is added to the DAE system. During the motion some mass–points might collide with the obstacle or lift off, this leads to jump discontinuities in the solution.
The solution of an ODE with Lipschitz–continuous right hand side depends continuously on perturbations of the initial values and the right hand side, i.e. small errors in the initial values and in the equations lead to small errors in the solution, see e.g. [27]. In the case of a DAE the sensitivity of the solution also depends on derivatives of the perturbations which might be large compared to the perturbation itself, see e.g. [3], [4], [11]. In this paper we derive error estimates for the solution of the equations of motion in the presence of unilateral constraints. While the solution of a DAE is smooth and standard analytical tools can be used to derive error estimates, we have to account for jump discontinuities in the presence of unilateral constraints. We study the special case of one inelastic collision or one lift–off of a single mass–point without tangential friction forces, and we consider perturbations in the dynamic equations, in the global equality constraints, and in the unilateral constraints. This leads to the basic theoretical results: It turns out that the estimates still include second order derivatives of the perturbations in the equality constraints and in the unilateral constraints, but there are no derivatives of the perturbations in the dynamic equations. This result corresponds to the “pure” DAE case without unilateral constraints.

2 Mechanical systems with constraints

In this section we set up the equations of motion for constrained mechanical systems that consist of d discrete mass–points (or particles). The coordinate (or position) vector of the system at time $t \in [0, T]$, $T > 0$ is denoted by $q(t) \in \mathbb{R}^{3d}$ and the velocity is given by $u(t) = \dot{q}(t) \in \mathbb{R}^{3d}$. Usually, the mass–points interact with each other and there are constraints in the system which reduce the degrees of freedom of the system. We assume that these constraints are given by relations between the coordinates

$$g(t, q) = 0, \quad g : (-\varepsilon, T_\varepsilon) \times \mathbb{R}^{3d} \to \mathbb{R}^m \tag{1}$$

with some constants $\varepsilon > 0$, $T_\varepsilon > T$. Constraints that can be expressed in this fashion are called holonomic. Nonholonomic constraints which also involve the velocity u are not considered here, see e.g. [13], [14].
In most practical applications the interaction of two or more bodies that either collide or slide on each other has to be considered, typical examples are robots interacting with the environment, gears in engines, and wheel–rail systems, see e.g. [4], [12], [20], [21], [23], [24]. We consider the special case that the motion of the system of mass–points is constrained by an obstacle which is given by the hyperplane

$$\mathcal{H} := \{Q \in \mathbb{R}^3 : Q_1 = K\}, \;\; K > 0. \tag{2}$$

With the set $J := \{1, 4, 7, \ldots, 3d-2\} \subset I := \{1 \ldots 3d\}$ we find the unilateral constraints

$$q_j(t) - K \geq 0 \quad \text{for all } j \in J. \tag{3}$$

A constraint $j \in J$ is called "active" at time t if the corresponding mass–point $q_j(t)$ is in contact with the obstacle $\mathcal{H}$, i.e. $(q_j(t) - K) = 0$. We define the "active set" by

$$J_A(t) := \{j \in J : q_j(t) = K\}, \quad a(t) := \#J_A(t), \tag{4}$$

and the corresponding "passive set" $J_P(t) := J \backslash J_A(t)$ contains the indices of all "passive" constraints, i.e. $(q_j(t) - K) > 0$ for all $j \in J_P(t)$. Notice that every active constraint corresponds to a holonomic constraint of the form (1). Newton's second law together with d'Alembert's principle leads to Lagrange's equations of motion (see e.g. [8])

$$\begin{aligned} \dot{q}(t) &= u(t) && (5)\\ M\dot{u}(t) &= f(t, q(t), u(t)) + G(t, q(t))^T \nu(t) + H^T \lambda(t) && (6)\\ 0 &= g(t, q(t)) && (7) \end{aligned}$$

with the complementarity conditions (Hertz–Signorini–Moreau 1963, see [17])

$$(q_j(t) - K) \geq 0, \quad \lambda_j(t) \geq 0, \quad (q_j(t) - K)\lambda_j(t) = 0 \quad \text{for all } j \in J, \tag{8}$$

see e.g. [7], [13], [20] for a general discussion. Here $H^T\lambda$ are the contact forces, where

$$\lambda = \left(\lambda_1, \lambda_4, \lambda_7, \ldots, \lambda_{3d-2}\right)^T \in \mathbb{R}^d, \quad H \in \mathbb{R}^{d,3d}, \ H_{ji} = \begin{cases} 1, & \text{if } i = 3j-2 \\ 0, & \text{otherwise} \end{cases}.$$

The vector $\nu \in \mathbb{R}^m$ in (6) contains the m Lagrange multipliers that correspond to the global holonomic constraints (7) and the diagonal mass–matrix

$$M := \text{diag}(m_1, m_2, \ldots, m_{3d}) \in \mathbb{R}^{3d,3d} \tag{9}$$

contains the lumped masses $m_{3i-2} = m_{3i-1} = m_{3i} > 0$ of the mass–points $i = 1 \ldots d$ as diagonal entries. The applied forces are given by $f(t, q, u)$ and the forces of constraint that maintain the global holonomic constraints (7) are given by $G(t, q)^T\nu$, where $G(t, q) := (\partial g/\partial q)(t, q) \in \mathbb{R}^{m,3d}$.

The complementarity conditions (8) can be explained as follows: The frictionless contact of a single mass–point with the obstacle $\mathcal{H}$ is characterized by the three facts that the mass–point cannot penetrate the obstacle, the mass–point cannot pull on the obstacle (no adhesion), and either the mass-point presses on the obstacle or it is separated from the obstacle. The geometric condition of "no penetration" is given by the first inequality in (8), and the second inequality in (8) expresses the "no adhesion" condition. The equality condition of complementarity in (8) refers to the fact that at time t the mass–point is either distant and there is no contact force, i.e. $(q_j(t) - K) \neq 0$ and $\lambda_j(t) = 0$, or it is in contact with the obstacle and there is an interaction by a contact force, i.e. $(q_j(t) - K) = 0$ and $\lambda_j(t) \neq 0$. When the mass–point "touches" the obstacle, i.e. both $(q_j(t) - K)$ and $\lambda_j(t)$ are zero, we find a more complicated contact situation , cf. Sect. 2.3. We will use $K = 0$ in the following, and we state a well–known result which will be used frequently (see e.g. [9] for a proof):

Lemma 2.1 *Let $M \in \mathbb{R}^{3d,3d}$ and $\mathcal{G} \in \mathbb{R}^{s,3d}$ be given such that rank$(\mathcal{G}) = s$ and M is positive definite. Then the block matrix*

$$N := \begin{pmatrix} M & \mathcal{G}^T \\ \mathcal{G} & 0 \end{pmatrix} \in \mathbb{R}^{3d+s,3d+s}$$

as well as its Schur complement $-\mathcal{G}M^{-1}\mathcal{G}^T \in \mathbb{R}^{s,s}$ is invertible.

2.1 Smooth motion

In time intervals where the active and passive sets are constant the equations of motion (5)-(7) together with the active constraints in (8) result to

$$\begin{aligned} \dot{q}(t) &= u(t) && (10)\\ M\dot{u}(t) &= f(t,q(t),u(t)) + G(t,q(t))^T\nu(t) + A(t)^T(A(t)H^T\lambda(t)) && (11)\\ 0 &= g(t,q(t)) && (12)\\ 0 &= A(t)q(t) && (13) \end{aligned}$$

$$A(t) \in \mathbb{R}^{a(t),3d}, \;\; A_{ji}(t) = \begin{cases} 1, & \text{if } i = k_j \in J_A(t) = \{k_1,\dots,k_{a(t)}\} \\ 0, & \text{otherwise} \end{cases}. \qquad (14)$$

The $a(t)$ active constraints are included as holonomic constraints with Lagrange multipliers $\Lambda(t) := A(t)H^T\lambda(t) \in \mathbb{R}^{a(t)}$, and (10)-(13) is a semi–explicit DAE system for (q,u,ν,Λ), see e.g. [4], [5], [7], [11]. The remaining components of $\lambda(t)$ are obtained from the complementarity conditions (8) as $\lambda_j(t) = 0$ for $j \notin J_A(t)$. Differentiating (12), (13) once and twice with respect to t yields the first and second order "hidden" constraints which are also fulfilled by a smooth solution of the system (10)-(13):

$$0 = g_t(t,q) + G(t,q)u, \;\; 0 = A(t)u, \quad \text{and} \qquad (15)$$

$$\begin{aligned} 0 &= \widetilde{g}(t,q,u) + G(t,q)M^{-1}\left[f(t,q,u) + G(t,q)^T\nu + A(t)^T(A(t)H^T\lambda(t))\right], \\ 0 &= A(t)M^{-1}\left[f(t,q,u) + G(t,q)^T\nu + A(t)^T(A(t)H^T\lambda(t))\right], \end{aligned} \qquad (16)$$

$$\text{where} \quad \widetilde{g}(t,q,u) := g_{tt}(t,q) + 2G_t(t,q)u + G_q(t,q)(u,u). \qquad (17)$$

For (t,q) fixed we have $g_{tt}(t,q) \in \mathbb{R}^m$, $G_t(t,q) \in \mathbb{R}^{3d,m}$, and $G_q(t,q) : \mathbb{R}^{3d} \times \mathbb{R}^{3d} \to \mathbb{R}^m$ is a bilinear mapping. We assume that the functions f and g have a sufficient number of continuous derivatives, and we set $\mathcal{V} := \mathbb{R}^{3d} \times \mathbb{R}^{3d} \times \mathbb{R}^m \times \mathbb{R}^d$. For $a < b$ and $s \geq 1$ we denote the left and right limits of a function $\psi \in \mathcal{C}((a,b),\mathbb{R}^s)$ at time $c \in (a,b)$ by $\psi^-(c)$ and $\psi^+(c)$, respectively. Furthermore, $J_A^-(c)$ and $J_A^+(c)$ denote the active sets before and after a collision or a lift–off. The corresponding passive sets are defined by $J_P^-(c) = J\backslash J_A^-(c)$ and $J_P^+(c) = J\backslash J_A^+(c)$.

It turns out that the model (5)-(8) is not yet complete: The active and passive sets are time–dependent and can alter in the course of motion. In case of a collision a passive constraint suddenly becomes active, and an active constraint becomes passive if the corresponding mass–point lifts off the obstacle. Events like this can cause discontinuities in the system and the equations of motion are no longer valid. Therefore jump conditions have to be derived that describe the behavior of the system if such an event occurs.

2.2 The jump conditions

When a collision occurs, the mass–point usually approaches the obstacle with nonzero normal velocity, i.e. nonzero velocity in the direction normal to the obstacle, and the distance between the mass–point and the obstacle becomes zero. In a partly or completely elastic impact the mass–point will separate from the body immediately after the collision, and it will stay in contact with the obstacle only if the collision is inelastic. Since the obstacle is assumed to be rigid, the duration of the collision is infinitesimal. The forces needed to prevent the mass–point from penetrating the obstacle are impulsive and there is a jump discontinuity in the normal velocity. However, in reality there are no rigid bodies. The collision starts with a compression phase where the obstacle is deformed and elastic energy is stored. At the point of maximum compression the normal velocity becomes zero, changes sign and the extension (or restitution) phase starts. In this phase some of the stored elastic energy is returned as kinetic energy and the mass–point lifts off. The simplest model to approximate this process is given by Newton's impact law, which relates the normal velocity $u_p^+(c)$ after the impact of a mass–point $p \in J_P^-(c)$ at time $c > 0$ to the normal velocity $u_p^-(c)$ before the impact:

$$q_p(c) = 0,\ u_p^-(c) < 0 \quad \Longrightarrow \quad u_p^+(c) = -\varepsilon u_p^-(c) \tag{18}$$

The constant $\varepsilon \in [0, 1]$ is the coefficient of restitution. The value $\varepsilon = 1$ corresponds to the completely elastic case, and only for $\varepsilon = 0$ the collision is inelastic and we have $p \in J_A^+(c)$. Another collision model is given by Poisson's law, it relates the impulse delivered during the compression phase to the impulse delivered during the extension phase. In the frictionless case Newton's and Poisson's law are equivalent, for a deeper discussion see e.g. [12], [18], [20].

The jump discontinuity in the normal velocity of the colliding mass–point is caused by an impulsive force (δ–distribution) which is applied at $t = c$, it approximates a large force with a short duration. At the time of impact there is not only an impulsive contact force $I^p = I^p(c) \in \mathbb{R}$ in the constraint $p \in J_P^-(c)$ that suddenly becomes active, we also have impulsive constraint forces $I^\lambda = I^\lambda(c) \in \mathbb{R}^{a^-(c)}$ and $I^\nu = I^\nu(c) \in \mathbb{R}^m$ caused by the constraints in $J_A^-(c)$ that are already active as well as the global holonomic constraints (7). These impulses lead to jump discontinuities in the velocities of all mass–points in the system and by local conservation of impulses we find the jump condition

$$M\Big(u^+(c) - u^-(c)\Big) = G(c, q(c))^T I^\nu + A^-(c)^T I^\lambda + e_p I^p, \tag{19}$$

where $e_p \in \mathbb{R}^{3d}$ is the pth unit vector. This equation is used at $t = c$ instead of the equations of motion (5)-(8), for a derivation of (19) see e.g. [10]. In [2] it is shown that (6) is still valid at $t = c$ in the sense of distributions if (19) holds.

Additional conditions are derived by the fact that the first order hidden constraints (15) have to be fulfilled after the impact, i.e. consistent initial values are needed after the impact, see Sect. 2.3. For the global holonomic constraints (7) and the constraints in $J_A^-(c)$ that are already active (15) leads to

$$g_t(c, q(c)) + G(c, q(c))u^+(c) = 0, \quad A^-(c)u^+(c) = 0. \tag{20}$$

Obviously, there might be a whole subset of indices in $J_P^-(c)$ such that all mass–points with indices in this subset collide with the obstacle at the same time. Newton's impact law (18) for inelastic collisions ($\varepsilon = 0$), the condition for the impulses (19), and the conditions for the first order hidden constraints (20) lead to the system

$$\begin{pmatrix} M & G(c,q(c))^T & A^+(c)^T \\ G(c,q(c)) & 0 & 0 \\ A^+(c) & 0 & 0 \end{pmatrix} \begin{pmatrix} u^+(c) \\ -I^\nu \\ -I_C^\lambda \end{pmatrix} = \begin{pmatrix} Mu^-(c) \\ -g_t(c,q(c)) \\ 0 \end{pmatrix} \tag{21}$$

with the impulses $I_C^\lambda = I_C^\lambda(c) \in \mathbb{R}^{a^+(c)}$. If the rank condition

$$\operatorname{rank} \begin{pmatrix} G(t,q) \\ H \end{pmatrix} = m + d \tag{22}$$

is fulfilled at $(t,q) = (c, q(c))$, Lemma 2.1 implies that (21) can be solved uniquely for $u^+(c)$, I^ν, and I_C^λ. After the collision the Lagrange multipliers are determined uniquely by the second order hidden constraints (16) with the new active set $J_A^+(c)$. These also involve $u^+(c)$ and $A^+(c)$, and therefore we also find jump discontinuities in the Lagrange multipliers, see [2] for details.

During a lift–off the solution (q,u,ν,λ) is continuous, but discontinuities in $\dot\nu$ and $\dot\lambda$ are possible. Usually, the contact force λ_p of an active constraint $p \in J_A^-(c)$ becomes zero at $t = c$ with $\dot\lambda_p^-(c) < 0$ and the corresponding mass–point lifts off the obstacle. Because of the complementarity conditions (8) the contact force has to be zero after the lift–off, so there is a jump discontinuity in $\dot\lambda_p$ and this also affects $\dot\nu$ and $\dot\lambda_j$ for $j \in J_A^+(c) = J_A^-(c)\backslash\{p\}$ (details can be found in [2]):

$$\begin{pmatrix} \dot\nu^+(c) - \dot\nu^-(c) \\ A^+(c)H^T(\dot\lambda^+(c) - \dot\lambda^-(c)) \end{pmatrix} = D(c,q(c))^{-1} \begin{pmatrix} -G(c,q(c))M^{-1}e_p\dot\lambda_p^-(c) \\ -A^+(c)M^{-1}e_p\dot\lambda_p^-(c) \end{pmatrix},$$

$$\text{where} \quad D(t,q(t)) := \begin{pmatrix} G(t,q) \\ A(t) \end{pmatrix} M^{-1} \begin{pmatrix} G(t,q) \\ A(t) \end{pmatrix}^T. \tag{23}$$

2.3 Existence and uniqueness of a piecewise smooth solution

The DAE system (10)-(13) has a unique and smooth solution provided that the rank condition (22) holds and the initial values are consistent up to the first order hidden constraints, i.e. they fulfill (12), (13), and (15). Using Lemma 2.1 this can be shown by a well–known reduction process (see e.g. Ch. 1 in [4]). However, in the presence of unilateral constraints we find jump discontinuities and therefore we extend the notion of solution to piecewise smooth solutions.

For a countable (but possibly infinite) subset $\mathcal{U} \subset [0,T)$ with $T > 0$ we define the function spaces of piecewise smooth solutions

$$\begin{aligned} \mathcal{C}_s^l([0,T)\backslash\mathcal{U},\mathbb{R}^d) \;&:=\; \big\{\psi \in \mathcal{C}^l([0,T)\backslash\mathcal{U},\mathbb{R}^d) \mid \psi^{(r)}(c) := (\psi^{(r)})^-(c) < \infty,\ (\psi^{(r)})^+(c) < \infty \\ &\qquad \text{for } c \in \mathcal{U} \text{ and } r = 0\ldots l\big\} \\ \mathcal{K}^l([0,T),\mathcal{U};\mathcal{V}) \;&:=\; \big\{(q,u,\nu,\lambda) : [0,T) \to \mathcal{V} \mid q \in \mathcal{C}^l([0,T),\mathbb{R}^{3d}) \cap \mathcal{C}_s^3([0,T)\backslash\mathcal{U},\mathbb{R}^{3d}); \\ &\qquad u \in \mathcal{C}_s^2([0,T)\backslash\mathcal{U},\mathbb{R}^{3d});\ \nu \in \mathcal{C}_s^1([0,T)\backslash\mathcal{U},\mathbb{R}^m);\ \lambda \in \mathcal{C}_s^1([0,T)\backslash\mathcal{U},\mathbb{R}^d)\big\} \end{aligned}$$

for $0 \le l < \infty$ and $0 < d < \infty$. Notice that the existence of left and right limits of the piecewise smooth solutions is assured here. Functions $\psi \in \mathcal{C}_s([0,T)\backslash\mathcal{U},\mathbb{R})$ with a finite event set $\mathcal{U} = \{c_1,\dots,c_N\}$, $0 < c_1 < \cdots < c_N < T$ are Lebesgue integrable, and the Gronwall–Lemma still holds for piecewise functions of this type, see [2]. Furthermore, a function $\Psi \in \mathcal{C}([0,T],\mathbb{R}) \cap \mathcal{C}_s^1([0,T]\backslash\mathcal{U},\mathbb{R})$ with $\Psi' = \psi \in \mathcal{C}_s([0,T]\backslash\mathcal{U},\mathbb{R})$ is absolutely continuous and of bounded variation, and the fundamental theorem still holds.
In general, we expect solutions (q,u,ν,λ) of (5)-(8) to be in $\mathcal{K}^0([0,T),\mathcal{U};\mathcal{V})$ since collisions lead to discontinuities in u, ν, and λ. However, if there are only lift–off events and no collisions in $[0,T)$ we find $(q,u,\nu,\lambda) \in \mathcal{K}^2([0,T),\mathcal{U};\mathcal{V})$, i.e. $u = \dot{q} \in \mathcal{C}^1([0,T),\mathbb{R}^{3d})$, $\dot{u} \in \mathcal{C}([0,T),\mathbb{R}^{3d})$, and therefore also $\nu, \lambda \in \mathcal{C}([0,T),\mathbb{R}^{3d})$ by (6).

Definition 2.1 *$(q,u,\nu,\lambda) : [0,T) \to \mathcal{V}$ with $T > 0$ is a* **piecewise solution** *of the equations of motion (5)-(8), (21) with global holonomic and unilateral constraints if there exists a countable event set $\mathcal{U} \subset [0,T)$ such that $(q,u,\nu,\lambda) \in \mathcal{K}^0([0,T),\mathcal{U};\mathcal{V})$ and the following holds:*
For any closed subset $[a,b] \subset [0,T)$ equations (5)–(8) hold in $[a,b]\backslash\mathcal{U}_a^b$, $\mathcal{U}_a^b := [a,b] \cap \mathcal{U}$ and the jump conditions (21) hold for all elements $c \in \mathcal{U}_a^b$ in the following sense: Let $(u^+(c), I^\nu, I_C^\lambda)$ be the unique solution of (21) with $q(c) = \overline{q}(c)$ and $u^-(c) = \overline{u}^-(c)$, then $u^+(c) = \overline{u}^+(c)$ results.

Notice that $\mathcal{U}$ might be infinite and there might be an infinite number of finite subsets $\mathcal{U}_a^b \subset \mathcal{U}$. An example of this type is a mass point that collides with the obstacle at $t = (1/2)T$, lifts off at $t = (3/4)T$, collides again at $t = (7/8)T$, and so on. This example is strongly related to an elastic ball bouncing on a horizontal plane with a coefficient of restitution $\varepsilon \in (0,1)$, see e.g. [18], [25].
A global existence and uniqueness result can be found in [2]. For technical reasons we leave out the proof and only discuss the construction of the piecewise smooth solution: We find a unique and smooth solution of the DAE system (10)-(13) in time intervals where no collision or lift–off occurs. However, if such an event occurs we have to modify the active and passive sets, and in the case of a collision new initial values have to be calculated. The initial values $q(c)$ and $u^+(c)$ determined by the jump conditions (21) are consistent up to the first order hidden constraints, and therefore existence and uniqueness of a solution of the DAE system (10)-(13) with the new active set $J_A^+(c)$ is guaranteed. We can assemble the DAE solutions on the subintervals and this leads to a unique piecewise solution of (5)-(8), (21). It exists at least in a short time interval until we find

- an accumulation point of collision/lift–off events
- an incompatibility in the active/passive sets, i.e. we find a negative Lagrange multiplier that corresponds to an active constraint
- a more complicated contact situation where q_j, λ_j, and some of their derivatives are zero for an index $j \in J$

In the second case the solution can be extended if we find a consistent active set such that the complementarity conditions (8) are fulfilled. This is usually done by solving a

linear complementarity problem (LCP), see e.g. [6], [13], [20]. Then the improved active set consists only of those active constraints that will not become passive in infinitesimal time after the impact. In the presence of global holonomic constraints we find a "mixed" LCP which can be transformed to a "standard" LCP. However, uniqueness of a solution of the resulting LCP is not assured in this case, see [2] for details.
In [16] as well as in [19] and [25] the contact problem is stated in terms of differential measures. It is proved that there exists at least one solution under the additional assumption that f does not depend on the velocity and is globally bounded. Furthermore, the formulation is extended to contacts with isotropic tangential friction (Coulomb's friction law) and the existence of a solution is also shown in that case. However, uniqueness is not treated in [16] and there seems to be no discussion of stability in this general situation. Notice that a piecewise solution of our original problem also solves this generalized problem.

3 Perturbation analysis

It is expected that small errors in the equations caused by inexact measurements or uncareful modeling have only little influence on the solution of the system, and in this section it will be analyzed in which sense the influence is small. In the DAE case a perturbation theory is well known, the solution is smooth and standard analytical tools can be used to derive error estimates, see e.g. [3], [4], [11]. However, in the presence of unilateral constraints a more subtle analysis is needed since in general the solution shows jump discontinuities. In order to analyze the sensitivity of the piecewise solution of the system (5)-(8) we consider the perturbed system

$$\begin{aligned} \dot q &= u && (24)\\ M\dot u &= f(t,q,u) + G(t,q)^T\nu + H^T\lambda + \delta && (25)\\ \eta &= g(t,q) && (26) \end{aligned}$$

$$q_j \geq \theta_j, \quad \lambda_j \geq 0, \quad (q_j - \theta_j)\lambda_j = 0 \quad \text{for all } j \in J. \qquad (27)$$

We study only single and isolated collisions, and Newton's law (18) for an inelastic impact of a mass–point $p \in J_P^-(c)$ at time $c > 0$ reads in the presence of perturbations

$$q_p(c) = \theta_p(c),\ u_p^-(c) < \dot\theta_p(c) \quad \Longrightarrow \quad u_p^+(c) = \dot\theta_p(c),\ p \in J_A^+(c).$$

Using $A(t)(q(t) - H^T\theta(t)) = 0$ for the active constraints in (27), the jump conditions (21) in the special case of a single collision result to

$$\begin{pmatrix} M & G(c,q(c))^T & A^+(c)^T \\ G(c,q(c)) & 0 & 0 \\ A^+(c) & 0 & 0 \end{pmatrix} \begin{pmatrix} u^+(c) \\ -I^\nu \\ -I_p^\lambda \end{pmatrix} = \begin{pmatrix} Mu^-(c) \\ -g_t(c,q(c)) + \dot\eta(c) \\ A^+(c)H^T\dot\theta(c) \end{pmatrix}. \qquad (28)$$

After the impact the Lagrange multipliers have to fulfill the second order hidden constraints with the active set $J_A^+(c) = J_A^-(c) \cup \{p\}$. In the presence of perturbations (16) leads to

$$\begin{pmatrix} \nu^+ \\ \Lambda^+ \end{pmatrix} = D^+(c,q)^{-1} \begin{pmatrix} \ddot{\eta}(c) - \widetilde{g}(c,q,u^+) - G(c,q)M^{-1}\Big(f(c,q,u^+) + \delta\Big) \\ A^+(c)H^T\ddot{\theta} - A^+(c)M^{-1}\Big(f(c,q,u^+) + \delta\Big) \end{pmatrix}, \qquad (29)$$

where $\Lambda^+ := A^+(c)H^T\lambda^+$ and $D^+(c,q)$ defined in (23) is invertible by Lemma 2.1 if the rank condition (22) is fulfilled. In time intervals where the active set is constant the system (24)-(27) leads to the DAE system

$$\dot{q} = u \qquad (30)$$

$$M\dot{u} = f(t,q,u) + G(t,q)^T\nu + A(t)^T\left(A(t)H^T\lambda\right) + \delta \qquad (31)$$

$$\eta = g(t,q) \qquad (32)$$

$$A(t)H^T\theta = A(t)q \qquad (33)$$

For given event sets $\overline{U} = \{\overline{c}\}$ and $\widehat{U} = \{\widehat{c}\}$ of the original system (5)-(8) and the perturbed system (24)-(27) we set $c_{\min} := \min\{\overline{c}, \widehat{c}\}$ and $c_{\max} := \max\{\overline{c}, \widehat{c}\}$ in the following. Furthermore, we use the abbreviations $J_A^0 := J_A(0)$ and $J_P^0 := J_P(0)$, and $\|\cdot\|$ denotes the maximum norm. For the perturbation analysis in the next section we will need the following auxiliary lemma (see e.g. [26] for a proof):

Lemma 3.1 *Suppose that $T \in \mathcal{C}^1(U, \mathbb{R}^s)$ with the open set $U \subset \mathbb{R}^s$ and constants $\rho, \sigma, \kappa > 0$ are given such that with the set $S := \{\epsilon \in \mathbb{R}^s : \|\epsilon\| \le \rho\}$ the following conditions hold:*

(a) $\|T'(0)^{-1}\| \le \sigma$, *(b)* $\|T'(0) - T'(\epsilon)\| \le \kappa < \frac{1}{\sigma}$ *for* $\epsilon \in S$, *(c)* $\|T(0)\| \le (\frac{1}{\sigma} - \kappa)\rho$.

Then the system $T(\epsilon) = 0$ has a unique solution $\bar{\epsilon} \in S$ and the following estimate holds for all $\epsilon_1, \epsilon_2 \in S$:

$$\|\epsilon_1 - \epsilon_2\| \le \frac{\sigma}{1 - \sigma\kappa}\|T(\epsilon_1) - T(\epsilon_2)\|$$

3.1 Sensitivity of differential–algebraic equations

While the solution of an ODE with Lipschitz–continuous right hand side depends continuously on the perturbations (see e.g. [27]), the solution of a DAE also depends on the derivatives of the perturbations. This well–known result can be found for example in [7] and [11]. We need rather precise error bounds for the solution of the system (10)-(13), see [2] for a proof of the following theorem:

Theorem 3.1 *Suppose that $(\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda}) \in \mathcal{C}^{2,1,0,0}([0,T], \mathcal{V})$ with $T > 0$ is a solution of (10)-(13) and let the rank condition (22) hold for all $(t,q) \in (-\varepsilon, T_\varepsilon) \times \mathbb{R}^{3d}$ with $\varepsilon > 0$, $T_\varepsilon > T$. Then there exist constants $K, C_1^*, C_2^*, K_s > 0$ such that the following holds: For perturbations $\delta \in \mathcal{C}([0,T_\varepsilon), \mathbb{R}^{3d})$, $\eta \in \mathcal{C}^2([0,T_\varepsilon), \mathbb{R}^m)$, $\theta \in \mathcal{C}^2([0,T_\varepsilon), \mathbb{R}^d)$ with*

$$\max_{l=0,1,2}\|\eta^{(l)}(t)\|, \max_{l=0,1,2}\|\theta^{(l)}(t)\|, \|\delta(t)\| \le K \quad \textit{for } t \in [0, T_\varepsilon) \qquad (34)$$

and consistent initial values $(\widehat{q}^0, \widehat{u}^0, \widehat{\nu}^0, \widehat{\lambda}^0) \in \mathcal{V}$ *with*

$$\theta_j(0) = \widehat{q}_j^0, \;\; \dot{\theta}_j(0) = \widehat{u}_j^0 \;\; \textit{for } j \in J_A^0, \quad \eta(0) = g(0, \widehat{q}^0), \;\; \dot{\eta}(0) = g_t(0, \widehat{q}^0) + G(0, \widehat{q}^0)\widehat{u}^0,$$
$$\Psi^0 := \|\widehat{q}^0 - \overline{q}^0\| + \|\widehat{u}^0 - \overline{u}^0\| \le K. \tag{35}$$

there exists a unique solution $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda}) \in \mathcal{C}^{2,1,0,0}([0,\tau), \mathcal{V})$ *of the perturbed system (30)-(33) with* $T < \tau \le T_\varepsilon$. *Furthermore, the estimates*

$$\|\widehat{q}(t) - \overline{q}(t)\| + \|\widehat{u}(t) - \overline{u}(t)\| \le C_1^* \left(\Psi^0 + \int_0^t \Big(\|\ddot{\eta}(\xi)\| + \|\ddot{\theta}(\xi)\| + \|\delta(\xi)\| \Big) \, d\xi \right) \tag{36}$$

$$\|\widehat{\nu}(t) - \overline{\nu}(t)\| + \|\widehat{\lambda}(t) - \overline{\lambda}(t)\| \le C_2^* \Big(\Psi^0 + \max_{0 \le \xi \le t} \|\ddot{\eta}(\xi)\| + \max_{0 \le \xi \le t} \|\ddot{\theta}(\xi)\| + \max_{0 \le \xi \le t} \|\delta(\xi)\| \Big) \tag{37}$$

hold for $t \in [0, T]$ *and the solution is bounded, i.e.*

$$\|\widehat{q}(t)\| + \|\widehat{u}(t)\| + \|\dot{\widehat{u}}(t)\| + \|\widehat{\nu}(t)\| + \|\widehat{\lambda}(t)\| \le K_s. \tag{38}$$

Remark 3.1
1.) Suppose that $\delta \in \mathcal{C}^1([0, T_\varepsilon), \mathbb{R}^{3d})$, $\eta \in \mathcal{C}^3([0, T_\varepsilon), \mathbb{R}^m)$, $\theta \in \mathcal{C}^3([0, T_\varepsilon), \mathbb{R}^d)$ and $(\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda}), (\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda}) \in \mathcal{C}^{3,2,1,1}([0,T], \mathcal{V})$. Furthermore, let $\|\eta^{(3)}(t)\|, \|\theta^{(3)}(t)\| \le K$ for $t \in [0, T_\varepsilon)$ with K from (34) and (35). Then we easily derive $\|\dot{\widehat{\nu}}(t) - \dot{\overline{\nu}}(t)\| = \mathcal{O}(K)$, $\|\dot{\widehat{\lambda}}(t) - \dot{\overline{\lambda}}(t)\| = \mathcal{O}(K)$, $\|\dot{\widehat{u}}(t) - \dot{\overline{u}}(t)\| = \mathcal{O}(K)$, and $\|\ddot{\widehat{u}}(t) - \ddot{\overline{u}}(t)\| = \mathcal{O}(K)$. These results will be needed in the proof of Theorem 3.4.
2.) A detailed sensitivity analysis and a systematic classification of DAEs is given by the perturbation index, see [11]. It is defined as the smallest integer $pi \ge 1$ such that an error estimate for the solution holds, which contains derivatives of the perturbations up to order $pi - 1$ on the right hand side ($pi = 0$ corresponds to an ODE). In (36) and (37) we find derivatives of the perturbations up to second order and therefore the system (10)-(13) has perturbation index 3. Sharper error bounds can be found in [3], [4].

3.2 Perturbation analysis of a collision

In this section we consider the special case of one isolated inelastic collision of a single mass–point in the system. The main result (Theorem 3.3) will be a pointwise error estimate for the position and the velocity vectors after the collision. We begin by estimating the velocities right after the jump discontinuities (Lemma 3.2) and continue with the Lagrange multipliers in time intervals where the piecewise solution is smooth (Lemma 3.3). These results finally lead to the desired error bounds for the position and velocity vectors.

Lemma 3.2 *Suppose that* $(\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda}) \in \mathcal{C}^{2,1,0,0}([0, \overline{c}], \mathcal{V})$ *solves the DAE system (10)-(13) with* $\overline{c} > 0$ *and constant active set* $J_A^0 \subset J$ *(and* J_P *defined by* $J \backslash J_A^0$*) such that*

- *there exists a unique* $p \in J_P^0$ *with* $\overline{q}_p(t) > 0$ *for all* $t \in [0, \overline{c})$ *and* $\overline{q}_p(\overline{c}) = 0$
- $\overline{\lambda}_j > 0$ *for all* $j \in J_A^0$ *and* $\overline{q}_j > 0$ *for all* $j \in J_P^0 \backslash \{p\}$ *in* $[0, \overline{c}]$

and let $\overline{u}^*(\overline{c})$, $\overline{\nu}^*(\overline{c})$, *and* $\overline{\lambda}_j^*(\overline{c}) > 0$ *for* $j \in J_A^0 \cup \{p\}$ *be the unique solution of (21) and (16). Furthermore, let* $K_s > 0$ *be given and suppose that the rank condition (22) holds for all* $(t,q) \in (-\varepsilon, c_\varepsilon) \times \mathbb{R}^{3d}$ *with* $\varepsilon > 0$, $c_\varepsilon > \overline{c}$. *Then there exist constants* $K, C^* > 0$ *such that the following holds:*
Let $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda}) \in \mathcal{C}^{2,1,0,0}([0,\widehat{c}], \mathcal{V})$ *with* $0 < \widehat{c} < c_\varepsilon$ *and* $|\widehat{c} - \overline{c}| \le K$ *be a solution of the perturbed DAE system (30)-(33) with active set* J_A^0 *and perturbations* $\theta \in \mathcal{C}^2([0,\widehat{c}], \mathbb{R}^d)$, $\eta \in \mathcal{C}^2([0,\widehat{c}], \mathbb{R}^m)$, $\delta \in \mathcal{C}([0,\widehat{c}], \mathbb{R}^{3d})$ *with*

$$\max_{l=1,2} \|\theta^{(l)}(t)\|, \max_{l=1,2} \|\eta^{(l)}(t)\| \le K \quad \text{for } t \in [0,\widehat{c}].$$

Furthermore, suppose that

- $\widehat{q}_p(t) > \theta_p(t)$ *for all* $t \in [0,\widehat{c})$ *and* $\widehat{q}_p(\widehat{c}) = \theta_p(\widehat{c})$
- $\widehat{\lambda}_j > 0$ *for all* $j \in J_A^0$ *and* $\widehat{q}_j > \theta_j$ *for all* $j \in J_P^0 \backslash \{p\}$ *in* $[0,\widehat{c}]$
- $\|\widehat{q}(t) - \overline{q}(t)\| \le K$ *for* $t \in [0, c_{min}]$ *and* $\|\widehat{q}(t)\| + \|\widehat{u}(t)\| + \|\dot{\widehat{u}}(t)\| \le K_s$ *for* $t \in [0,\widehat{c}]$

and let $\widehat{u}^*(\widehat{c})$, $\widehat{\nu}^*(\widehat{c})$, *and* $\widehat{\lambda}_j^*(\widehat{c}) > 0$ *for* $j \in J_A^0 \cup \{p\}$ *be the unique solution of (28) and (29). Then the following estimate holds for all* $i \in I$*:*

$$\begin{aligned} |\widehat{u}_i^*(\widehat{c}) - \overline{u}_i^*(\overline{c})| \quad &\le \quad C^* \Big(|\widehat{c} - \overline{c}| + \sum_{j \in I} (|\widehat{q}_j(c_{min}) - \overline{q}_j(c_{min})| + |\widehat{u}_j(c_{min}) - \overline{u}_j(c_{min})|) \\ &\qquad + \sum_{k=1}^{m} |\dot{\eta}_k(c_{min})| + \sum_{j \in J_A^0 \cup \{p\}} |\dot{\theta}_j(c_{min})| \Big). \end{aligned} \tag{39}$$

Proof
The jump conditions (21) and (28) for the original and the perturbed system yield

$$\Gamma(\overline{c}, \overline{q}(\overline{c})) \begin{pmatrix} \overline{u}^*(\overline{c}) \\ -\overline{I}_\nu \\ -\overline{I}_p^\lambda \end{pmatrix} = \begin{pmatrix} M\overline{u}(\overline{c}) \\ -g_t(\overline{c}, \overline{q}(\overline{c})) \\ 0 \end{pmatrix}, \quad \Gamma(\widehat{c}, \widehat{q}(\widehat{c})) \begin{pmatrix} \widehat{u}^*(\widehat{c}) \\ -\widehat{I}_\nu \\ -\widehat{I}_p^\lambda \end{pmatrix} = \begin{pmatrix} M\widehat{u}(\widehat{c}) \\ -g_t(\widehat{c}, \widehat{q}(\widehat{c})) + \dot{\eta}(\widehat{c}) \\ A^*(0) H^T \dot{\theta}(\widehat{c}) \end{pmatrix},$$

$$\text{where} \quad \Gamma(t,q) := \begin{pmatrix} M & G(t,q)^T & A^*(0)^T \\ G(t,q) & 0 & 0 \\ A^*(0) & 0 & 0 \end{pmatrix} \tag{40}$$

and $A^*(0)$ corresponds to the active set $J_A^0 \cup \{p\}$. The rank condition (22) holds, by Lemma 2.1 the matrix $\Gamma(t,q)$ is nonsingular for $(t,q) = (\overline{c}, \overline{q}(\overline{c}))$ and $(t,q) = (\widehat{c}, \widehat{q}(\widehat{c}))$, and subtracting the left from the right equation in (40) yields for $\widehat{c} \le \overline{c}$

$$\begin{aligned} \begin{pmatrix} w^* \\ \overline{I}_\nu - \widehat{I}_\nu \\ \overline{I}_p^\lambda - \widehat{I}_p^\lambda \end{pmatrix} \quad &= \quad \Gamma^{-1}(\widehat{c}, \widehat{q}(\widehat{c})) \begin{pmatrix} M\widehat{u}(\widehat{c}) \\ -g_t(\widehat{c}, \widehat{q}(\widehat{c})) + \dot{\eta}(\widehat{c}) \\ A^*(0) H^T \dot{\theta}(\widehat{c}) \end{pmatrix} - \Gamma^{-1}(\overline{c}, \overline{q}(\overline{c})) \begin{pmatrix} M\widehat{u}(\widehat{c}) \\ -g_t(\widehat{c}, \widehat{q}(\widehat{c})) + \dot{\eta}(\widehat{c}) \\ A^*(0) H^T \dot{\theta}(\widehat{c}) \end{pmatrix} \\ &\quad + \Gamma^{-1}(\overline{c}, \overline{q}(\overline{c})) \begin{pmatrix} M\widehat{u}(\widehat{c}) \\ -g_t(\widehat{c}, \widehat{q}(\widehat{c})) + \dot{\eta}(\widehat{c}) \\ A^*(0) H^T \dot{\theta}(\widehat{c}) \end{pmatrix} - \Gamma^{-1}(\overline{c}, \overline{q}(\overline{c})) \begin{pmatrix} M\overline{u}(\overline{c}) \\ -g_t(\overline{c}, \overline{q}(\overline{c})) \\ 0 \end{pmatrix} \end{aligned}$$

with $w^* := \widehat{u}^*(\widehat{c}) - \overline{u}^*(\overline{c})$. Since $\overline{q}(t)$ and $\widehat{q}(t)$ are bounded in $[0, \overline{c}]$ and $[0, \widehat{c}]$, respectively, they are contained in a compact subset $Q \subset \mathbb{R}^{3d}$. In $[c_{\min}, c_{\max}] \times Q$ the functions g_t, G, and therefore also Γ have continuous derivatives and are thus bounded and Lipschitz–continuous. We find

$$\begin{aligned}\|\Gamma(\widehat{c}, \widehat{q}(\widehat{c})) - \Gamma(\overline{c}, \overline{q}(\overline{c}))\| &\leq C_1 (|\widehat{c} - \overline{c}| + \|\widehat{q}(\widehat{c}) - \overline{q}(\overline{c})\|) \\ &\leq C_2 (|\widehat{c} - \overline{c}| + \|\widehat{q}(\widehat{c}) - \overline{q}(\widehat{c})\|) = \mathcal{O}(K)\end{aligned}$$

with constants $C_1, C_2 > 0$, and a perturbation lemma (see e.g. [22]) also yields

$$\|\Gamma^{-1}(\widehat{c}, \widehat{q}(\widehat{c})) - \Gamma^{-1}(\overline{c}, \overline{q}(\overline{c}))\| \leq C_3 (|\widehat{c} - \overline{c}| + \|\widehat{q}(\widehat{c}) - \overline{q}(\widehat{c})\|)$$

with $C_3 > 0$. Using all the properties listed above we find a constant $C > 0$ such that the following estimate holds for all $i \in I$:

$$|w_i^*| \leq C\Big(|\widehat{c} - \overline{c}| + \sum_{j \in I} |\widehat{q}_j(\widehat{c}) - \overline{q}_j(\widehat{c})| + \sum_{j \in I} |\widehat{u}_j(\widehat{c}) - \overline{u}_j(\widehat{c})| + \sum_{k=1}^{m} |\dot{\eta}_k(\widehat{c})| + \sum_{j \in J_A^0 \cup \{p\}} |\dot{\theta}_j(\widehat{c})|\Big)$$

This finally gives estimate (39), and in case $\overline{c} < \widehat{c}$ we find a similar result.

□

Lemma 3.3 *Suppose that* $(\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda}) \in \mathcal{C}^{2,1,0,0}([\alpha, \beta], \mathcal{V})$ *solves the DAE system (10)-(13) with* $0 \leq \alpha < \beta$ *and the constant active set* $J_A^\alpha \subset J$ *(and* J_P^α *defined by* $J \backslash J_A^\alpha$*) such that* $\overline{\lambda}_j \geq 0$ *for all* $j \in J_A^\alpha$ *and* $\overline{q}_j \geq 0$ *for all* $j \in J_P^\alpha$ *in* $[\alpha, \beta]$*. Furthermore, suppose that the rank condition (22) holds for all* $(t, q) \in (\alpha_\varepsilon, \beta_\varepsilon) \times \mathbb{R}^{3d}$ *with* $\alpha_\varepsilon < \alpha$, $\beta_\varepsilon > \beta$*. Then there exist constants* $K, C^* > 0$ *such that the following holds:*
Let $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda}) \in \mathcal{C}^{2,1,0,0}([\alpha, \beta], \mathcal{V})$ *be a solution of the perturbed DAE system (30)-(33) with the active set* J_A^α *and the perturbations* $\theta \in \mathcal{C}^2([\alpha, \beta], \mathbb{R}^d)$, $\eta \in \mathcal{C}^2([\alpha, \beta], \mathbb{R}^m)$, $\delta \in \mathcal{C}([\alpha, \beta], \mathbb{R}^{3d})$ *with*

$$\max_{l=1,2} \|\theta^{(l)}(t)\|, \max_{l=1,2} \|\eta^{(l)}(t)\| \leq K \quad \textit{for } t \in [\alpha, \beta].$$

Furthermore, suppose that $\widehat{\lambda}_j \geq 0$ *for* $j \in J_A^\alpha$ *and* $\widehat{q}_j \geq \theta_j$ *for* $j \in J_P^\alpha$ *in* $[\alpha, \beta]$*, and let*

$$\|\widehat{q}(t) - \overline{q}(t)\| + \|\widehat{u}(t) - \overline{u}(t)\| \leq K \quad \textit{for } t \in [\alpha, \beta]. \tag{41}$$

Then the following estimate holds for all $t \in [\alpha, \beta]$, $j \in J$, *and* $k = 1 \ldots m$:

$$|\widehat{\lambda}_j(t) - \overline{\lambda}_j(t)|,\ |\widehat{\nu}_k(t) - \overline{\nu}_k(t)| \leq C^* \Big(\sum_{i \in I} (|\widehat{q}_i(t) - \overline{q}_i(t)| + |\widehat{u}_i(t) - \overline{u}_i(t)|) + h(t)\Big), \tag{42}$$

$$\textit{where} \quad h(t) = \sum_{i \in J_A^\alpha} |\ddot{\theta}_i(t)| + \sum_{k=1}^{m} |\ddot{\eta}_k(t)| + \sum_{i \in I} |\delta_i(t)| \tag{43}$$

Proof
To simplify notations we set $w_i := \widehat{q}_i - \overline{q}_i$ and $\dot{w}_i := \widehat{u}_i - \overline{u}_i$ for $i \in I$. For $t \in [\alpha, \beta]$ the second order hidden constraints yield

$$\widetilde{g}(t, \widehat{q}, \widehat{u}) + G(t, \widehat{q})\dot{\widehat{u}} - \widetilde{g}(t, \overline{q}, \overline{u}) - G(t, \overline{q})\dot{\overline{u}} = \ddot{\eta}, \quad A(\alpha)\ddot{w} = A(\alpha)H^T\ddot{\theta}$$

with $\widetilde{g}$ defined in (17), and using $0 = G(t,\widehat{q})\dot{\overline{u}} - G(t,\widehat{q})\dot{\overline{u}}$ in the first equation yields

$$\begin{pmatrix} G(t,\widehat{q}) \\ A(\alpha) \end{pmatrix} \ddot{w} + \begin{pmatrix} \widetilde{g}(t,\widehat{q},\widehat{u}) - \widetilde{g}(t,\overline{q},\overline{u}) \\ 0 \end{pmatrix} + \begin{pmatrix} G(t,\widehat{q}) - G(t,\overline{q}) \\ 0 \end{pmatrix} \dot{\overline{u}} = \begin{pmatrix} \ddot{\eta} \\ A(\alpha)H^T\ddot{\theta} \end{pmatrix}.$$

Substituting

$$\begin{aligned} \ddot{w} \;=\; & M^{-1}\Big(f(t,\widehat{q},\widehat{u}) - f(t,\overline{q},\overline{u}) + G(t,\widehat{q})^T\widehat{\nu} - G(t,\overline{q})^T\overline{\nu} \\ & + G(t,\overline{q})^T\overline{\nu} - G(t,\widehat{q})^T\overline{\nu} + A(\alpha)^T(\widehat{\lambda} - \overline{\lambda}) + \delta\Big) \end{aligned}$$

immediately gives

$$\begin{aligned} & \begin{pmatrix} G(t,\widehat{q}) \\ A(\alpha) \end{pmatrix} \Big[M^{-1}A(\alpha)^T(\widehat{\lambda} - \overline{\lambda}) + M^{-1}G(t,\widehat{q})^T(\widehat{\nu} - \overline{\nu})\Big] \\ & = \begin{pmatrix} \ddot{\eta} \\ A(\alpha)H^T\ddot{\theta} \end{pmatrix} - \begin{pmatrix} \widetilde{g}(t,\widehat{q},\widehat{u}) - \widetilde{g}(t,\overline{q},\overline{u}) \\ 0 \end{pmatrix} - \begin{pmatrix} G(t,\widehat{q}) - G(t,\overline{q}) \\ 0 \end{pmatrix} \dot{\overline{u}} \\ & \quad - \begin{pmatrix} G(t,\widehat{q}) \\ A(\alpha) \end{pmatrix} M^{-1}\Big[f(t,\widehat{q},\widehat{u}) - f(t,\overline{q},\overline{u}) + (G(t,\widehat{q}) - G(t,\overline{q}))^T\,\overline{\nu} + \delta\Big]. \end{aligned}$$

The matrix $D(t,\widehat{q})$ defined in (23) is nonsingular by Lemma 2.1 and we find

$$\begin{aligned} \begin{pmatrix} \overline{\nu} - \widehat{\nu} \\ \widehat{\lambda} - \overline{\lambda} \end{pmatrix} \;=\; & D^{-1}(t,\widehat{q})\Bigg[\begin{pmatrix} \ddot{\eta} \\ A(\alpha)H^T\ddot{\theta} \end{pmatrix} \\ & - \begin{pmatrix} \widetilde{g}(t,\widehat{q},\widehat{u}) - \widetilde{g}(t,\overline{q},\overline{u}) \\ 0 \end{pmatrix} - \begin{pmatrix} G(t,\widehat{q}) - G(t,\overline{q}) \\ 0 \end{pmatrix} \dot{\overline{u}} \\ & - \begin{pmatrix} G(t,\widehat{q}) \\ A(\alpha) \end{pmatrix} M^{-1}\Big[f(t,\widehat{q},\widehat{u}) - f(t,\overline{q},\overline{u}) + (G(t,\widehat{q}) - G(t,\overline{q}))^T\,\overline{\nu} + \delta\Big]\Bigg]. \end{aligned} \tag{44}$$

Since $\overline{q}$, $\overline{u}$, $\overline{\nu}$, and $\dot{\overline{u}}$ are bounded in $[\alpha,\beta]$ we find $K_s > 0$ such that $\|\widehat{q}(t)\| + \|\widehat{u}(t)\| \leq K_s$ for all $t \in [\alpha,\beta]$ by (41), and thus $(\overline{q},\overline{u})$ and $(\widehat{q},\widehat{u})$ are contained in a compact subset $Q \subset \mathbb{R}^{3d} \times \mathbb{R}^{3d}$. In $[\alpha,\beta] \times Q$ the functions f, g, $\widetilde{g}$, G, and therefore also D have continuous derivatives and are thus bounded and Lipschitz–continuous. Applying a perturbation lemma (see e.g. [22]) yields the boundedness of $\|D^{-1}(t,\widehat{q}(t))\|$ since

$$\|D(t,\widehat{q}(t)) - D(t,\overline{q}(t))\| \leq C\|\widehat{q}(t) - \overline{q}(t)\| = \mathcal{O}(K)$$

by (41) with some constant $C > 0$. All properties listed above finally lead to estimate (42) by taking norms in (44) and using $\widehat{\lambda}_j = \overline{\lambda}_j = 0$ in $[\alpha,\beta]$ for $j \in J_P^{\alpha}$.

□

The following important auxiliary lemma for real functions will be needed in Theorem 3.3 and Theorem 3.5 to estimate different contact and lift–off times:

Lemma 3.4 *Suppose that $\overline{\varphi} \in \mathcal{C}^2([0,\tau],\mathbb{R})$ with $\tau > 0$ and $\overline{c} \in (0,\tau)$ are given such that $\overline{\varphi}(t) > 0$ for all $t \in [0,\overline{c})$, $\overline{\varphi}(\overline{c}) = 0$, $\dot{\overline{\varphi}}(\overline{c}) < 0$, and $\overline{\varphi}(t) < 0$ for all $t \in (\overline{c},\tau]$. Then there* *exist constants $K, \rho, C^* > 0$ such that for any $\widehat{\varphi} \in \mathcal{C}^1([0,\tau],\mathbb{R})$ and any perturbation $\zeta \in \mathcal{C}^1([0,\tau],\mathbb{R})$ with*

$$|\widehat{\varphi}(t) - \overline{\varphi}(t)| + |\dot{\widehat{\varphi}}(t) - \dot{\overline{\varphi}}(t)| \le K, \quad |\zeta(t)|, |\dot{\zeta}(t)| \le K \quad \text{for } t \in [0,\tau]$$

the following holds: There exists a unique $\widehat{c} \in [\overline{c}-\rho, \overline{c}+\rho] \subset (0,\tau)$ such that $\widehat{\varphi}(t) > \zeta(t)$ for all $t \in [0,\widehat{c})$, $\widehat{\varphi}(\widehat{c}) = \zeta(\widehat{c})$, $\dot{\widehat{\varphi}}(\widehat{c}) < \dot{\zeta}(\widehat{c})$, and $\widehat{\varphi}(t) < \zeta(t)$ for all $t \in (\widehat{c}, \overline{c}+\rho]$. Furthermore, the following estimate holds:

$$|\widehat{c} - \overline{c}| \le C^*(|\widehat{\varphi}(c_{min}) - \overline{\varphi}(c_{min})| + |\zeta(c_{min})|). \tag{45}$$

Proof

In order to apply Lemma 3.1 we show that (a)-(c) hold. It is sufficient to have $\zeta \equiv 0$, since one can then apply the result to $\widehat{\varphi} - \zeta$. With a constant $\rho > 0$ we define the set $S := \{\epsilon : |\epsilon| \le \rho\}$ and the function $T(\epsilon) := \widehat{\varphi}(\overline{c} + \epsilon)$, and for ρ small enough we obtain $\overline{c} + \epsilon \in (0,\tau)$ and $\dot{\overline{\varphi}}(\overline{c}+\epsilon) < 0$ for all $\epsilon \in S$.

(a) For $K < (1/4)|\dot{\overline{\varphi}}(\overline{c})|$ we find

$$|\dot{\widehat{\varphi}}(\overline{c}) - \dot{\overline{\varphi}}(\overline{c})| < \frac{1}{4}|\dot{\overline{\varphi}}(\overline{c})| \implies \frac{5}{4}\dot{\overline{\varphi}}(\overline{c}) < \dot{\widehat{\varphi}}(\overline{c}) < \frac{3}{4}\dot{\overline{\varphi}}(\overline{c}) < 0 \implies 0 < \frac{1}{4}|\dot{\overline{\varphi}}(\overline{c})| < |\dot{\widehat{\varphi}}(\overline{c})|.$$

Therefore there exists a $\sigma > 0$ such that $|T'(0)^{-1}| = |\dot{\widehat{\varphi}}(\overline{c})|^{-1} < \sigma$ with $\sigma := 4/|\dot{\overline{\varphi}}(\overline{c})|$.

(b) For ρ and K small enough we find constants $\kappa, C > 0$ (where $C > 0$ is a constant that depends only on $\ddot{\overline{\varphi}}$) such that

$$\begin{aligned} |\dot{T}(0) - \dot{T}(\epsilon)| &= |\dot{\widehat{\varphi}}(\overline{c}) - \dot{\widehat{\varphi}}(\overline{c}+\epsilon)| \\ &\le |\dot{\widehat{\varphi}}(\overline{c}) - \dot{\overline{\varphi}}(\overline{c})| + |\dot{\overline{\varphi}}(\overline{c}) - \dot{\overline{\varphi}}(\overline{c}+\epsilon)| + |\dot{\overline{\varphi}}(\overline{c}+\epsilon) - \dot{\widehat{\varphi}}(\overline{c}+\epsilon)| \\ &\le C\,(\epsilon + K) \le \kappa < \frac{1}{\sigma}. \end{aligned}$$

(c) Using $\overline{\varphi}(\overline{c}) = 0$ we immediately find for K small enough

$$|T(0)| = |\widehat{\varphi}(\overline{c})| = |\widehat{\varphi}(\overline{c}) - \overline{\varphi}(\overline{c})| \le \left(\frac{1}{\sigma} - \kappa\right)\rho.$$

The assumptions (a)-(c) of Lemma 3.1 are fulfilled and we find a unique $\widehat{c} \in (0,\tau)$ with $(\widehat{c} - \overline{c}) \in S$ and $T(\widehat{c} - \overline{c}) = \widehat{\varphi}(\widehat{c}) = 0$. Furthermore, the estimate

$$|\widehat{c} - \overline{c}| \le \frac{\sigma}{1-\sigma\kappa}|T(0) - T(\widehat{c}-\overline{c})| = \frac{\sigma}{1-\sigma\kappa}|\widehat{\varphi}(\overline{c})| = \frac{\sigma}{1-\sigma\kappa}\left(|\widehat{\varphi}(\overline{c}) - \overline{\varphi}(\overline{c})|\right)$$

holds with $\overline{\varphi}(\overline{c}) = 0$ for $\widehat{c} \ge \overline{c}$, and for $\widehat{c} < \overline{c}$ we find

$$\overline{\varphi}(\overline{c}) = \overline{\varphi}(\widehat{c}) + \dot{\overline{\varphi}}(\xi)(\overline{c} - \widehat{c}) \quad \implies \quad |\overline{c} - \widehat{c}| \le C\left(|\overline{\varphi}(\widehat{c}) - \widehat{\varphi}(\widehat{c})|\right)$$

with $\xi \in [\widehat{c}, \overline{c}]$ and a constant $C > 0$, where again $\overline{\varphi}(\overline{c}) = 0$ and $\overline{\varphi}(\widehat{c}) = \overline{\varphi}(\widehat{c}) - \widehat{\varphi}(\widehat{c})$ were used. For $K \le (1/4)\inf_{\xi\in[0,\overline{c}-\rho)}\overline{\varphi}(\xi)$ we find

$$|\widehat{\varphi}(t) - \overline{\varphi}(t)| \le \frac{1}{4}\overline{\varphi}(t) \quad \implies \quad 0 < \frac{3}{4}\overline{\varphi}(t) \le \widehat{\varphi}(t) \le \frac{5}{4}\overline{\varphi}(t) \quad \implies \quad 0 < \frac{1}{4}\overline{\varphi}(t) < \widehat{\varphi}(t)$$

for $t \in [0, \overline{c} - \rho)$, i.e. $\widehat{\varphi} > 0$ in $[0, \overline{c} - \rho)$. In a similar way we find $\dot{\widehat{\varphi}} < 0$ in $[\overline{c} - \rho, \overline{c} + \rho]$ and $\widehat{\varphi} < 0$ in $(\overline{c} + \rho, \tau]$ for K small enough. The function $\widehat{\varphi}$ is continuous and $\widehat{c}$ is unique in $[\overline{c} - \rho, \overline{c} + \rho]$, this finally gives $\widehat{\varphi} > 0$ in $[\overline{c} - \rho, \widehat{c})$ and $\widehat{\varphi} < 0$ in $(\widehat{c}, \overline{c} + \rho]$.

□

The next theorem states the following: Suppose that we have a piecewise solution of the original system (5)-(8) with exactly one collision. Then there exists a unique piecewise solution of the perturbed system (24)-(27) with the same properties if the perturbations are small enough.

Theorem 3.2 *Suppose that* $(\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda}) \in \mathcal{K}^0([0, T], \overline{\mathcal{U}}; \mathcal{V})$ *is a piecewise solution of the original system (5)-(8), (21) with* $T > 0$*, active and passive sets* J_A*,* J_P*, and event set* $\overline{\mathcal{U}} = \{\overline{c}\}$*,* $0 < \overline{c} < T$ *such that*

(a1) there exists a unique $p \in J_P^0$ *with* $\overline{q}_p(t) > 0$ *for all* $t \in [0, \overline{c})$*,* $\overline{u}_p^-(\overline{c}) < 0$*,* $\overline{\lambda}_p^+(\overline{c}) > 0$*, and* $\overline{\lambda}_p(t) > 0$ *for all* $t \in (\overline{c}, T]$

(a2) $\overline{\lambda}_j^-(\overline{c}) > 0$*,* $\overline{\lambda}_j^+(\overline{c}) > 0$*, and* $\overline{\lambda}_j(t) > 0$ *holds for all* $j \in J_A^0$ *and* $t \in [0, T]$

(a3) $\overline{q}_j(t) > 0$ *holds for all* $j \in J_P^0 \backslash \{p\}$ *and* $t \in [0, T]$

Furthermore, suppose that the rank condition (22) holds for all $(t, q) \in (-\varepsilon, T_\varepsilon) \times \mathbb{R}^{3d}$ *with* $\varepsilon > 0$*,* $T_\varepsilon > T$*. Then there exist constants* $K, K_s, C^* > 0$ *such that for perturbations* $\theta \in \mathcal{C}^2([0, T_\varepsilon], \mathbb{R}^d)$*,* $\eta \in \mathcal{C}^2([0, T_\varepsilon], \mathbb{R}^m)$*,* $\delta \in \mathcal{C}([0, T_\varepsilon], \mathbb{R}^{3d})$ *with*

$$\max_{l=0,1,2} \|\theta^{(l)}(t)\|, \max_{l=0,1,2} \|\eta^{(l)}(t)\|, \|\delta(t)\| \leq K \quad \textit{for } t \in [0, T_\varepsilon] \tag{46}$$

and consistent initial values $(\widehat{q}^0, \widehat{u}^0, \widehat{\nu}^0, \widehat{\lambda}^0) \in \mathcal{V}$ *with (35) there exists a unique* $\widehat{c} \in (0, T)$ *and a unique piecewise solution* $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda}) \in \mathcal{K}^0([0, \tau), \widehat{\mathcal{U}}; \mathcal{V})$ *of the perturbed system (24)-(27), (28) with* $T < \tau \leq T_\varepsilon$ *and event set* $\widehat{\mathcal{U}} = \{\widehat{c}\}$*. It has the properties*

(p1) $\widehat{q}_p(t) > \theta_p(t)$ *for* $t \in [0, \widehat{c})$*,* $\widehat{u}_p^-(\widehat{c}) < \dot{\theta}_p(\widehat{c})$*,* $\widehat{\lambda}_p^+(\widehat{c}) > 0$*, and* $\widehat{\lambda}_p(t) > 0$ *for* $t \in (\widehat{c}, T]$

(p2) $\widehat{\lambda}_j^-(\widehat{c}) > 0$*,* $\widehat{\lambda}_j^+(\widehat{c}) > 0$*, and* $\widehat{\lambda}_j(t) > 0$ *holds for all* $j \in J_A^0$ *and* $t \in [0, T]$

(p3) $\widehat{q}_j(t) > \theta_j(t)$ *holds for all* $j \in J_P^0 \backslash \{p\}$ *and* $t \in [0, T]$

Furthermore, the following estimates and bounds hold for $t \in [0, T]$*:*

$$\|\widehat{q}(t) - \overline{q}(t)\| + \|\widehat{u}(t) - \overline{u}(t)\| + \|\widehat{\nu}(t) - \overline{\nu}(t)\| + \|\widehat{\lambda}(t) - \overline{\lambda}(t)\| = \mathcal{O}(K) \tag{47}$$

$$\|\widehat{q}(t)\| + \|\widehat{u}(t)\| + \|\dot{\widehat{u}}(t)\| + \|\widehat{\nu}(t)\| + \|\widehat{\lambda}(t)\| \leq K_s \tag{48}$$

$$|\widehat{c} - \overline{c}| \leq C^* \Big(|\widehat{q}_p(c_{min}) - \overline{q}_p(c_{min})| + |\theta_p(c_{min})| \Big) = \mathcal{O}(K) \tag{49}$$

Proof
The solution $(\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda}) \in \mathcal{K}^0([0, T], \overline{\mathcal{U}}; \mathcal{V})$ with $\overline{\lambda}_j = 0$ for $j \in J_P^0$ solves the unperturbed DAE system (10)-(13) with the active set J_A^0 in $[0, \overline{c})$, it can be extended continuously to $[0, \overline{c}]$. By the local existence and uniqueness theorem of Picard–Lindelöf there exists a constant $\rho^R > 0$ such that the solution of the DAE can be extended (to the right)

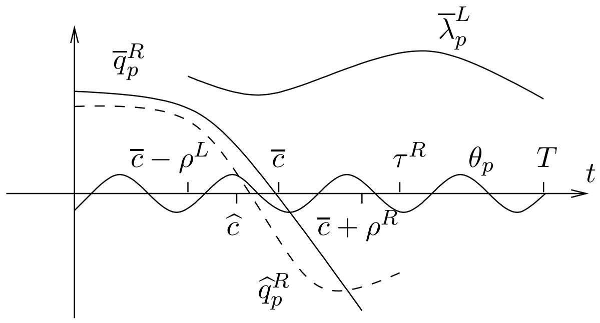

Figure 1: Sketch corresponding to the proof of Theorem 3.2.

to $[0,\overline{c}+\rho^R]$ with $(\overline{q}^R,\overline{u}^R,\overline{\nu}^R,\overline{\lambda}^R)=(\overline{q},\overline{u},\overline{\nu},\overline{\lambda})$ in $[0,\overline{c})$. Similarly, $(\overline{q},\overline{u},\overline{\nu},\overline{\lambda})$ solves (10)-(13) with the active set $J_A^0\cup\{p\}$ in $(\overline{c},T]$, the solution can be extended (to the left) to $[\overline{c}-\rho^L,T]$ with a constant $\rho^L>0$ and we find $(\overline{q}^L,\overline{u}^L,\overline{\nu}^L,\overline{\lambda}^L)=(\overline{q},\overline{u},\overline{\nu},\overline{\lambda})$ in $(\overline{c},T]$, see Figure 1. For $\rho'\leq\min\{\rho^R,\rho^L\}$ small enough we find $\overline{\lambda}_j^R,\overline{\lambda}_j^L>0$ for $j\in J_A^0$ and $\overline{q}_j^R,\overline{q}_j^L>0$ for $j\in J_P^0\backslash\{p\}$ in $[\overline{c}-\rho',\overline{c}+\rho']$ by assumptions (a2),(a3). Furthermore, we find $\overline{\lambda}_p^L>0$, $\overline{u}_p^R<0$ in $[\overline{c}-\rho',\overline{c}+\rho']$ by assumption (a1) and this immediately yields $\overline{q}_p^R<0$ in $(\overline{c},\overline{c}+\rho']$ by continuity if ρ' is small enough.
From Theorem 3.1 we obtain constants $K^R,K_s^R>0$ such that for $K\leq K^R$ there exists a unique solution $(\widehat{q}^R,\widehat{u}^R,\widehat{\nu}^R,\widehat{\lambda}^R)$ of the perturbed DAE system (30)-(33) with the active set J_A^0 in $[0,\tau^R)$, where $\overline{c}+\rho^R<\tau^R\leq T_\varepsilon$. Furthermore, the estimate

$$\|\widehat{q}^R(t)-\overline{q}^R(t)\|+\|\widehat{u}^R(t)-\overline{u}^R(t)\|+\|\widehat{\nu}^R(t)-\overline{\nu}^R(t)\|+\|\widehat{\lambda}^R(t)-\overline{\lambda}^R(t)\|=\mathcal{O}(K)\quad(50)$$

holds for $t\in[0,\overline{c}+\rho^R]$, and $(\widehat{q}^R,\widehat{u}^R,\widehat{\nu}^R,\widehat{\lambda}^R)$ as well as $\dot{\widehat{u}}^R$ are bounded by K_s^R. As in the proof of Lemma 3.4 estimate (50) yields $\widehat{\lambda}_j^R>0$ for $j\in J_A^0$ and $\widehat{q}_j^R>\theta_j$ for $j\in J_P\backslash\{p\}$ in $[0,\overline{c}+\rho']$.
For K small enough Lemma 3.4 with $\overline{\varphi}=\overline{q}_p^R$, $\widehat{\varphi}=\widehat{q}_p^R$, $\zeta=\theta_p$, and $\tau=\overline{c}+\rho'$ yields a constant $\rho>0$ with $\rho<\rho'$ and a unique $\widehat{c}\in[\overline{c}-\rho,\overline{c}+\rho]$ such that $\widehat{q}_p^R>\theta_p$ in $[0,\widehat{c})$, $\widehat{q}_p^R(\widehat{c})=\theta_p(\widehat{c})$, $\widehat{u}_p^R(\widehat{c})<\dot{\theta}_p(\widehat{c})$, and $\widehat{q}_p^R<\theta_p$ in $(\widehat{c},\overline{c}+\rho]$. Furthermore, we have the estimate (49), where $|\widehat{c}-\overline{c}|=\mathcal{O}(K)$ follows from (50). We set $(\widehat{q},\widehat{u},\widehat{\nu},\widehat{\lambda}):=(\widehat{q}^R,\widehat{u}^R,\widehat{\nu}^R,\widehat{\lambda}^R)$ in $[0,\widehat{c})$ and with the active set $J_A^*:=J_A\cup\{p\}$ we determine $\widehat{u}^*(\widehat{c})$ by (28) and $\widehat{\nu}^*(\widehat{c})$, $\widehat{\lambda}^*(\widehat{c})$ by (29). This yields

$$\begin{aligned}
\|\widehat{q}(\widehat{c})-\overline{q}^L(\widehat{c})\| &\leq \|\widehat{q}(\widehat{c})-\overline{q}^R(\widehat{c})\|+\|\overline{q}^R(\widehat{c})-\overline{q}^L(\widehat{c})\|\\
&\leq \mathcal{O}(K)+\|\overline{q}^R(\overline{c})-\overline{q}^L(\overline{c})\|+C_1|\widehat{c}-\overline{c}|=\mathcal{O}(K)\\
\|\widehat{u}^*(\widehat{c})-\overline{u}^L(\widehat{c})\| &\leq \|\widehat{u}^*(\widehat{c})-\overline{u}^+(\overline{c})\|+C_2|\widehat{c}-\overline{c}|=\mathcal{O}(K)
\end{aligned}$$

with some constants $C_1,C_2>0$, where estimates (49), (50) and (39) from Lemma 3.2 were used. Now Theorem 3.1 can be applied again: There exist constants $K^L,K_s^L>0$ such that for $K\leq K^L$ we find a unique solution $(\widehat{q},\widehat{u},\widehat{\nu},\widehat{\lambda})$ of the perturbed DAE system (30)-(33) with the active set J_A^* in $[\widehat{c},\tau)$, where $T<\tau\leq T_\varepsilon$. Furthermore, $(\widehat{q},\widehat{u},\widehat{\nu},\widehat{\lambda})$ and $\dot{\widehat{u}}$ are bounded by K_s^L and (48) holds with $K_s:=\max\{K_s^R,K_s^L\}$. Finally, estimate

$$\|\widehat{q}(t)-\overline{q}^L(t)\|+\|\widehat{u}(t)-\overline{u}^L(t)\|+\|\widehat{\nu}(t)-\overline{\nu}^L(t)\|+\|\widehat{\lambda}(t)-\overline{\lambda}^L(t)\|=\mathcal{O}(K)$$

holds for all $t \in [\widehat{c}, T]$ and as in the proof of Lemma 3.4 we can show that $\widehat{q}_j > \theta_j$ for $j \in J_P^0 \backslash \{p\}$ and $\widehat{\lambda}_j > 0$ for $j \in J_A^0 \cup \{p\}$ in $[\widehat{c}, T]$ by the above estimate.
Now suppose that $(\widetilde{q}, \widetilde{u}, \widetilde{\nu}, \widetilde{\lambda}) \in \mathcal{K}^0([0, \widetilde{\tau}), \widetilde{\mathcal{U}}; \mathcal{V})$ with $\widetilde{\tau} > 0$ and event set $\widetilde{\mathcal{U}}$ is another piecewise solution of the perturbed system (24)-(27), (28) with consistent initial values $(\widetilde{q}^0, \widetilde{u}^0, \widetilde{\nu}^0, \widetilde{\lambda}^0) = (\widehat{q}^0, \widehat{u}^0, \widehat{\nu}^0, \widehat{\lambda}^0) \in \mathcal{V}$. Since $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda})$ is the unique solution of a DAE system in $[0, \widehat{c})$ and properties (p1)–(p3) hold we find $(\widetilde{q}, \widetilde{u}, \widetilde{\nu}, \widetilde{\lambda}) = (\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda})$ in $[0, \widehat{c})$ and $\widehat{c}$ is the smallest element in $\widetilde{\mathcal{U}}$. The new initial values $\widetilde{u}^*(\widehat{c}) = \widehat{u}^*(\widehat{c})$, $\widetilde{\nu}^*(\widehat{c}) = \widehat{\nu}^*(\widehat{c})$, and $\widetilde{\lambda}^*(\widehat{c}) = \widehat{\lambda}^*(\widehat{c})$ are determined uniquely by (28) and (29). Since $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda})$ is the unique solution of a DAE system in $(\widehat{c}, \tau)$ and properties (p1)–(p3) hold we finally obtain $(\widetilde{q}, \widetilde{u}, \widetilde{\nu}, \widetilde{\lambda}) = (\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda})$ in $(\widehat{c}, \tau)$ and $\widetilde{\mathcal{U}} = \widehat{\mathcal{U}} = \{\widehat{c}\}$, $\widetilde{\tau} = \tau$.

□

Now existence and uniqueness of a piecewise solution of the perturbed system (24)-(27) with the desired properties is guaranteed and we derive pointwise error estimates for the position and velocity vectors after the collision. In the critical region $(c_{\min}, c_{\max})$ we can only show that the error of the velocity vector is bounded. However, this region becomes small as the perturbations tend to zero (see estimate (49) in Theorem 3.2), and this leads to a pointwise estimate for the position vector in $(c_{\min}, c_{\max})$.

Theorem 3.3 *Suppose that* $(\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda}) \in \mathcal{K}^0([0, T], \overline{\mathcal{U}}; \mathcal{V})$ *is a piecewise solution of the original system (5)-(8), (21) with* $T > 0$*, active and passive sets* J_A*,* J_P*, and event set* $\overline{\mathcal{U}} = \{\overline{c}\}$*,* $0 < \overline{c} < T$ *such that the assumptions (a1)-(a3) in Theorem 3.2 hold. Furthermore, suppose that the rank condition (22) holds for all* $(t, q) \in (-\varepsilon, T_\varepsilon) \times \mathbb{R}^{3d}$ *with* $\varepsilon > 0$*,* $T_\varepsilon > T$*.*
Then there exist constants $K, C^* > 0$ *such that for initial values* $(\widehat{q}^0, \widehat{u}^0, \widehat{\nu}^0, \widehat{\lambda}^0) \in \mathcal{V}$ *that fulfill the consistency conditions (35) and for perturbations* $\theta \in \mathcal{C}^2([0, T_\varepsilon], \mathbb{R}^d)$*,* $\eta \in \mathcal{C}^2([0, T_\varepsilon], \mathbb{R}^m)$*,* $\delta \in \mathcal{C}([0, T_\varepsilon], \mathbb{R}^{3d})$ *with (46) there exists a unique piecewise solution* $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda}) \in \mathcal{K}^0([0, T], \widehat{\mathcal{U}}; \mathcal{V})$ *of the perturbed system (24)-(27), (28) with the event set* $\widehat{\mathcal{U}} = \{\widehat{c}\}$*,* $0 < \widehat{c} < T$ *(see Theorem 3.2). Furthermore, defining*

$$h(t) := \sum_{i \in J_A^\alpha} |\ddot{\theta}_i(t)| + \sum_{k=1}^{m} |\ddot{\eta}_k(t)| + \sum_{i \in I} |\delta_i(t)| \quad \text{for } t \in [0, T] \tag{51}$$

$$\Phi(t) := \sum_{i \in I} (|\widehat{q}_i^0 - \overline{q}_i^0| + |\widehat{u}_i^0 - \overline{u}_i^0|) + \int_0^t h(\xi)\, d\xi \quad \text{for } t \in [0, c_{min}] \tag{52}$$

the following estimates hold with $c_{min} = \min\{\overline{c}, \widehat{c}\}$*,* $c_{max} = \max\{\overline{c}, \widehat{c}\}$*:*

$$t \in [0, c_{min}) : \quad \sum_{i \in I} (|\widehat{q}_i(t) - \overline{q}_i(t)| + |\widehat{u}_i(t) - \overline{u}_i(t)|) \le C^* \Phi(t) \tag{53}$$

$$t \in (c_{min}, c_{max}) : \quad \sum_{i \in I} |\widehat{q}_i(t) - \overline{q}_i(t)| \le C^* \Big(\Phi(c_{min}) + |\theta_p(c_{min})| + \sum_{j \in J_A^0} \int_{c_{min}}^t |\ddot{\theta}_j(\xi)|\, d\xi + \sum_{j \in J_P^0} \int_{c_{min}}^t |\delta_j(\xi)|\, d\xi \Big) \tag{54}$$

$$t \in (c_{min}, c_{max}) : \quad \sum_{i \in I} |\widehat{u}_i(t) - \overline{u}_i(t)| \le C^* \tag{55}$$

$$
\begin{aligned}
t \in (c_{max}, T]: \quad & \sum_{i \in I} \left(|\widehat{q}_i(t) - \overline{q}_i(t)| + |\widehat{u}_i(t) - \overline{u}_i(t)| \right) \\
& \le C^* \Bigg(\Phi(c_{min}) + |\theta_p(c_{min})| + |\dot{\theta}_p(c_{min})| + \sum_{k=1}^{m} |\dot{\eta}_k(c_{min})| \\
& \quad + \sum_{j \in J_A^0} \int_{c_{min}}^{c_{max}} |\ddot{\theta}_j(\xi)| \, d\xi + \sum_{j \in J_P^0} \int_{c_{min}}^{c_{max}} |\delta_j(\xi)| \, d\xi + \int_{c_{max}}^{t} h(\xi) \, d\xi \Bigg)
\end{aligned} \tag{56}
$$

Proof
By Theorem 3.2 the piecewise solution of the perturbed system exists and is unique in $[0, T]$. It has the properties (p1)-(p3) and (47)-(49) hold. Estimate (53) immediately follows from Theorem 3.1. To simplify the notations we set $w_i := \widehat{q}_i - \overline{q}_i$ with $w_i^0 := w_i(0)$ and $\dot{w}_i := \widehat{u}_i - \overline{u}_i$ with $\dot{w}_i^0 := \dot{w}_i(0)$ for $i \in I$. Furthermore, we omit the arguments of the perturbations and the solutions $(\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda})$, $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda})$ in the integrals and define $\overline{I}_i := \overline{u}_i^+(\overline{c}) - \overline{u}_i^-(\overline{c})$, $\widehat{I}_i := \widehat{u}_i^+(\widehat{c}) - \widehat{u}_i^-(\widehat{c})$ for $i \in I$. In case $\overline{c} < \widehat{c}$ we find

$$
\begin{aligned}
\dot{w}_i(T) &= \dot{w}_i^+(\widehat{c}) + \int_{\widehat{c}}^{T} \ddot{w}_i dt = \dot{w}_i^-(\widehat{c}) + \widehat{I}_i + \int_{\widehat{c}}^{T} \ddot{w}_i dt \\
&= \dot{w}_i^+(\overline{c}) + \int_{\overline{c}}^{\widehat{c}} \ddot{w}_i dt + \widehat{I}_i + \int_{\widehat{c}}^{T} \ddot{w}_i dt \\
&= \dot{w}_i^-(\overline{c}) - \overline{I}_i + \widehat{I}_i + \int_{\overline{c}}^{\widehat{c}} \ddot{w}_i dt + \int_{\widehat{c}}^{T} \ddot{w}_i dt \\
&= \dot{w}_i^0 - \overline{I}_i + \widehat{I}_i + \int_{0}^{\overline{c}} \ddot{w}_i dt + \int_{\overline{c}}^{\widehat{c}} \ddot{w}_i dt + \int_{\widehat{c}}^{T} \ddot{w}_i dt \\
w_i(T) &= w_i^0 + \int_0^T \dot{w}_i dt \\
&= w_i^0 + \int_0^{\overline{c}} \left(\dot{w}_i^0 + \int_0^t \ddot{w}_i d\xi \right) dt \\
&\quad + \int_{\overline{c}}^{\widehat{c}} \left(\dot{w}_i^+(\overline{c}) + \int_{\overline{c}}^{t} \ddot{w}_i d\xi \right) dt + \int_{\widehat{c}}^{T} \left(\dot{w}_i^+(\widehat{c}) + \int_{\widehat{c}}^{t} \ddot{w}_i d\xi \right) dt \\
&= w_i^0 + \overline{c}\dot{w}_i^0 + (\widehat{c} - \overline{c})\dot{w}_i^+(\overline{c}) + (T - \widehat{c})\dot{w}_i^+(\widehat{c}) + \Psi_i(T) \\
&= w_i^0 + \overline{c}\dot{w}_i^0 + (\widehat{c} - \overline{c}) \left(\dot{w}_i^-(\overline{c}) - \overline{I}_i \right) + (T - \widehat{c}) \left(\dot{w}_i^-(\widehat{c}) + \widehat{I}_i \right) + \Psi_i(T) \\
&= w_i^0 + \overline{c}\dot{w}_i^0 + (\widehat{c} - \overline{c}) \left(\dot{w}_i^0 + \int_0^{\overline{c}} \ddot{w}_i dt - \overline{I}_i \right) \\
&\quad + (T - \widehat{c}) \left(\dot{w}_i^0 - \overline{I}_i + \int_0^{\overline{c}} \ddot{w}_i dt + \int_{\overline{c}}^{\widehat{c}} \ddot{w}_i dt + \widehat{I}_i \right) + \Psi_i(T) \\
&= w_i^0 + T\dot{w}_i^0 - (T - \overline{c})\overline{I}_i + (T - \widehat{c})\widehat{I}_i + (T - \overline{c}) \int_0^{\overline{c}} \ddot{w}_i dt + (T - \widehat{c}) \int_{\overline{c}}^{\widehat{c}} \ddot{w}_i dt + \Psi_i(T)
\end{aligned}
$$

$$
\text{where} \quad \Psi_i(T) := \int_0^{\overline{c}} \int_0^t \ddot{w}_i d\xi dt + \int_{\overline{c}}^{\widehat{c}} \int_{\overline{c}}^{t} \ddot{w}_i d\xi dt + \int_{\widehat{c}}^{T} \int_{\widehat{c}}^{t} \ddot{w}_i d\xi dt
$$

and in case $\overline{c} = \widehat{c}$ as well as $\overline{c} < \widehat{c}$ we find the same result by reversing the roles of $\widehat{c}$ and $\overline{c}$. Taking absolute values and adding both equations yields

$$
|w_i(T)| + |\dot{w}_i(T)| \le |w_i^0| + (1 + T)|\dot{w}_i^0| + (1 + T + \overline{c})|\widehat{I}_i - \overline{I}_i| + |\overline{c} - \widehat{c}||\widehat{I}_i|
$$

$$+(1+T)\int_0^{c_{\min}} |\ddot{w}_i|dt + (1+T-c_{\min})\int_{c_{\min}}^{c_{\max}} |\ddot{w}_i|dt$$
$$+(1+T-c_{\max})\int_{c_{\max}}^{T} |\ddot{w}_i|dt \tag{57}$$

with a constant $C > 0$, where $\overline{c}\widehat{I}_i - \overline{c}\widehat{I}_i = 0$ was used. Now we proceed by estimating the integrals on the right hand side of (57): Using the active constraints we find

$$\int_0^{c_{\min}} |\ddot{w}_i|dt = \int_0^{c_{\min}} |\ddot{\theta}_i|dt \quad \text{for } i \in J_A^0, \qquad \int_{c_{\max}}^{T} |\ddot{w}_i|dt = \int_{c_{\max}}^{T} |\ddot{\theta}_i|dt \quad \text{for } i \in J_A^0 \cup \{p\}.$$

By (48) the solutions $(\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda})$ and $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda})$ are bounded and therefore included in a compact subset of $\mathcal{V}$. The functions f, g, and their derivatives are continuous and therefore bounded and Lipschitz–continuous on this compact subset. Using estimate (42) from Lemma 3.3 with $\alpha = 0$, $\beta = c_{\min}$ we find for $i \in J_P^0$

$$\begin{aligned}
\int_0^{c_{\min}} |\ddot{w}_i|dt \;&\leq\; \frac{1}{m_i}\int_0^{c_{\min}} \Big(|f_i(t,\widehat{q},\widehat{u}) - f_i(t,\overline{q},\overline{u})| + |G(t,\widehat{q})^T(\widehat{\nu}-\overline{\nu})|_i \\
&\qquad + |(G(t,\widehat{q}) - G(t,\overline{q}))^T\overline{\nu}|_i + |\delta_i|\Big)dt \\
&\leq\; C_1\int_0^{c_{\min}} \left(\sum_{j\in I}(|w_j| + |\dot{w}_j|) + h(t) + \sum_{j\in I}|w_j| + |\delta_i|\right) dt \\
&\leq\; C_2\int_0^{c_{\min}} \left(h(t) + \sum_{j\in I}(|w_j| + |\dot{w}_j|)\right) dt
\end{aligned}$$

with constants $C_1, C_2 > 0$. Similarly, applying Lemma 3.3 with $\alpha = c_{\max}$, $\beta = T$ yields for $i \in J_P^0 \backslash \{p\}$

$$\int_{c_{\max}}^{T} |\ddot{w}_i|dt \;\leq\; C\int_{c_{\max}}^{T} \left(h(t) + \sum_{j\in I}(|w_j| + |\dot{w}_j|)\right) dt$$

with a constant $C > 0$. Furthermore, using the active constraints and estimate (49) from Theorem 3.2 gives

$$\begin{aligned}
\int_{c_{\min}}^{c_{\max}} |\ddot{w}_i|dt \;&=\; \int_{c_{\min}}^{c_{\max}} |\ddot{\theta}_i|dt \quad \text{for } i \in J_A^0, \\
\int_{c_{\min}}^{c_{\max}} |\ddot{w}_p|dt \;&\leq\; \frac{1}{m_p}\int_{c_{\min}}^{c_{\max}} \Big(|f_p(t,\widehat{q},\widehat{u}) - f_p(t,\overline{q},\overline{u})| + |\delta_p| \\
&\qquad + |G(\widehat{q})^T\widehat{\nu}|_p + |G(\overline{q})^T\overline{\nu}|_p + |\widehat{\lambda}_p| + |\overline{\lambda}_p|\Big)dt \\
&\leq\; C_1\left[\int_{c_{\min}}^{c_{\max}} \left(\sum_{j\in I}(|w_j| + |\dot{w}_j|) + |\delta_p|\right) dt + (c_{\max} - c_{\min})\right] \\
&\leq\; C_2\left[\int_{c_{\min}}^{c_{\max}} \left(\sum_{j\in I}(|w_j| + |\dot{w}_j|) + |\delta_p|\right) dt + |w_p^0| + \int_0^{c_{\min}} |\dot{w}_p|dt + |\theta_p(c_{\min})|\right]
\end{aligned}$$

with $C_1, C_2 > 0$, and for $i \in J_P^0 \backslash \{p\}$ we find in a similar way

$$\begin{aligned}
\int_{c_{\min}}^{c_{\max}} |\ddot{w}_i| dt &\leq \frac{1}{m_i} \int_{c_{\min}}^{c_{\max}} \left(|f_i(t, \widehat{q}, \widehat{u}) - f_i(t, \overline{q}, \overline{u})| + |\delta_i| + |G(\widehat{q})^T \widehat{\nu}|_i + |G(\overline{q})^T \overline{\nu}|_i \right) dt \\
&\leq C \left[\int_{c_{\min}}^{c_{\max}} \left(\sum_{j \in I} (|w_j| + |\dot{w}_j|) + |\delta_i| \right) dt + |w_p^0| + \int_0^{c_{\min}} |\dot{w}_p| dt + |\theta_p(c_{\min})| \right].
\end{aligned}$$

We proceed by estimating the terms $|\widehat{I}_i - \overline{I}_i|$ in (57) and for $j \in J_A^0$ we immediately find $\overline{I}_j(\overline{c}) = \widehat{I}_j(\widehat{c}) = 0$. Then estimates (39) from Lemma 3.2 and (49) from Theorem 3.2 yield for $i \in J_P^0$

$$\begin{aligned}
|\widehat{I}_i - \overline{I}_i| &= |\widehat{u}_i^+(\widehat{c}) - \widehat{u}_i^-(\widehat{c}) + \overline{u}_i^-(\overline{c}) - \overline{u}_i^+(\overline{c})| \\
&\leq |\widehat{u}_i^+(\widehat{c}) - \overline{u}_i^+(\overline{c})| + C_1(|\dot{w}_i^-(c_{\min})| + |\widehat{c} - \overline{c}|) \\
&\leq C_2 \Bigg(|\widehat{c} - \overline{c}| + \sum_{j \in I} (|w_j(c_{\min})| + |\dot{w}_j^-(c_{\min})|) \\
&\qquad + \sum_{k=1}^m |\dot{\eta}_k(c_{\min})| + \sum_{j \in J_A^0 \cup \{p\}} |\dot{\theta}_j(c_{\min})| \Bigg) \\
&\leq C_3 \Bigg(|\theta_p(c_{\min})| + \sum_{j \in I} (|w_j^0| + |\dot{w}_j^0|) + \sum_{j \in I} \int_0^{c_{\min}} (|\dot{w}_j| + |\ddot{w}_j|) \, dt \\
&\qquad + \sum_{k=1}^m |\dot{\eta}_k(c_{\min})| + |\dot{\theta}_p(c_{\min})| + \sum_{j \in J_A^0} \int_0^{c_{\min}} |\ddot{\theta}_j| dt \Bigg)
\end{aligned}$$

with some constants $C_1, C_2, C_3 > 0$, where $\dot{w}_j^0 = \dot{\theta}_j(0)$ for $j \in J_A^0$ was used. Taking sums on both sides of (57) and substituting the estimates for the integrals and the terms $|\widehat{I}_i - \overline{I}_i|$ we find

$$\begin{aligned}
\sum_{i \in I} (|w_i(T)| + |\dot{w}_i(T)|) &\leq C_1 \left(\sum_{i \in I} (|w_i^0| + |\dot{w}_i^0|) + \sum_{i \in J_P} (|\widehat{I}_i - \overline{I}_i|) \right) \\
&\quad + C_2 \left(\sum_{i \in I} \int_0^{c_{\min}} |\ddot{w}_i| dt + \int_{c_{\min}}^{c_{\max}} |\ddot{w}_i| dt + \int_{c_{\max}}^T |\ddot{w}_i| dt \right) \\
&\leq C_3 \psi(T) + C_4 \int_0^T \sum_{i \in I} (|w_i| + |\dot{w}_i|) \, dt \qquad (58)
\end{aligned}$$

$$\begin{aligned}
\text{where} \quad \psi(\tau) &:= \Phi(c_{\min}) + |\theta_p(c_{\min})| + |\dot{\theta}_p(c_{\min})| + \sum_{k=1}^m |\dot{\eta}_k(c_{\min})| \\
&\quad + \sum_{j \in J_A^0} \int_{c_{\min}}^{c_{\max}} |\ddot{\theta}_j| dt + \sum_{j \in J_P^0} \int_{c_{\min}}^{c_{\max}} |\delta_j| dt + \int_{c_{\max}}^{\tau} h(t) dt
\end{aligned}$$

is defined for $\tau \in (c_{\max}, T]$. Here $C_1, C_2, C_3, C_4 > 0$ are constants, and $\Phi(c_{\min})$ is defined in (52). In fact, following the proof we find the same estimate (58) with T replaced by

$\tau \in (c_{\max}, T]$. Moreover, defining

$$
\begin{aligned}
\psi(\tau) &:= \Phi(\tau) \quad \text{for } \tau \in [0, c_{\min}) \\
\psi(\tau) &:= \Phi(c_{\min}) + \sum_{i \in J_P^0} |\mathcal{I}_i| + \sum_{j \in J_A^0} \int_{c_{\min}}^{\tau} |\ddot{\theta}_j| dt + \sum_{j \in J_P^0} \int_{c_{\min}}^{\tau} |\delta_j| dt \quad \text{for } \tau \in (c_{\min}, c_{\max}) \\
\mathcal{I}_i &:= \begin{cases} \widehat{I}_i \ \text{for } c_{\min} = \widehat{c}, \\ \overline{I}_i \ \text{for } c_{\min} = \overline{c} \end{cases}
\end{aligned}
$$

the above estimate holds in the whole interval, i.e.

$$
\sum_{i \in I} (|w_i(\tau)| + |\dot{w}_i(\tau)|) \leq C_1 \psi(\tau) + C_2 \int_0^{\tau} \sum_{i \in I} (|w_i| + |\dot{w}_i|) \, dt
$$

for $\tau \in [0, T] \backslash \{\widehat{c}, \overline{c}\}$ with constants $C_1, C_2 > 0$. Notice that ψ is discontinuous and we have $\psi(\tau) = \mathcal{O}(1)$ for $\tau \in (c_{\min}, c_{\max})$. Now Gronwall's inequality leads to

$$
\sum_{i \in I} (|w_i(\tau)| + |\dot{w}_i(\tau)|) \leq C_1 \psi(\tau) + C_2 \int_0^{\tau} \psi(\xi) e^{C_2(\tau - \xi)} d\xi \tag{59}
$$

for $\tau \in [0, T]$. For $\tau \in (c_{\min}, c_{\max})$ we find

$$
\sum_{i \in I} (|w_i(\tau)| + |\dot{w}_i(\tau)|) \leq (C_1 + e^{C_2 \tau} - 1) \psi(\tau) \tag{60}
$$

with $\psi(\tau) \geq \psi(t)$ for all $t \in [0, \tau] \backslash \{c_{\min}\}$. The boundedness of $|\mathcal{I}_i|$, $i \in J_p^0$ also yields the bound for $|\dot{w}_i(\tau)|$, $i \in I$ in (55).
For $|w_i(\tau)|$, $i \in I$, $\tau \in (c_{\min}, c_{\max})$ we find with estimates (53) and (60)

$$
\begin{aligned}
\sum_{i \in I} |w_i(\tau)| &= \sum_{i \in I} \left(|w_i^0| + \int_0^{\tau} |\dot{w}_i(t)| dt \right) \leq \sum_{i \in I} \left(|w_i^0| + \int_0^{c_{\min}} |\dot{w}_i(t)| dt + \int_{c_{\min}}^{\tau} |\dot{w}_i(t)| dt \right) \\
&\leq C_3 \Phi(c_{\min}) + (\tau - c_{\min})(C_1 + e^{C_2 \tau} - 1) \psi(\tau) \\
&\leq C_4 \left(\Phi(c_{\min}) + (\tau - c_{\min}) \sum_{i \in J_P^0} |\mathcal{I}_i| + \sum_{j \in J_A^0} \int_{c_{\min}}^{\tau} |\ddot{\theta}_j| dt + \sum_{j \in J_P^0} \int_{c_{\min}}^{\tau} |\delta_j| dt \right),
\end{aligned}
$$

where $C_3, C_4 > 0$. Estimates (49) from Theorem 3.2 and (53) immediately give

$$
(\tau - c_{\min}) \leq C_5 (|w_p(c_{\min})| + |\theta_p(c_{\min})|) \leq C_6 (\Phi(c_{\min}) + |\theta_p(c_{\min})|)
$$

with $C_5, C_6 > 0$, and this finally leads to (54). For $\tau \in (c_{\max}, T]$ estimate (59) yields

$$
\begin{aligned}
\sum_{i \in I} (|w_i(\tau)| + |\dot{w}_i(\tau)|) &\leq C_1 \psi(\tau) + C_2 \int_0^{c_{\min}} \psi(\xi) e^{C_2(\tau - \xi)} d\xi \\
&\quad + C_2 \int_{c_{\min}}^{c_{\max}} \psi(\xi) e^{C_2(\tau - \xi)} d\xi + C_2 \int_{c_{\max}}^{\tau} \psi(\xi) e^{C_2(\tau - \xi)} d\xi \\
&\leq C_7 \Big(\psi(\tau) + (c_{\max} - c_{\min}) \sum_{i \in J_P^0} |\mathcal{I}_i| \Big) \\
&\leq C_8 (\psi(\tau) + |w_p(c_{\min})| + |\theta_p(c_{\min})|) \leq C_9 \psi(\tau)
\end{aligned}
$$

with constants $C_7, C_8, C_9 > 0$. Here estimates (49), (53), $\psi(\tau) \geq \psi(t)$ for all $t \in [0, c_{\min}) \cup (c_{\max}, \tau]$, and

$$\psi(t) \leq \psi(\tau) + \sum_{i \in J_P^0} |\mathcal{I}_i|$$

for $t \in (c_{\min}, c_{\max})$ were used. This finally leads to (56).

□

3.3 Perturbation analysis of a lift–off

We proceed as in the previous section and first prove the following: Suppose that we have a piecewise solution of the original unperturbed system (5)-(8) with exactly one lift–off of a single mass–point. Then there exists a unique piecewise solution of the perturbed system (24)-(27) with the same properties if the perturbations are small enough. The proof is similar to that of Theorem 3.2, but we apply Lemma 3.4 with $\overline{\varphi} = \overline{\lambda}_p$ instead of $\overline{\varphi} = \overline{q}_p$. Furthermore, the boundedness of the third order derivatives of the perturbations in the global holonomic and the unilateral constraints is required here, cf. Remark 3.1.

Theorem 3.4 *Suppose that $(\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda}) \in \mathcal{K}^2([0,T], \overline{\mathcal{U}}; \mathcal{V})$ is a piecewise solution of the original system (5)-(8) with $T > 0$, active and passive sets J_A, J_P, and event set $\overline{\mathcal{U}} = \{\overline{c}\}$, $0 < \overline{c} < T$ such that*

(a1) there exists a unique $p \in J_A^0$ such that $\overline{\lambda}_p(t) > 0$ for all $t \in [0, \overline{c})$, $\overline{\lambda}_p(\overline{c}) = 0$, $\dot{\overline{\lambda}}_p^-(\overline{c}) < 0$, $\ddot{\overline{u}}_p^+(\overline{c}) > 0$, and $\overline{q}_p(t) > 0$ for all $t \in (\overline{c}, T]$

(a2) $\overline{\lambda}_j(t) > 0$ holds for all $j \in J_A^0 \backslash \{p\}$ and $t \in [0, T]$

(a3) $\overline{q}_j(t) > 0$ holds for all $j \in J_P^0$ and $t \in [0, T]$

Furthermore, suppose that the rank condition (22) holds for all $(t, q) \in (-\varepsilon, T_\varepsilon) \times \mathbb{R}^{3d}$ with $\varepsilon > 0$, $T_\varepsilon > T$. Then there exist constants $K, K_s, C^* > 0$ such that for perturbations $\theta \in \mathcal{C}^2([0, T_\varepsilon], \mathbb{R}^d)$, $\eta \in \mathcal{C}^2([0, T_\varepsilon], \mathbb{R}^m)$, $\delta \in \mathcal{C}([0, T_\varepsilon], \mathbb{R}^{3d})$ with

$$\max_{l=0,1,2,3} \|\theta^{(l)}(t)\|, \max_{l=0,1,2,3} \|\eta^{(l)}(t)\|, \|\delta(t)\| \leq K \quad \textit{for } t \in [0, T_\varepsilon] \tag{61}$$

and consistent initial values $(\widehat{q}^0, \widehat{u}^0, \widehat{\nu}^0, \widehat{\lambda}^0) \in \mathcal{V}$ with (35) there exists a unique $\widehat{c} \in (0, T)$ and a unique piecewise solution $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda}) \in \mathcal{K}^2([0, \tau), \widehat{\mathcal{U}}; \mathcal{V})$ of the perturbed system (24)-(27) with $T < \tau \leq T_\varepsilon$ and event set $\widehat{\mathcal{U}} = \{\widehat{c}\}$. It has the properties

(p1) $\widehat{\lambda}_p(t) > 0$ for $t \in [0, \widehat{c})$, $\widehat{\lambda}_p(\widehat{c}) = 0$, $\dot{\widehat{\lambda}}_p^-(\widehat{c}) < 0$, and $\widehat{q}_p(t) > \theta_p(t)$ for $t \in (\widehat{c}, T]$

(p2) $\widehat{\lambda}_j(t) > 0$ holds for all $j \in J_A^0 \backslash \{p\}$ and $t \in [0, T]$

(p3) $\widehat{q}_j(t) > \theta_j(t)$ holds for all $j \in J_P^0$ and $t \in [0, T]$

Furthermore, the following estimates holds:

$$\|\widehat{q}(t)\| + \|\widehat{u}(t)\| + \|\widehat{\nu}(t)\| + \|\widehat{\lambda}(t)\| \leq K_s \tag{62}$$

$$|\widehat{c} - \overline{c}| \leq C^* \left(|\widehat{\lambda}_p(c_{min}) - \overline{\lambda}_p(c_{min})| \right) = \mathcal{O}(K) \tag{63}$$

Proof
As in the proof of Theorem 3.2 the solution $(\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda})$ can be extended uniquely to the intervals $[0, \overline{c} + \rho^R]$ and $[\overline{c} - \rho^L, T]$ with some constants $\rho^R, \rho^L > 0$ such that $(\overline{q}^R, \overline{u}^R, \overline{\nu}^R, \overline{\lambda}^R) = (\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda})$ in $[0, \overline{c}]$ and $(\overline{q}^L, \overline{u}^L, \overline{\nu}^L, \overline{\lambda}^L) = (\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda})$ in $[\overline{c}, T]$, respectively. For $\rho' \le \min\{\rho^R, \rho^L\}$ small enough we find $\overline{\lambda}_j^R, \overline{\lambda}_j^L > 0$ for $j \in J_A^0 \backslash \{p\}$ and $\overline{q}_j^R, \overline{q}_j^L > 0$ for $j \in J_P^0$ in $[\overline{c} - \rho', \overline{c} + \rho']$. Furthermore, we obtain $\dot{\overline{\lambda}}_p^R < 0$ and $\ddot{\overline{u}}_p^L > 0$ in $[\overline{c} - \rho', \overline{c} + \rho']$ if ρ' is small enough, and by continuity this also gives $\overline{\lambda}_p^R < 0$ in $(\overline{c}, \overline{c} + \rho']$. By Theorem 3.1 there exist constants $K^R, K_s^R > 0$ such that for $K \le K^R$ there exists a unique solution $(\widehat{q}^R, \widehat{u}^R, \widehat{\nu}^R, \widehat{\lambda}^R)$ of the perturbed DAE system (30)-(33) with the active set J_A^0 in $[0, \tau^R)$, where $\overline{c} + \rho^R < \tau^R \le T_\varepsilon$. Furthermore, the estimate

$$\begin{aligned} &\|\widehat{q}^R(t) - \overline{q}^R(t)\| + \|\widehat{u}^R(t) - \overline{u}^R(t)\| \\ &\quad + \|\widehat{\nu}^R(t) - \overline{\nu}^R(t)\| + \|\widehat{\lambda}^R(t) - \overline{\lambda}^R(t)\| + \|\dot{\widehat{\lambda}}^R(t) - \dot{\overline{\lambda}}^R(t)\| = \mathcal{O}(K) \end{aligned} \quad (64)$$

holds for $t \in [0, \overline{c} + \rho^R]$ (see Remark 3.1) and $(\widehat{q}^R, \widehat{u}^R, \widehat{\nu}^R, \widehat{\lambda}^R)$ are bounded by K_s^R. As in the proof of Lemma 3.4 estimate (64) yields $\widehat{\lambda}_j^R > 0$ for $j \in J_A^0 \backslash \{p\}$ and $\widehat{q}_j^R > \theta_j$ for $j \in J_P^0$ in $[0, \overline{c} + \rho']$ if K is small enough.
We apply Lemma 3.4 with $\overline{\varphi} = \overline{\lambda}_p^R$, $\widehat{\varphi} = \widehat{\lambda}_p^R$, $\zeta = 0$, and $\tau = \overline{c} + \rho'$. For K small enough we find a constant $\rho > 0$ with $\rho \le \rho'$ and a unique $\widehat{c} \in [\overline{c} - \rho, \overline{c} + \rho]$ such that $\widehat{\lambda}_p^R > 0$ in $[0, \widehat{c})$, $\widehat{\lambda}_p^R(\widehat{c}) = 0$, $\dot{\widehat{\lambda}}_p^R(\widehat{c}) < 0$, and $\widehat{\lambda}_p^R < 0$ in $(\widehat{c}, \overline{c} + \rho]$. Furthermore, we estimate (63) follows, and (64) leads to $|\widehat{c} - \overline{c}| = \mathcal{O}(K)$. With $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda}) := (\widehat{q}^R, \widehat{u}^R, \widehat{\nu}^R, \widehat{\lambda}^R)$ in $[0, \widehat{c}]$ we immediately find

$$\|\widehat{q}(\widehat{c}) - \overline{q}^L(\widehat{c})\| \le \|\widehat{q}(\widehat{c}) - \overline{q}^R(\widehat{c})\| + \|\overline{q}^R(\widehat{c}) - \overline{q}^L(\widehat{c})\| \le \mathcal{O}(K) + C|\widehat{c} - \overline{c}| = \mathcal{O}(K)$$

with a constant $C > 0$, and similarly $\|\widehat{u}(\widehat{c}) - \overline{u}^L(\widehat{c})\| = \mathcal{O}(K)$. Now Theorem 3.1 can be applied again and if K is small enough there exists a unique solution $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda})$ of the perturbed DAE system (30)-(33) with the active set $J_A^0 \backslash \{p\}$ in $[\widehat{c}, \tau]$, where $T < \tau \le T_\varepsilon$. Furthermore, $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda})$ are bounded by a constant $K_s^L > 0$, the bound (62) holds with $K_s := \{K_s^R, K_s^L\}$, and the estimate

$$\|\widehat{q}(t) - \overline{q}^L(t)\| + \|\widehat{u}(t) - \overline{u}^L(t)\| + \|\widehat{\nu}(t) - \overline{\nu}^L(t)\| + \|\widehat{\lambda}(t) - \overline{\lambda}^L(t)\| = \mathcal{O}(K) \quad (65)$$

holds for all $t \in [\widehat{c}, T]$. By the 3rd and 4th order hidden constraints also

$$\|\dot{\widehat{u}}(t) - \dot{\overline{u}}^L(t)\| + \|\ddot{\widehat{u}}(t) - \ddot{\overline{u}}^L(t)\| = \mathcal{O}(K) \quad (66)$$

holds for $t \in [\widehat{c}, T]$, see Remark 3.1.
As in the proof of Lemma 3.4 we show that estimate (65) implies $\widehat{q}_j > \theta_j$ for $j \in J_P^0$ and $\widehat{\lambda}_j > 0$ for $j \in J_A^0 \backslash \{p\}$ in $[\widehat{c}, T]$. For $\widehat{c} > \overline{c}$ we also find $\widehat{q}_p > \theta_p$ in $[\widehat{c}, T]$ by (65) if K is small enough. If $\widehat{c} \le \overline{c}$ estimate (66) yields $\ddot{\widehat{u}}_p > \theta^{(3)}$ in $[\widehat{c}, \widehat{c} + \rho']$ if K is small enough, and Taylor expansion gives

$$\widehat{q}_p(t) = \widehat{q}_p(\widehat{c}) + (t - \widehat{c})\widehat{u}_p(\widehat{c}) + \frac{1}{2}(t - \widehat{c})^2 \dot{\widehat{u}}_p(\widehat{c}) + \frac{1}{2}\int_{\widehat{c}}^t (t - \xi)^2 \ddot{\widehat{u}}_p(\xi) d\xi > \theta_p(t)$$

for $t \in [\widehat{c}, \widehat{c}+\rho']$ since $\widehat{q}_p^{(l)}(\widehat{c}) = \theta_p^{(l)}(\widehat{c})$ for $l = 0, 1, 2$ by continuity. Estimate (65) finally yields $\widehat{q}_p > \theta_p$ in $[\widehat{c}+\rho', T]$, and uniqueness of the piecewise solution is shown as in the proof of Theorem 3.2.

□

Now we derive error estimates for the position and velocity vectors as in Theorem 3.3. Notice that in this situation the solution is continuous, and therefore we find pointwise and uniform estimates for both the position and the velocity vector in the whole time interval.

Theorem 3.5 *Suppose that $(\overline{q}, \overline{u}, \overline{\nu}, \overline{\lambda}) \in \mathcal{K}^2([0,T], \overline{\mathcal{U}}; \mathcal{V})$ is a piecewise solution of the original system (5)-(8) with $T > 0$, active and passive sets J_A, J_P, and event set $\overline{\mathcal{U}} = \{\overline{c}\}$, $0 < \overline{c} < T$ such that assumptions (a1)-(a3) of Theorem 3.4 hold. Furthermore, suppose that the rank condition (22) holds for all $(t, q) \in (-\varepsilon, T_\varepsilon) \times \mathbb{R}^{3d}$ with $\varepsilon > 0$, $T_\varepsilon > T$.*
Then there exist constants $K, C^* > 0$ such that for initial values $(\widehat{q}^0, \widehat{u}^0, \widehat{\nu}^0, \widehat{\lambda}^0) \in \mathcal{V}$ that fulfill the consistency conditions (35) and for perturbations $\theta \in \mathcal{C}^2([0,T_\varepsilon], \mathbb{R}^d)$, $\eta \in \mathcal{C}^2([0,T_\varepsilon], \mathbb{R}^m)$, $\delta \in \mathcal{C}([0,T_\varepsilon], \mathbb{R}^{3d})$ with (61) there exists a unique piecewise solution $(\widehat{q}, \widehat{u}, \widehat{\nu}, \widehat{\lambda}) \in \mathcal{K}^2([0,T], \widehat{\mathcal{U}}; \mathcal{V})$ of the perturbed system (24)-(27) with the event set $\widehat{\mathcal{U}} = \{\widehat{c}\}$, $0 < \widehat{c} < T$ (see Theorem 3.4). Furthermore, the following estimate holds for all $t \in [0,T]$ with h defined in (43) (see Lemma 3.3) and $c_{min} = \min\{\overline{c}, \widehat{c}\}$, $c_{max} = \max\{\overline{c}, \widehat{c}\}$:

$$\begin{aligned}\sum_{i\in I}(|\widehat{q}_i(t) - \overline{q}_i(t)| + |\widehat{u}_i(t) - \overline{u}_i(t)|) \qquad (67)\\ \leq C^* \left(\sum_{i\in I} \left(|\widehat{q}_i^0 - \overline{q}_i^0| + |\widehat{u}_i^0 - \overline{u}_i^0|\right) + h(c_{min}) + \int_0^{c_{min}} h(\xi)d\xi \right.\\ \left. + \sum_{j\in J_A^0\setminus\{p\}} \int_{c_{min}}^{c_{max}} |\ddot{\theta}_j(\xi)|d\xi + \sum_{j\in J_P^0\cup\{p\}} \int_{c_{min}}^{c_{max}} |\delta_j(\xi)|d\xi + \int_{c_{max}}^{\max\{c_{max},t\}} h(\xi)d\xi \right)\end{aligned}$$

Proof
By Theorem 3.4 the piecewise solution of the perturbed system with the desired properties (p1)-(p4) exists, is bounded, and unique. To simplify notations we set $w_i := \widehat{q}_i - \overline{q}_i$ with $w_i^0 := w_i(0)$ and $\dot{w}_i := \widehat{u}_i - \overline{u}_i$ with $\dot{w}_i^0 := \dot{w}_i(0)$ for $i \in I$. Since the solutions of the exact and the perturbed systems are continuous, also w_i, $\dot{w}_i$, and

$$\ddot{w}_i = \frac{1}{m_i}\left[f_i(t,\widehat{q},\widehat{u}) - f_i(t,\overline{q},\overline{u}) + \left(G(t,\widehat{q})^T\widehat{\nu} - G(t,\overline{q})^T\overline{\nu}\right)_i + \left(H^T(\widehat{\lambda} - \overline{\lambda})\right)_i + \delta_i\right]$$

are continuous and we immediately find

$$\begin{aligned}\dot{w}_i(T) &= \dot{w}_i^0 + \int_0^T \ddot{w}_i(t)dt\\ w_i(T) &= w_i^0 + \int_0^T \left(\dot{w}_i^0 + \int_0^t \ddot{w}_i(\xi)d\xi\right)dt = w_i^0 + T\dot{w}_i^0 + \int_0^T\int_0^t \ddot{w}_i(\xi)d\xi dt\end{aligned}$$

Taking absolute values and adding both equations gives

$$|w_i(T)| + |\dot{w}_i(T)| \leq C\left(|w_i^0| + (1+T)|\dot{w}_i^0| + (1+T)\int_0^T |\ddot{w}_i(t)|dt\right) \qquad (68)$$

with a constant $C > 0$ and we proceed by estimating the integral on the right hand side. Using the active constraints we immediately find

$$\int_0^T |\ddot{w}_i(t)|dt = \int_0^T |\ddot{\theta}_i(t)|dt \quad \text{for } i \in J_A^0\backslash\{p\} \quad \text{and} \quad \int_0^{c_{\min}} |\ddot{w}_p(t)|dt = \int_0^{c_{\min}} |\ddot{\theta}_p(t)|dt.$$

Since $\widehat{q}$ and $\widehat{u}$ are bounded, they are contained in a compact subset of $\mathbb{R}^{3d}$, where f and g have continuous derivatives and are thus bounded and Lipschitz–continuous. Using estimate (42) from Lemma 3.3 with $\alpha = 0$, $\beta = c_{\min}$ we find for $i \in J_P^0$

$$\begin{aligned}
\int_0^{c_{\min}} |\ddot{w}_i(t)|dt \quad &\leq \quad \frac{1}{m_i}\int_0^{c_{\min}} \big(|f_i(t,\widehat{q},\widehat{u}) - f_i(t,\overline{q},\overline{u})| \\
&\qquad +|G(t,\widehat{q})^T(\widehat{\nu}-\overline{\nu})|_i + |(G(t,\widehat{q}) - G(t,\overline{q}))^T\overline{\nu}|_i + |\delta_i|\big)\, dt \\
&\leq \quad C_1 \int_0^{c_{\min}} \left(\sum_{j\in I}(|w_j(t)| + |\dot{w}_j(t)|) + h(t)\right) dt \qquad (69)
\end{aligned}$$

with some constant $C_1 > 0$, and similarly, using estimate (42) from Lemma 3.3 with $\alpha = c_{\max}$, $\beta = T$ we find for $i \in J_P^0 \cup \{p\}$

$$\int_{c_{\max}}^T |\ddot{w}_i| \leq C_2 \int_{c_{\max}}^T \left(\sum_{j\in I}(|w_j(t)| + |\dot{w}_j(t)|) + h(t)\right) dt$$

with some constant $C_2 > 0$. Estimates (63) from Theorem 3.4, (42) from Lemma 3.3 (with $\alpha = 0$, $\beta = c_{\min}$) and (69) from above lead to

$$\begin{aligned}
(c_{\max} - c_{\min}) &\leq C_1|\widehat{\lambda}_p(c_{\min}) - \overline{\lambda}_p(c_{\min})| \leq C_2 \left(\sum_{j\in I}(|w_j(c_{\min})| + |\dot{w}_j(c_{\min})|) + h(c_{\min})\right) \\
&\leq C_4 \left[\sum_{j\in I}(|w_j^0| + |\dot{w}_j^0|) + \int_0^{c_{\min}} \left(\sum_{j\in I}(|w_j(t)| + |\dot{w}_j(t)|) + h(t)\right) dt + h(c_{\min})\right].
\end{aligned}$$

with $C_1, C_2, C_3, C_4 > 0$ for K small enough. For $i \in J_P^0 \cup \{p\}$ this result leads to

$$\begin{aligned}
\int_{c_{\min}}^{c_{\max}} |\ddot{w}_i(t)|dt \quad &\leq \quad \frac{1}{m_i}\int_{c_{\min}}^{c_{\max}} \big(|f_i(t,\widehat{q},\widehat{u}) - f_i(t,\overline{q},\overline{u})| \\
&\qquad +|G(t,\widehat{q})^T\widehat{\nu}|_i + |G(t,\overline{q})^T\overline{\nu}|_i + |\widehat{\lambda}_i| + |\overline{\lambda}_i| + |\delta_i|\big)\, dt \\
&\leq \quad C_1 \left[\int_{c_{\min}}^{c_{\max}} \left(\sum_{j\in I}(|w_j(t)| + |\dot{w}_j(t)|) + |\delta_i(t)|\right) dt + (c_{\max} - c_{\min})\right] \\
&\leq \quad C_2\Big[\sum_{j\in I}(|w_j^0| + |\dot{w}_j^0|) + \int_0^{c_{\max}} \left(\sum_{j\in I}(|w_j(t)| + |\dot{w}_j(t)|)\right) dt \\
&\qquad + \int_0^{c_{\min}} h(t)dt + \int_{c_{\min}}^{c_{\max}} |\delta_i(t)|dt + h(c_{\min})\Big]
\end{aligned}$$

with some constants $C_1, C_2 > 0$, where $\widehat{\lambda}_i = \overline{\lambda}_i = 0$ for $i \in J_P^0$ and $\widehat{\lambda}_p$, $\overline{\lambda}_p$ are bounded in $[c_{\min}, c_{\max}]$. Taking sums on both sides of (68) and substituting the estimates for the integrals leads to

$$\sum_{i\in I}(|w_i(T)| + |\dot{w}_i(T)|) \leq C_1\psi(T) + C_2 \int_0^T \sum_{i\in I}(|w_i(t)| + |\dot{w}_i(t)|)dt,$$

$$\begin{aligned} \text{where} \quad \psi(\tau) \quad := \quad & \sum_{i\in I}(|w_i^0| + |\dot{w}_i^0|) + h(c_{\min}) + \sum_{j\in J_P^0\cup\{p\}} \int_{c_{\min}}^{c_{\max}} |\delta_j(t)|dt \\ & + \sum_{j\in J_A^0\setminus\{p\}} \int_{c_{\min}}^{c_{\max}} |\ddot{\theta}_j(t)|dt + \int_0^{c_{\min}} h(t)dt + \int_{c_{\max}}^{\max\{c_{\max},\tau\}} h(t)dt \end{aligned}$$

is defined for $\tau \in [0, T]$ and $C_1, C_2 > 0$ are constants. Following the proof we find the same estimate

$$\sum_{i\in I}(|w_i(\tau)| + |\dot{w}_i(\tau)|) \leq C_1\psi(\tau) + C_2 \int_0^\tau \sum_{i\in I}(|w_i(t)| + |\dot{w}_i(t)|)\, dt$$

for $\tau \in [0, T]$. Gronwall's inequality gives

$$\sum_{i\in I}(|w_i(\tau)| + |\dot{w}_i(\tau)|) \leq C_1\psi(\tau) + C_2 \int_0^\tau \psi(\xi)e^{C_2(\tau-\xi)}d\xi \leq (C_1 + e^{C_2\tau} - 1)\psi(\tau)$$

with $\psi(\tau) \geq \psi(t)$ for all $t \in [0, \tau]$, and this finally leads to the desired estimate (67).

□

Estimates (67) and (56) for $t > c_{\max}$ are quite similar: We find integrals of $|\ddot{\theta}_j|$ in $[c_{\min}, c_{\max}]$ for all constraints j that always remain active, and integrals of $|\delta_j|$ in $[c_{\min}, c_{\max}]$ for $j = p$ and all constraints j that always remain passive. The integrals of h in $[0, c_{\min}]$ and $[c_{\max}, T]$ result from (42) in Lemma 3.3, where the piecewise solution is smooth. Derivatives of the perturbations occur up to second order, this agrees with the error estimates (36) we found in Theorem 3.1 for the equations of motion without unilateral constraints which have perturbation index 3. Notice that the value $h(c_{\min})$ in estimate (67) includes second order derivatives of the perturbations in contrast to the values $|\theta_p(c_{\min})|$, $|\dot{\theta}_p(c_{\min})|$, and $|\dot{\eta}_k(c_{\min})|$ for $k = 1 \ldots m$ in estimate (56). This indicates that in case of a lift–off the piecewise solution of the system (5)-(8), (21) might be much more sensitive subject to small perturbations. However, there is no guarantee that the estimates are sharp, so this conclusion cannot be drawn without further ado.

4 A numerical example

To illustrate the theoretical results we consider a biomechanical application: the hydrostatic skeleton. In physical terms it is realized as an incompressible fluid enclosed by an elastic body wall. Contraction or relaxation of muscles embedded in the body wall leads to changes in the shape of the body by means of a pressure applied to the internal fluid.

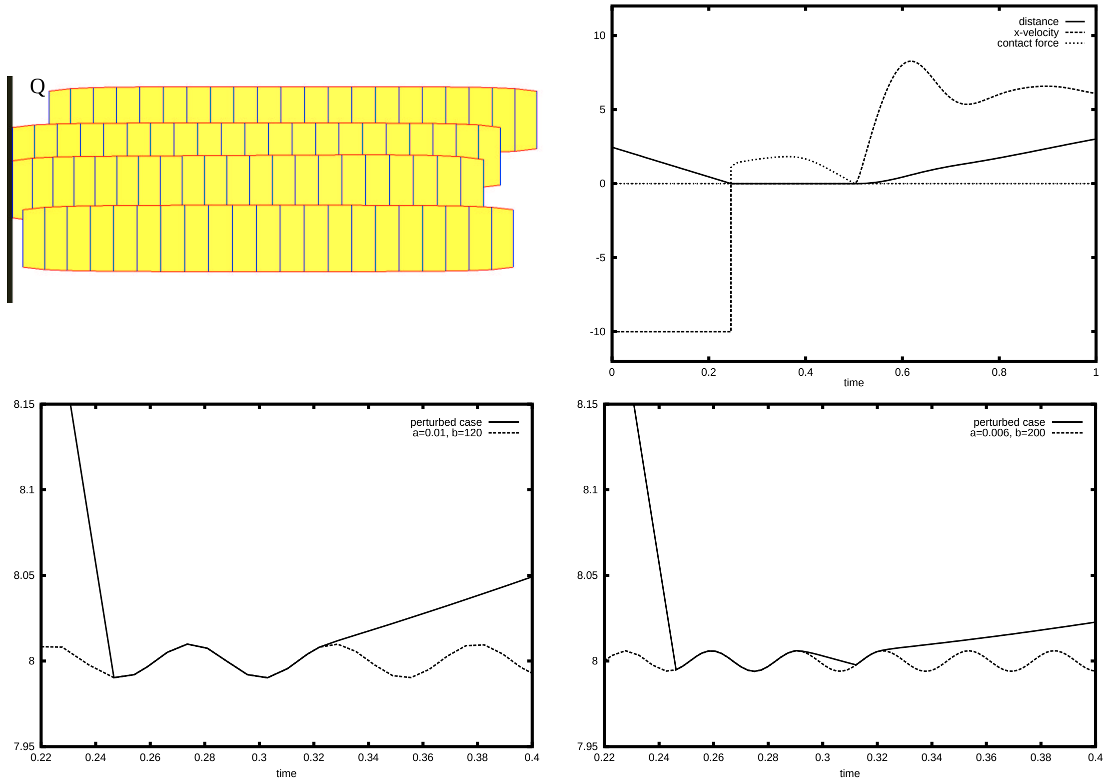

Figure 2: Upper left: Collision of the model with the obstacle $\mathcal{H}$ in the unperturbed case. We use the initial velocity $(-10, 0, 0)^T$ for all corners and $K = 8$ for the obstacle $\mathcal{H}$ defined in (2). Upper right: Distance of the corner $Q \in \mathbb{R}^3$ from $\mathcal{H}$, velocity $\dot{Q}_1$ in (horizontal) x–direction, and contact force acting on Q versus time. Lower left: Horizontal coordinate Q_1 in the perturbed case with $a = 0.01$ and $b = 120$ versus time. Lower right: Same as lower left diagram, but parameters $a = 0.006$ and $b = 200$ are used.

In [1], [2] we developed a mathematical model which consists of a sequence of $N = 21$ hexahedral segments with elastic edges that act as damped springs. The total mass is concentrated in the $d = 4 * N + 4$ corners and the elastic forces are given by the term $f(t, q, u) \in \mathbb{R}^{3d}$ in (6), see [1], [2] for details. The model is based on the assumption that the total volume remains constant during the motion. This leads to the constraint

$$g(q) = V^{tot} - \sum_{k=1}^{N} V_k(q) = 0,$$

where $V^{tot} > 0$ is the prescribed total volume in the entire body ($m = 1$), and $V_k(q)$ is the actual volume of the kth segment that can be expressed in terms of the coordinates of the corners.

The equations of motion are solved by numerical methods implemented in the code MEXAX, see [15]. Figure 2 shows the collision of the model with the obstacle $\mathcal{H}$ defined in (2). Notice that the complementarity conditions (8) for the distance of the model to $\mathcal{H}$ and the contact force acting on $\mathcal{H}$ are fulfilled (see upper right diagram). We find nearly the same result in the presence of a small perturbation $\theta_j(t) = a\cos(bt)$ for all $j \in J$ in the obstacle except for the lift–off time. In the unperturbed case the model lifts off at $t \approx 0.5$ (see upper right diagram), while in the perturbed case with $a = 0.01$

and $b = 120$ it lifts off at $t \approx 0.32$, see lower left diagram. Decreasing the amplitude of the perturbation to $a = 0.006$ and increasing the frequency to $b = 200$ leads to a lift–off at $t \approx 0.29$, the model collides again with the obstacle at $t \approx 0.31$ and finally lifts off at $t \approx 0.32$, see lower right diagram in Figure 2. This example reflects the theoretical results in Sect. 3, and it shows the influence of the second derivative of the perturbations on the solution: $||\dot{\theta}_j(t)|| = ab$ remains the same for both parameter configurations, but $||\ddot{\theta}_j(t)|| = ab^2$ differs in the two perturbed cases.

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