Numerical approximation of homoclinic chaos

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Abstract

Transversal homoclinic orbits of maps are known to generate a Cantor set on which a power of the map conjugates to the Bernoulli shift on two symbols. This conjugacy may be regarded as a coding map, which for example assigns to a homoclinic symbol sequence a point in the Cantor set that lies on a homoclinic orbit of the map with a prescribed number of humps. In this paper we develop a numerical method for evaluating the conjugacy at periodic and homoclinic symbol sequences. The approach combines our previous method for computing the primary homoclinic orbit with the constructive proof of Smale's Theorem given by Palmer. It is shown that the resulting nonlinear systems are well conditioned uniformly with respect to the characteristic length of the symbol sequence and that Newton's method converges uniformly too when started at a proper pseudo orbit. For the homoclinic symbol sequences an error analysis is given. The method works in arbitrary dimensions and it is illustrated by examples.

Key words. Dynamical systems, numerical methods, homoclinic points for maps, multihumped homoclinic orbits, chaos

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1 Introduction

The original idea in Smale's Homoclinic Theorem was geometric in nature (cf. [9] and [8], [6]). A transversal homoclinic point of a diffeomorphism implies a horseshoe which may be regarded as a compact invariant set on which a power of the map is conjugated to the full shift on two symbols. Since then, more analytically oriented proofs have been developed (see [7] and the references therein) which also generalize to non–diffeomorphisms (cf. [4], [10]). These approaches seem to be better suited for developing numerical approximations of the invariant set and the conjugacy. In this paper we aim at such a method and we will combine the elegant proof of Palmer [7] (which employs the Shadowing Lemma) with our previous method [2] for computing a primary homoclinic orbit.

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In what follows we will outline the basic idea of the paper in non-technical terms and refer to the corresponding sections for the details.

We consider a smooth diffeomorphism f on $\mathbb{R}^k$ with a hyperbolic fixed point $\xi \in \mathbb{R}^k$. We assume that there exists a transversal homoclinic orbit $(x_n)_{n\in\mathbb{Z}}$ based at ξ, i.e.

$$x_{n+1} = f(x_n), \qquad n \in \mathbb{Z}$$
$$\lim_{n\to\pm\infty} x_n = \xi$$

and the stable and unstable manifolds of ξ intersect transversally at x_0, and hence at each point x_n, $n \in \mathbb{Z}$.

In [2] we have analyzed the numerical approximation of such orbits by solving a finite boundary value problem

$$y_{n+1} = f(y_n) \quad , \quad n = n_-, \dots, n_+ - 1 \tag{1}$$
$$b(y_{n_-}, y_{n_+}) = 0 \tag{2}$$

where $n_\pm$ are assumed to be large and (2) represents a set of k boundary conditions. Our convergence analysis as $n_\pm \to \pm\infty$ applies to two important cases: *periodic boundary conditions* given by

$$b(y_{n_-}, y_{n_+}) = y_{n_-} - y_{n_+} = 0$$

and *projection boundary conditions* of the form

$$b(y_{n_-}, y_{n_+}) = (b_-(y_{n_-}), b_+(y_{n_+})) = 0,$$

where the zero sets of b_- and b_+ are linear approximations to the local unstable and stable manifold of ξ respectively.

According to Palmer [7] the Cantor set Λ in Smale's Theorem can be constructed as follows. Consider the compact hyperbolic set $\Lambda_{THO} = \{\xi\} \cup \{x_n : n \in \mathbb{Z}\}$ and take any $\delta, \varepsilon > 0$ such that δ-pseudo orbits in Λ_{THO} can be uniquely ε-shadowed by f–orbits in $\mathbb{R}^k$ and such that the closed balls $B_\varepsilon(x_0)$ and $B_\varepsilon(\xi)$ do not intersect. Then choose $L \in \mathbb{N}$ so large that $x_{-L}, x_{L+1} \in B_{\delta/2}(\xi)$ holds and consider the orbit segments of length $p = 2L+1$ given by

$$C_0 = (\xi, \dots, \xi),\ C_1 = (x_{-L}, \dots, x_L).$$

To any binary sequence $a \in \Sigma_2 := \{0,1\}^{\mathbb{Z}}$, i.e.

$$a = (\dots, a_{-1}, a_0, a_1, \dots), \quad a_i \in \{0,1\}, \tag{3}$$

associate the δ-pseudo orbit

$$(\dots, C_{a_{-1}}, C_{a_0}, C_{a_1}, \dots) \tag{4}$$

and let $(y_n)_{n\in\mathbb{Z}}$ be its unique ε-shadowing orbit. The conjugacy Ψ is then defined by $\Psi(a) = y_0$ and the set Λ by $\Lambda = \Psi(\Sigma_2)$. By uniqueness of the shadowing construction it is easy to see that

$$\Psi \circ \sigma = f^p \circ \Psi, \quad p = 2L+1$$

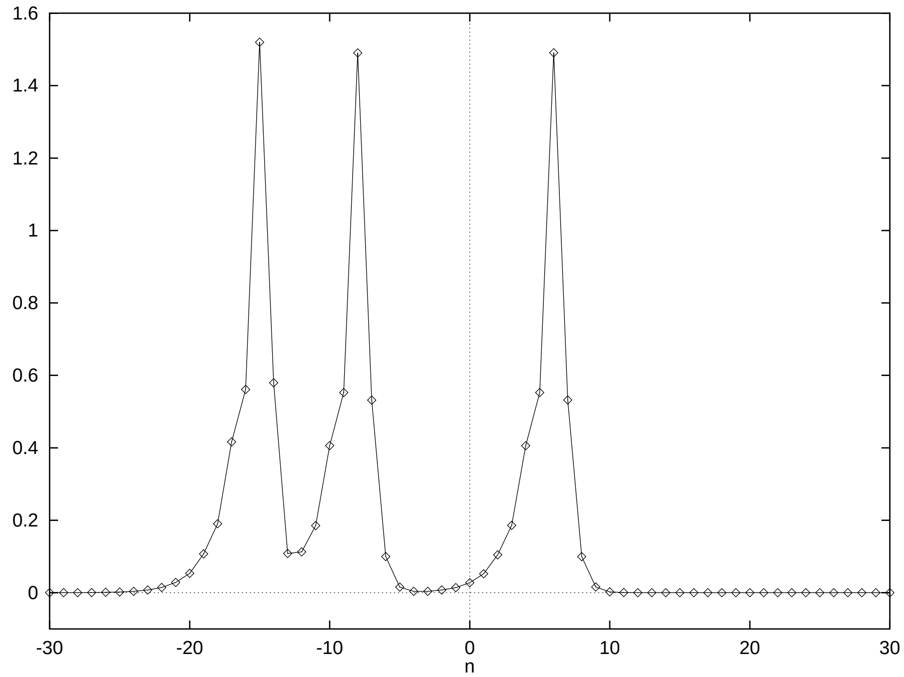

Figure 1: $\|y_n - \xi\|$ versus n for the code $(\ldots 0110100 \ldots)$ with $L = 3$ and Hénon's map.

holds for the Bernoulli shift σ on Σ_2.

Our approach will be to mimic this construction numerically for periodic sequences, i.e. $a_i = a_{i+2m-1}$ for all $i \in \mathbb{Z}$ and some $m \in \mathbb{N}$, and homoclinic sequences, i.e. $a_i = 0$ for $|i| > m$ and some $m \in \mathbb{N}$. For convenience we choose an odd characteristic length $2m - 1$ for these sequences.

First we compute an orbit segment $(x_{n_-}, \ldots, x_{n_+})$ as in [2] by solving a nonlinear system (1), (2) to get an approximate segment C_1. For a given periodic or homoclinic symbol sequence we then form the pseudo orbit (4). This orbit is taken as a starting vector for Newton's method applied to an enlarged system (1) with $n_+ - n_- = (2m-1)(2L+1)$ subject to either periodic or projection boundary conditions. This procedure and its results will be illustrated in Section 5. For example, a homoclinic orbit which belongs to the sequence

$$(\ldots 0\,1\,1\,0\,1\,0\,0 \ldots)$$

consists of three humps as shown in Figure 1 and it contributes just one point y_0 to the Cantor set Λ.

Our aim in the theory is to show that for L sufficiently large a Newton method started in this way converges and yields the desired point in the Cantor set Λ in the periodic case or a sufficiently good approximation to it in the homoclinic case.

Moreover, we prove in Section 4 that the large nonlinear system (1), (2) is well conditioned uniformly with respect to L and the characteristic length $2m - 1$ of the coding sequence. In particular, in the homoclinic case we show that the remaining error due to the finite boundary condition (2) can be made small by enlarging $-n_-, n_+$ by an amount which is independent of m and L.

In order to derive such a result we prove in Section 3 a general approximation theorem for homoclinic orbits in a given hyperbolic set converging to the same fixed point. This generalizes our previous result from [2] in such a way that it applies to homoclinic orbits with an arbitray number of humps. The underlying hyperbolic set is obtained as a set of all shadowing orbits which is bounded but not necessarily closed. Therefore, we summarize in Section 2 some basic facts about exponential dichotomies and hyperbolic sets avoiding compactness whenever possible.

Let us finally remark that the Cantor set Λ constructed above is not a maximal invariant set of f with respect to some neighborhood of the homoclinic. However, in a suitable neighborhood of $\{\xi\} \cup \{x_0\}$ it is maximally invariant with respect to f^p (see [6, §2]). An appropriate symbolic description of the maximal invariant set under f in some neighborhood of $\{\xi\} \cup \{x_n : n \in \mathbb{Z}\}$ is given in [4], [10]. The basic difference is that now the length of the trivial segment $C_0 = (\xi, \ldots, \xi)$ in (4) can vary at each occurrence. There seems to be no difficulty in using such pseudo orbits in our numerical approach but we have not carried out the details for this case.

2 Exponential dichotomies and the Shadowing Lemma

In this section we review some basic tools from the theory of exponential dichotomies [7] as well as some facts about hyperbolic sets which are bounded but not necessarily closed. Consider a homogeneous difference equation in $\mathbb{R}^k$

$$u_{n+1} = A_n u_n, \quad n \in \mathbb{Z}, \quad A_n \in \mathbb{R}^{k,k} \text{ nonsingular} \tag{5}$$

with solution operator

$$\Phi(n,m) = \begin{cases} A_{n-1} \cdot \ldots \cdot A_m & \text{if } n > m \\ I & \text{if } n = m \\ A_n^{-1} \cdot \ldots \cdot A_{m-1}^{-1} & \text{if } n < m \end{cases}.$$

In the following let

$$J = \{n \in \mathbb{Z} : n_- - 1 < n < n_+ + 1\}, \quad n_\pm \in \mathbb{Z} \cup \{\pm\infty\}, \quad n_- \le n_+$$

be any interval in $\mathbb{Z}$. If no confusion with real intervals arises we simply write $J = [n_-, n_+]$. We also make frequent use of the Banach space of bounded sequences on J given by

$$S_J = \{u_J = (u_n)_{n \in J} \subset (\mathbb{R}^k)^J : \|u_J\|_\infty := \sup_{n \in J} \|u_n\| < \infty\}.$$

Definition 2.1 The difference equation (5) has an *exponential dichotomy on J* if there exist projectors P_n, $n \in J$ in $\mathbb{R}^k$ and constants $K, \alpha > 0$ such that

$$P_n \Phi(n,m) = \Phi(n,m) P_m \quad \text{for all } n, m \in J$$

and

$$\begin{array}{rcl}\|\Phi(n,m)P_m\| & \leq & Ke^{-\alpha(n-m)} \\ \|\Phi(m,n)(I-P_n)\| & \leq & Ke^{-\alpha(n-m)}\end{array} \qquad \text{for all } n \geq m \text{ in } J.$$

For brevity we say that (5) has an exponential dichotomy on J with data (K,α,P_n).

If $J = \mathbb{N}$ the ranges of the projectors are uniquely determined as

$$\mathcal{R}(P_0) = \{\mu \in \mathbb{R}^k : \sup_{n\geq 0} \|\Phi(n,0)\mu\| < \infty\}, \tag{6}$$

and in case $J = -\mathbb{N}$ the nullspaces are unique with

$$\mathcal{N}(P_0) = \{\mu \in \mathbb{R}^k : \sup_{n\leq 0} \|\Phi(n,0)\mu\| < \infty\}. \tag{7}$$

If $J = \mathbb{Z}$ the projectors are uniquely determined and both (6) and (7) hold (cf. [7, Proposition 2.3]).

By [7, Lemma 2.7] we have the following result concerning inhomogeneous equations.

Lemma 2.2 *Let the linear difference equation* (5) *have an exponential dichotomy on* $\mathbb{Z}$ *with data* (K,α,P_n). *Then for every* $r_\mathbb{Z} \in S_\mathbb{Z}$ *the inhomogeneous difference equation*

$$u_{n+1} = A_n u_n + r_n$$

has a unique solution $u_\mathbb{Z} \in S_\mathbb{Z}$. *Moreover*

$$\|u_\mathbb{Z}\|_\infty \leq K\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\|r_\mathbb{Z}\|_\infty \tag{8}$$

holds.

Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a C^1 diffeomorphism.

Definition 2.3 An invariant set $\Lambda \subset \mathbb{R}^k$ of f, i.e. $f(\Lambda) = \Lambda$, is called *hyperbolic* if there exist constants $K,\alpha > 0$ and $l \in \mathbb{N}$ such that for every $x \in \Lambda$ the variational equation along the orbit $(f^n(x))_{n\in\mathbb{Z}}$

$$u_{n+1} = f'(f^n(x))u_n$$

has an exponential dichotomy on $\mathbb{Z}$ with constants K,α and projectors $P_n(x)$, $n \in \mathbb{Z}$ of rank l.

In this case the solution operator is

$$\Phi(n,m)(x) = Df^{n-m}(f^m(x)), \quad n,m \in \mathbb{Z}.$$

Let us emphasize that we do not assume a hyperbolic set to be compact as in [7]. Nevertheless, the same proof as in [7, Proposition 3.2] may be used to characterize hyperbolicity in terms of the standard splitting into growing and decaying modes.

Proposition 2.4 *Let Λ be an invariant set of f. Then Λ is hyperbolic iff there exist constants $K, \alpha > 0$ and a projector valued function $\mathcal{P} : \Lambda \to \mathbb{R}^{k,k}$ of constant rank such that for all $x \in \Lambda$ and $n \in \mathbb{N}$*

$$\mathcal{P}(f(x))Df(x) = Df(x)\mathcal{P}(x), \tag{9}$$

$$\|Df^n(x)\mathcal{P}(x)\| \leq Ke^{-\alpha n}, \tag{10}$$

$$\|Df^{-n}(x)(I - \mathcal{P}(x))\| \leq Ke^{-\alpha n}. \tag{11}$$

In this proposition the projector valued function $\mathcal{P}$ and the projectors $P_n(x)$ from Definiton 2.3 are related by (cf. [7])

$$\mathcal{P}(f^n(x)) = P_n(x) \quad \text{for all } x \in \Lambda, n \in \mathbb{Z}$$

and the function $\mathcal{P}$ is uniquely determined by (9)–(11).

The next result is again the same as in [7, Proposition 3.3]. But we do not assume Λ to be compact and so we need a different proof.

Proposition 2.5 *Let Λ, $\mathcal{P}$ and $P_n(x)$, $x \in \Lambda$ as above. Then the map*

$$\mathcal{P} : \begin{array}{ccc} \Lambda & \to & \mathbb{R}^{k,k} \\ x & \mapsto & P_0(x) \end{array}$$

is continuous.

Proof. Assume that there exists an $\varepsilon > 0$ and a sequence $(x_\nu)_{\nu \in \mathbb{N}}$ in the hyperbolic set Λ such that for some $x \in \Lambda$

$$\lim_{\nu \to \infty} x_\nu = x \qquad \text{and} \qquad \|P_0(x_\nu) - P_0(x)\| \geq \varepsilon \quad \text{for all } \nu \in \mathbb{N}.$$

Then there is a sequence $\xi_\nu \in \mathbb{R}^k$, $\nu \in \mathbb{N}$ satisfying

$$\|\xi_\nu\| = 1, \quad \|(P_0(x_\nu) - P_0(x))\xi_\nu\| \geq \varepsilon.$$

Since the $P_0(y), y \in \Lambda$ are uniformly bounded by Proposition 2.4 we can assume

$$\lim_{\nu \to \infty} (P_0(x_\nu) - P_0(x))\xi_\nu = \eta, \quad \|\eta\| \geq \varepsilon.$$

For $\nu, n \in \mathbb{N}$ we have from the dichotomy and $\|\xi_\nu\| = 1$

$$\|\Phi(n,0)(x_\nu)(P_0(x_\nu) - P_0(x))\xi_\nu\| \leq Ke^{-\alpha n} + \|\Phi(n,0)(x_\nu)P_0(x)\|. \tag{12}$$

The map $\Phi(l,m)(y)$ is continuous in y for fixed $l, m \in \mathbb{Z}$ because $f \in C^1$. We let $\nu \to \infty$ in (12) and find

$$\|\Phi(n,0)(x)\eta\| \leq Ke^{-\alpha n} + \|\Phi(n,0)(x)P_0(x)\| \leq 2Ke^{-\alpha n}.$$

Therefore $\eta \in \mathcal{R}(P_0(x))$ holds by (6). In a similar way for $n, \nu \in \mathbb{N}$

$$\|\Phi(-n,0)(x_\nu)(P_0(x_\nu) - P_0(x))\xi_\nu\| \leq Ke^{-\alpha n} + \|\Phi(-n,0)(x_\nu)(I - P_0(x))\|$$

and $\|\Phi(-n,0)(x)\eta\| \leq 2Ke^{-\alpha n}$ which implies $\eta \in \mathcal{N}(P_0(x))$ by (7). Thus $\eta = 0$ follows which contradicts $\|\eta\| \geq \varepsilon$. ■

Definition 2.6 Let $\varepsilon, \delta \geq 0$. A sequence $y_{\mathbb{Z}} \in S_{\mathbb{Z}}$ is called a δ *pseudo orbit* of f if

$$\sup_{n\in\mathbb{Z}} \|y_{n+1} - f(y_n)\| \leq \delta.$$

An element $x_{\mathbb{Z}} \in S_{\mathbb{Z}}$ is called an ε *shadow* of a sequence $y_{\mathbb{Z}} \in S_{\mathbb{Z}}$ if

$$\sup_{n\in\mathbb{Z}} \|x_n - y_n\| \leq \varepsilon.$$

It is called an ε *shadowing orbit* if it is also an f-orbit, i.e. $f(x_n) = x_{n+1}$ for all $n \in \mathbb{Z}$.

Next we state the Shadowing Lemma in the version of [7, Theorem 3.5, Remark 3.6] which includes the uniqueness of the shadowing orbits as well as the exponential dichotomy of its variational equations.

Proposition 2.7 (Shadowing Lemma) *Let Λ be a compact hyperbolic set of f. Then there exist $K, \alpha, \varepsilon_0 > 0$ and $l \in \mathbb{N}$ with the following properties. For every $\varepsilon \in (0, \varepsilon_0]$ there exists $\delta = \delta(\varepsilon) > 0$ such that every δ pseudo orbit in Λ has a unique ε shadowing orbit. Moreover the variational equation along the shadowing orbit has an exponential dichotomy with the constants K, α and projectors of rank l.*

As an easy consequence of Proposition 2.7 we notice

Corollary 2.8 *Let the assumptions of Proposition* 2.7 *be satisfied and let $PO_\delta(\Lambda)$ be the set of all δ pseudo orbits in Λ. Then the bounded set*

$$\hat{\Lambda} = \{x \in \mathbb{R}^k : \text{ there exists } \varepsilon \in (0, \varepsilon_0] \text{ and } y_{\mathbb{Z}} \in PO_{\delta(\varepsilon)}(\Lambda) \text{ with } \sup_{n\in\mathbb{Z}} \|y_n - f^n(x)\| \leq \varepsilon\} \tag{13}$$

is hyperbolic in the sense of Definition 2.3.

In Section 4 the set $\hat{\Lambda}$ will provide the centers of the balls within which we find unique numerical solutions. We notice that $\hat{\Lambda}$, in general, is larger than the Cantor set Λ from the introduction, for which only a special type of pseudo orbits is chosen. $\hat{\Lambda}$ also contains the maximal invariant set of f constructed in the approach of [10].

3 Uniform approximation of homoclinic orbits in hyperbolic sets

In the following let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a C^1 diffeomorphism with a hyperbolic fixed point $\xi \in \mathbb{R}^k$. By [7, Section 4] this implies that $u_{n+1} = f'(\xi)u_n$ has an exponential dichotomy on $\mathbb{Z}$ with a constant projector P_s. Taking $X_s \oplus X_u$ as the decomposition of $\mathbb{R}^k$ into the stable and unstable subspace of $f'(\xi)$ the projector is given by

$$\mathcal{R}(P_s) = X_s, \qquad \mathcal{N}(P_s) = X_u. \tag{14}$$

Due to [7, Lemma 5.3] we have the following result.

Proposition 3.1 *There exist constants* $C, \beta, \rho > 0$ *such that*

$$\|f^n(x) - \xi\| \le Ce^{-\beta n}\|x - \xi\| \quad \textit{for all } n \in \mathbb{N}$$

if $f^n(x) \in B_\rho(\xi)$ *holds for all* $n \in \mathbb{N}$. *Similarly*

$$\|f^{-n}(x) - \xi\| \le Ce^{-\beta n}\|x - \xi\| \quad \textit{for all } n \in \mathbb{N}$$

if $f^{-n}(x) \in B_\rho(\xi)$ *holds for all* $n \in \mathbb{N}$.

Let $x_\mathbb{Z} \in S_\mathbb{Z}$ be a homoclinic orbit of f with respect to ξ, i.e. $x_{n+1} = f(x_n)$ for all $n \in \mathbb{Z}$ and $\lim_{n\to\pm\infty} x_n = \xi$. $x_\mathbb{Z}$ is called *transversal* if the variational equation $u_{n+1} = f'(x_n)u_n$ has an exponential dichotomy on $\mathbb{Z}$. By [7, Proposition 5.6]) this is equivalent to the standard condition of transversal intersection of stable and unstable manifolds at the point x_0. And by [2] it is also equivalent to the statement that $x_\mathbb{Z}$ is a regular zero of the operator $\Gamma(y_\mathbb{Z}) = (y_{n+1} - f(y_n))_{n\in\mathbb{Z}}$ in $S_\mathbb{Z}$.

We want to approximate homoclinic orbits by zeros of the operator $\Gamma_J : S_J \to S_J$ defined by

$$\Gamma_J(x_J) = (x_{n+1} - f(x_n)(n = n_-, \ldots, n_+ - 1), b(x_{n_-}, x_{n_+})), \quad x_J \in S_J \tag{15}$$

where $J = [n_-, n_+]$, $n_+ \ge n_-$ and $b : \mathbb{R}^{2k} \to \mathbb{R}^k$ is a given mapping.

In the following we give an approximation result for zeros of Γ_J lying in some bounded hyperbolic set M containing ξ. The result is a generalization of [2, Theorem 3.4] and the proof is quite similar. However, in our later applications there is a marked difference. Instead of a single homoclinic orbit we approximate all homoclinic orbits in the hyperbolic set M. This set M will be a shadowing type set $\hat\Lambda$ as in (13) and it contains contains infinitely many homoclinic orbits. This is true, for example, if $\hat\Lambda$ is constructed from the compact hyperbolic set

$$\Lambda_{THO} = \{\xi\} \cup \{x_n : n \in \mathbb{Z}\} \tag{16}$$

where $x_\mathbb{Z}$ is a transversal homoclinic orbit with respect to ξ. More generally, we can have

$$\Lambda = \{\xi\} \cup \bigcup_{j=1}^{N} \{x_n^j : n \in \mathbb{Z}\}$$

where $x_\mathbb{Z}^j$, $j = 1, \ldots, N$ are different transversal homoclinic orbits based at the same point ξ. This situation typically occurs in examples (see Section 5).

The equation $\Gamma_J(x) = 0$ is treated using the following approximate inverse function theorem (cf. [11, §3], [2, Proposition 3.3]).

Proposition 3.2 *Let* $F \in C^1(Y, Z)$ *with Banach spaces* Y, Z *and assume that* $F'(y_0)$ *is a homeomorphism for some* $y_0 \in Y$. *Further, let constants* $\mu, \nu, \tau > 0$ *be given with the following properties*

$$\begin{aligned} \|F'(y) - F'(y_0)\| &\le \nu < \tau \le \tfrac{1}{\|F'(y_0)^{-1}\|} \qquad \textit{for all } y \in B_\mu(y_0), \\ \|F(y_0)\| &\le (\tau - \nu)\mu. \end{aligned} \tag{17}$$

Then $F(y)=0$ has a unique solution in $B_\mu(y_0)$ and the following estimates hold

$$\begin{aligned} \|y_1-y_2\| &\le \frac{1}{\tau-\nu}\|F(y_1)-F(y_2)\| \quad \text{for all } y_1,y_2\in B_\mu(y_0), \\ \|F'(y)^{-1}\| &\le \frac{1}{\tau-\nu} \quad \text{for all } y\in B_\mu(y_0). \end{aligned} \tag{18}$$

The numerical approximation theorem is as follows.

Theorem 3.3 *Let $M\subset\mathbb{R}^k$ be a bounded hyperbolic set containing a hyperbolic fixed point ξ of a diffeomorphism $f\in C^1(\mathbb{R}^k,\mathbb{R}^k)$. Assume $b\in C^1(\mathbb{R}^{2k},\mathbb{R}^k)$ satisfies*

$$b(\xi,\xi)=0 \tag{19}$$

and the map $B\in L(X_s\oplus X_u,\mathbb{R}^k)$ defined by

$$B(x_s+x_u)=b'(\xi,\xi)(x_s,x_u) \tag{20}$$

is nonsingular.

Then there exist constants $\kappa,\sigma,C>0$ and $N\in\mathbb{N}$ such that Γ_J has a unique zero x_J in the closed ball

$$B_\sigma(\bar{x}_J)=\{x\in S_J : \|\bar{x}_J-x\|_\infty\le\sigma\}$$

for every $J=[n_-,n_+]$, $n_+-n_-\ge N$ and every finite orbit $\bar{x}_J=(f^n(\bar{x}_0))_{n\in J}$ with $\bar{x}_0\in M$ and $\bar{x}_{n_\pm}\in B_\kappa(\xi)$. Moreover, the following estimates hold

$$\|\Gamma'_J(y)^{-1}\|_\infty \le C \quad \text{for all } y\in B_\sigma(\bar{x}_J), \tag{21}$$

$$\|\bar{x}_J-x_J\|_\infty \le C\|b(\bar{x}_{n_-},\bar{x}_{n_+})\|. \tag{22}$$

Proof. We choose some orbit segment $\bar{x}_J=(f^n(\bar{x}_0))_{n\in J}$ in M with $J=[n_-,n_+]$, $n_+-n_-\ge N$ and $\bar{x}_{n_\pm}\in B_\kappa(\xi)$ and apply Proposition 3.2 with

$$Y=Z=S_J,\quad y_0=\bar{x}_J,\quad F=\Gamma_J.$$

We consider first

$$\Gamma'_J(\bar{x}_J)u_J=(u_{n+1}-f'(\bar{x}_n)u_n(n\in\tilde{J}),B_Ju_J)$$

where $\tilde{J}=[n_-,n_+-1]$ and

$$B_Ju_J=b'(\bar{x}_{n_-},\bar{x}_{n_+})(u_{n_-},u_{n_+})$$

and we show

$$\|\Gamma'_J(\bar{x}_J)^{-1}\|_\infty\le C. \tag{23}$$

This will be done in two steps.

Step 1: For any sequence $z_{\tilde{J}}\in S_{\tilde{J}}$ there exists $u_J\in S_J$ such that

$$u_{n+1}-f'(\bar{x}_n)u_n=z_n,\quad n\in\tilde{J} \tag{24}$$

and $\|u_J\|_\infty\le C\|z_{\tilde{J}}\|_\infty$.

Step 2: $\Gamma_J'(\bar{x}_J)v_J = (0,r)$ has a unique solution $v_J \in S_J$ for each $r \in \mathbb{R}^k$ and $\|v_J\|_\infty \le C\|r\|$.

Here $C, \kappa > 0$ and $N \in \mathbb{N}$ denote some generic constants depending only on the dichotomy data of the hyperbolic set M and the functions f and b with their derivatives. The constants C, κ, N are chosen appropriately within the proof.

Suppose we have accomplished the two steps above. Then for any given $z_{\tilde{J}}$ and $r \in \mathbb{R}^k$ we choose u_J as in step 1 and let v_J solve

$$\Gamma_J'(\bar{x}_J)v_J = (0, r - B_J u_J)$$

as in step 2. This implies

$$\Gamma_J'(\bar{x}_J)(u_J + v_J) = (z_{\tilde{J}}, r)$$

as well as

$$\begin{aligned} \|u_J + v_J\|_\infty &\le \|u_J\|_\infty + \|v_J\|_\infty \\ &\le C(\|z_{\tilde{J}}\|_\infty + \|r - B_J u_J\|) \\ &\le C(\|z_{\tilde{J}}\|_\infty + \|r\| + \|B_J\| \, \|u_J\|_\infty) \\ &\le C(\|z_{\tilde{J}}\|_\infty + \|r\|) \\ &\le C\|(z_{\tilde{J}}, r)\|_\infty. \end{aligned}$$

Thus we have shown the estimate (23).

Taking $\sigma > 0$ so small that

$$\|f'(y_1) - f'(z_1)\|, \; \|b'(y_1, y_2) - b'(z_1, z_2)\| \le \frac{1}{4C}$$

for all $z_1, z_2 \in M$ and $y_i \in B_\sigma(z_i), i = 1, 2$ we obtain

$$\begin{aligned} \|\Gamma_J'(y) - \Gamma_J'(\bar{x}_J)\|_\infty &\le \sup_{n \in J} \|f'(y_n) - f'(\bar{x}_n)\| + \|b'(y_{n_-}, y_{n_+}) - b'(\bar{x}_{n_-}, \bar{x}_{n_+})\| \\ &\le \frac{1}{2C} \end{aligned}$$

for all $y \in B_\sigma(\bar{x}_J)$. Setting $\tau = \frac{1}{C}$, $\nu = \frac{1}{2C}$ and $\mu = \sigma$ in (17) we finally obtain from assumption (19)

$$\|\Gamma_J(\bar{x}_J)\|_\infty = \|b(\bar{x}_{n_-}, \bar{x}_{n_+})\| \le \frac{\sigma}{2C}$$

for κ sufficiently small. Proposition 3.2 then yields the existence of the solution x_J in $B_\sigma(\bar{x}_J)$ and the estimate (22) follows from (18) by setting $y_1 = \bar{x}_J$, $y_2 = x_J$.

Proof of step 1. Let the constants K, α and projectors $\mathcal{P}(x)$, $x \in M$ be associated with the hyperbolic set M as in Proposition 2.4. We extend $z_{\tilde{J}}$ by setting $z_n = 0$ for $n \in \mathbb{Z} \setminus \tilde{J}$. Then we use Lemma 2.2 and solve

$$\Gamma'((f^n(\bar{x}_0))_{n \in \mathbb{Z}})u_{\mathbb{Z}} = z_{\mathbb{Z}}.$$

Obviously, $u_{|J}$ solves (24) and satisfies

$$\|u_{|J}\|_\infty \le \|u_{\mathbb{Z}}\|_\infty \le C\|z_{\mathbb{Z}}\|_\infty = C \sup_{n \in \tilde{J}} \|z_n\|$$

with $C = K\frac{1+e^{\alpha}}{1-e^{\alpha}}$ (see (8)).

Proof of step 2. From the uniqueness of the projectors on hyperbolic sets we have $\mathcal{P}(\xi) = P_s$ (see (14)) and continuity of $\mathcal{P}$ implies that for $x \in S$

$$\mathcal{P}(x) \to \mathcal{P}(\xi) \quad \text{as } x \to \xi.$$

So for $\|x-\xi\|$ small we have $\|\mathcal{P}(x) - \mathcal{P}(\xi)\| < 1$, hence the matrices $E(x) = I + \mathcal{P}(\xi) - \mathcal{P}(x)$ and $D(x) = I - \mathcal{P}(\xi) + \mathcal{P}(x)$ are nonsingular with

$$\|E(x)^{-1}\|, \|D(x)^{-1}\| \leq \frac{1}{1 - \|\mathcal{P}(x) - \mathcal{P}(\xi)\|} \tag{25}$$

and we find $\mathbb{R}^k = \mathcal{R}(\mathcal{P}(\xi)) \oplus \mathcal{N}(\mathcal{P}(x)) = \mathcal{N}(\mathcal{P}(\xi)) \oplus \mathcal{R}(\mathcal{P}(x))$. Therefore, $E(x) : \mathcal{R}(\mathcal{P}(x)) \to \mathcal{R}(\mathcal{P}(\xi)) = X_s$ is bijective and satisfies

$$\begin{aligned} \|(I - \mathcal{P}(\xi))E(x)^{-1}x_s\| &= \|(\mathcal{P}(x) - \mathcal{P}(\xi))E(x)^{-1}x_s\| \\ &\leq \frac{\|\mathcal{P}(x) - \mathcal{P}(\xi)\|}{1 - \|\mathcal{P}(x) - \mathcal{P}(\xi)\|}\|x_s\| \quad \text{for } x_s \in X_s. \end{aligned} \tag{26}$$

Similarly, $D(x) : \mathcal{N}(\mathcal{P}(x)) \to \mathcal{N}(\mathcal{P}(\xi)) = X_u$ is bijective and

$$\|\mathcal{P}(\xi)D(\xi)^{-1}x_u\| \leq \frac{\|\mathcal{P}(x) - \mathcal{P}(\xi)\|}{1 - \|\mathcal{P}(x) - \mathcal{P}(\xi)\|}\|x_u\| \quad \text{for } x_u \in X_u. \tag{27}$$

Using the solution operator $\Phi(n,m) = \Phi(n,m)(\bar{x}_0)$ for equation

$$u_{n+1} = f'(\bar{x}_n)u_n$$

we may write v_J as

$$v_n = \Phi(n, n_-)\eta_- + \Phi(n, n_+)\eta_+, \quad n \in J \tag{28}$$

where $\eta_- \in \mathcal{R}(\mathcal{P}(\bar{x}_{n_-}))$, $\eta_+ \in \mathcal{N}(\mathcal{P}(\bar{x}_{n_+}))$ are to be determined from the boundary conditions

$$\begin{aligned} r &= B_J v_J \\ &= D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})(\eta_- + \Phi(n_-, n_+)\eta_+) \\ &\quad + D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})(\eta_+ + \Phi(n_+, n_-)\eta_-). \end{aligned} \tag{29}$$

By the uniform exponential dichotomy in M we have

$$\|\Phi(n, n_\pm)\eta_\pm\| \leq Ke^{\pm\alpha(n-n_\pm)}\|\eta_\pm\|. \tag{30}$$

It is convenient to change coordinates in the ansatz (28) via

$$E(\bar{x}_{n_-})\eta_- = x_s \in X_s, \quad D(\bar{x}_{n_+})\eta_+ = x_u \in X_u. \tag{31}$$

We employ the linear map B from (20) and rewrite (29) as

$$r = B(x_s + x_u) + \rho_+ + \rho_- \tag{32}$$

where according to (25), (26), (27) and (30), (31)

$$
\begin{aligned}
\|\rho_-\| &= \|(D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+}) - D_1 b(\xi,\xi))x_s \\
&\quad + D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})((I - \mathcal{P}(\xi))E_{n_-}^{-1} x_s + \Phi(n_-, n_+)\eta_+)\| \\
&\le \left(\|D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+}) - D_1 b(\xi,\xi)\| + \|D_1 b(\bar{x}_{n_-}, \bar{x}_{n_+})\| \frac{\|\mathcal{P}(\bar{x}_{n_-}) - \mathcal{P}(\xi)\|}{1 - \|\mathcal{P}(\bar{x}_{n_-}) - \mathcal{P}(\xi)\|}\right)\|x_s\| \\
&\quad + \frac{K e^{-\alpha(n_+ - n_-)}}{1 - \|\mathcal{P}(\bar{x}_{n_+}) - \mathcal{P}(\xi)\|}\|x_u\| \\
&=: \varrho_-^s(\bar{x}_{n_-}, \bar{x}_{n_+})\|x_s\| + \varrho_-^u(\bar{x}_{n_+}, n_+ - n_-)\|x_u\| \\
\|\rho_+\| &\le \left(\|D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+}) - D_2 b(\xi,\xi)\| + \|D_2 b(\bar{x}_{n_-}, \bar{x}_{n_+})\| \frac{\|\mathcal{P}(\bar{x}_{n_+}) - \mathcal{P}(\xi)\|}{1 - \|\mathcal{P}(\bar{x}_{n_+}) - \mathcal{P}(\xi)\|}\right)\|x_u\| \\
&\quad + \frac{K e^{-\alpha(n_+ - n_-)}}{1 - \|\mathcal{P}(\bar{x}_{n_-}) - \mathcal{P}(\xi)\|}\|x_s\| \\
&=: \varrho_+^u(\bar{x}_{n_-}, \bar{x}_{n_+})\|x_u\| + \varrho_+^s(\bar{x}_{n_-}, n_+ - n_-)\|x_s\|
\end{aligned}
$$

Let $\|y_s + y_u\|_* := \|y_s\| + \|y_u\|$ for $y_s + y_u \in X_s \oplus X_u$. Due to assumption (20) we can choose κ and N such that

$$
\begin{aligned}
\|\mathcal{P}(z_1) - \mathcal{P}(\xi)\| &\le \frac{1}{2}, \\
\varrho_-^s(z_1, z_2), \varrho_-^u(z_1, m), \varrho_+^u(z_1, z_2), \varrho_+^s(z_1, m) &\le \frac{1}{4\|B^{-1}\|_*}
\end{aligned}
$$

for all $z_1, z_2 \in B_\kappa(\xi) \cap M$ and $m \ge N$. Therefore, the linear equation (32) has a unique solution (x_s, x_u) and

$$\|x_s\| + \|x_u\| \le 2\|B^{-1}\|_* \|r\|_* \le C\|r\|$$

with a constant $C > 0$ depending only on B and the chosen norms. We define η_-, η_+ by (31) and v_n by (28). From (25) and the exponential dichotomies we obtain the estimate of step 2. ■

Remark 3.4 From (18) we have the inverse Lipschitz estimate

$$\|y_1 - y_2\|_\infty \le C\|\Gamma_J(y_1) - \Gamma_J(y_2)\|_\infty \quad \text{for all } y_1, y_2 \in B_\sigma(\bar{x}_J)$$

which shows that the nonlinear equations are uniformly well conditioned.

The intention leading to Theorem 3.3 was to approximate homoclinic orbits in the hyperbolic set M, though we did not explicitly assume this for the finite orbits $\bar{x}_J$. However, if such an assumption is made we arrive at the following corollary

Corollary 3.5 *Let the assumptions of Theorem* 3.3 *hold. Then there exist* $\sigma > 0$ *as in Theorem* 3.3, $\kappa, \beta, C > 0$ *and* $N \in \mathbb{N}$ *such that the unique zero* x_J *of* Γ_J *in* $B_\sigma(\bar{x}_{|J})$ *satisfies*

$$\|x_J - \bar{x}_{|J}\|_\infty \le C e^{-\beta \min\{N_- - n_-, n_+ - N_+\}} \tag{33}$$

for every homoclinic orbit $\bar{x}_{\mathbb{Z}}$ *of* f *with respect to* ξ *in* M *and every* $J = [n_-, n_+]$ *with* $n_+ - n_- \geq N$ *and* $n_+ \geq N_+$, $n_- \leq N_-$ *where*

$$\begin{aligned} N_+ &= \inf\{n \in \mathbb{Z} \cup \{-\infty\} : \bar{x}_m \in B_\kappa(\xi) \text{ for all } m \geq n\}, \\ N_- &= \sup\{n \in \mathbb{Z} \cup \{\infty\} : \bar{x}_m \in B_\kappa(\xi) \text{ for all } m \leq n\} \end{aligned}$$

and $e^{-\infty} := 0$.

Proof. Let Theorem 3.3 hold with constants $\tilde{\kappa}, \sigma, \tilde{C} > 0$, $N \in \mathbb{N}$ and Proposition 3.1 with $\bar{C}, \beta, \rho > 0$. Set $\kappa = \min\{\tilde{\kappa}, \rho\}$. Applying Theorem 3.3 to $\bar{x}_{|J}$ we get

$$\|\bar{x}_{|J} - x_J\|_\infty \leq \tilde{C}\|b(\bar{x}_{n_-}, \bar{x}_{n_+})\| \tag{34}$$

for the unique zero x_J of Γ_J in $B_\sigma(\bar{x}_{|J})$. By a Taylor expansion there exists $\hat{C} > 0$ such that

$$\|b(x, y)\| \leq \hat{C} \min\{\|x - \xi\|, \|y - \xi\|\}$$

for all $x, y \in B_\kappa(\xi)$. Setting

$$C = \max\{\tilde{C}\hat{C}\bar{C}\kappa, \tilde{C}\}$$

Proposition 3.1, estimate (34) and $\|\bar{x}_{n_\pm} - \xi\| \leq \kappa$ imply (33). ■

Two types of boundary conditions that always satisfy the assumptions of Theorem 3.3 and Corollary 3.5 are *periodic boundary conditions*

$$b_{per}(x, y) = x - y, \quad x, y \in \mathbb{R}^k \tag{35}$$

and *projection boundary condition*

$$b_{proj}(x, y) = (Q_s^T(x - \xi), Q_u^T(y - \xi)), \quad x, y \in \mathbb{R}^k \tag{36}$$

where the columns of the matrices $Q_s \in \mathbb{R}^{k,k_s}$ and $Q_u \in \mathbb{R}^{k,k_u}$, $k = k_s + k_u$ provide a basis of the stable and unstable subspace of $f'(\xi)^T$. In the later case the convergence as $n_\pm \to \pm\infty$ in (33) is twice as fast as in the periodic case (see [2, Section 3] for more details on the error estimates).

The periodic boundary conditions are special in the sense that they can be used to compute any periodic orbit in the hyperbolic set, not just those which approximate a homoclinic orbit. The following theorem shows that the boundary value problems for general periodic orbits in M are uniformly well posed which is a more precise statement than the well known and trivial fact that periodic orbits in hyperbolic sets are hyperbolic.

Theorem 3.6 *Let* $M \subset \mathbb{R}^k$ *be a bounded hyperbolic set of a* C^1 *diffeomorphism* $f : \mathbb{R}^k \to \mathbb{R}^k$ *and let the operator* Γ_J *be defined by the periodic boundary condition* b_{per}. *Then there exist constants* $\sigma, C > 0$ *and an integer* $N \in \mathbb{N}$ *such that for any periodic orbit* $\bar{x}_{\mathbb{Z}}$ *of period* $m \geq N$ *in* M *the finite segment* $\bar{x}_{|J}$, $J = [n_-, n_- + m]$, $n_- \in \mathbb{Z}$ *is the only zero of* Γ_J *in* $B_\sigma(\bar{x}_{|J})$. *Moreover,* $\|\Gamma_J'(y)^{-1}\|_\infty \leq C$ *holds for all* $y \in B_\sigma(\bar{x}_{|J})$.

Proof. The proof is similar to that of Theorem 3.3. Let $K, \alpha > 0$ and a projector valued function $\mathcal{P} : \mathbb{R}^k \to \mathbb{R}^{k,k}$ be associated with the hyperbolic set M according to Proposition 2.4. At first we show in two steps that there exist $C > 0$ and $N \in \mathbb{N}$ such that

$$\|\Gamma'_J(\bar{x}_{|J})^{-1}\|_\infty \leq C \tag{37}$$

holds for every periodic orbit $\bar{x}_\mathbb{Z}$ in M of period $m \geq N$ where $J = [n_-, n_+]$, $n_+ - n_- = m$. By Lemma 2.2 we have as in the proof of Theorem 3.3

Step 1: For any sequence $z_{\tilde{J}} \in S_{\tilde{J}}$, $\tilde{J} = [n_-, n_+ - 1]$ there exists $u_J \in S_J$ such that

$$u_{n+1} - f'(\bar{x}_n)u_n = z_n, \quad n \in \tilde{J}$$

and $\|u_J\|_\infty \leq K\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\|z_{\tilde{J}}\|_\infty$.

If

$$N \geq \frac{\ln(4K)}{\alpha} \tag{38}$$

and $m \geq N$ we show

Step 2: $\Gamma'_J(\bar{x}_{|J})v_J = (0, r)$ has a unique solution $v_J \in S_J$ for each $r \in \mathbb{R}^k$ and $\|v_J\|_\infty \leq 8K^2\|r\|$ holds.

Set $P_n = \mathcal{P}(\bar{x}_n)$, $n \in \mathbb{Z}$. The periodicity of $\bar{x}_\mathbb{Z}$ implies $P_{n+m} = P_n$ for all $n \in \mathbb{Z}$, especially

$$P_{n_-} = P_{n_+}. \tag{39}$$

Let Φ be the solution operator of $u_{n+1} = f'(\bar{x}_n)u_n$. For $r \in \mathbb{R}^k$ every solution v_J of $\Gamma'_J(\bar{x}_{|J})v_J = (0, r)$ is given by $v_n = \Phi(n, n_-)\eta_- + \Phi(n, n_+)\eta_+$, $n \in J$ with some $\eta_- \in \mathcal{R}(P_{n_-})$, $\eta_+ \in \mathcal{N}(P_{n_+})$. By (39) this means

$$v_n = (\Phi(n, n_-)P_{n_-} - \Phi(n, n_+)(I - P_{n_+}))(\eta_- - \eta_+), \quad n \in J \tag{40}$$

and

$$r = v_{n_-} - v_{n_+} = (I - \Phi(n_+, n_-)P_{n_-} - \Phi(n_-, n_+)(I - P_{n_+}))(\eta_- - \eta_+) \tag{41}$$

follows. Using the dichotomies and (38) we get

$$\|\Phi(n_-, n_+)(I - P_{n_+}) + \Phi(n_+, n_-)P_{n_-}\| \leq 2Ke^{-\alpha(n_+ - n_-)} \leq \frac{1}{2}.$$

Thus the linear equation (41) has a unique solution $\eta_- - \eta_+ \in \mathcal{R}(P_{n_-}) \oplus \mathcal{N}(P_{n_+})$. Moreover,

$$\|\eta_- - \eta_+\| \leq 2\|r\|$$

and by (40)

$$\|v_n\| \leq 4K\|r\|$$

holds for all $n \in J$.

As in the proof of Theorem 3.3 these two steps imply that there exists $C = C(K, \alpha) > 0$ such that (37) holds. Since M is bounded we can take $\sigma > 0$ such that $\|f'(y) - f'(z)\| \le \frac{1}{2C}$ holds for all $z \in M$ and $y \in B_\sigma(Z)$. So

$$\|\Gamma_J'(y_J) - \Gamma_J'(\bar{x}_{|J})\|_\infty \le \frac{1}{2C}$$

follows for all $y_J \in B_\sigma(\bar{x}_{|J})$. Setting $\mu = \sigma$, $\nu = \frac{1}{2C}$ and $\tau = 2\nu$ the statement of the theorem follows from Proposition 3.2 applied to $Y = Z = S_J$, $y_0 = \bar{x}_J$ and $F = \Gamma_J$. ■

Our aim is to compute the zeros of Γ_J using Newton's method. Under the assumptions of Theorem 3.3 let f and b be of type C^2. Then there exists $C_1 > 0$ such that

$$\|\Gamma_J''(y_J)\|_\infty \le C_1$$

for all $y_J \in B_\sigma(\bar{x}_J)$. Choose $C_2 \in (0, \frac{1}{2CC_1})$ small such that $\rho := \frac{1-\sqrt{1-2CC_1C_2}}{CC_1} < \sigma$. Then there exist $\tilde{\sigma} \in (0, \sigma - \rho)$ and $\tilde{\kappa} \in (0, \kappa]$ such that for every orbit segment $\bar{x}_J$ in M with $n_+ - n_- > N$, $J = [n_-, n_+]$ and $\bar{x}_{n_\pm} \in B_{\tilde{\kappa}}(\xi)$ and every $y_J \in B_{\tilde{\sigma}}(\bar{x}_J)$ the estimate

$$\|\Gamma_J(y_J)\|_\infty \le \frac{C_2}{C}$$

holds. By the Theorem of Newton-Kantorovich (see [5, §12.6]) the iterates of Newton's method

$$y_J^{n+1} = y_J^n - \Gamma_J'(y_J^n)^{-1}\Gamma_J(y_J^n), \quad n \in \mathbb{N} \tag{42}$$

are well defined for every $y_J^0 \in B_{\tilde{\sigma}}(\bar{x}_J)$. They stay in $B_\rho(y_J^0)$ for all $n \in \mathbb{N}$ and converge to a zero of Γ_J in $B_\rho(y_J^0)$. This is the unique zero x_J of Γ_J in $B_\sigma(\bar{x}_J)$ given by Theorem 3.3 because the choice of $\tilde{\sigma}$ implies $B_\rho(y_J^0) \subset B_\sigma(\bar{x}_J)$. By [5, §12.6, (5)] we also get the uniform convergence rate of Newton's method

$$\|y_J^n - x_J\|_\infty \le \frac{(2CC_1C_2)^{2^n}}{CC_1 2^n}, \quad n \in \mathbb{N}$$

for every starting vector $y_J^0 \in B_{\tilde{\sigma}}(\bar{x}_J)$. The choice of C_2 implies $\tilde{\rho} := 2CC_1C_2 \in (0,1)$ and so we have the following result.

Corollary 3.7 *Assume $f \in C^2(\mathbb{R}^k, \mathbb{R}^k)$, $b \in C^2(\mathbb{R}^{2k}, \mathbb{R}^k)$ and let the assumptions of either Theorem 3.3 or Theorem 3.6 be satisfied. Then there exists a possibly smaller $\kappa > 0$ (in case of Theorem 3.3) and constants $\tilde{\sigma} \in (0, \sigma)$, $\tilde{\rho} \in (0,1)$ and $C > 0$ such that the Newton iterates (42) started at any $y_J^0 \in B_{\tilde{\sigma}}(\bar{x}_J)$ are well defined and converge to the unique zero x_J of Γ_J in $B_\sigma(\bar{x}_J)$. Moreover,*

$$\|y_J^n - x_J\|_\infty \le C\,\tilde{\rho}^{2^n}$$

and $y_J^n \in B_\sigma(\bar{x}_J)$ holds for all $n \in \mathbb{N}$. The finite orbits $\bar{x}_J$ either satisfy $\bar{x}_{n_\pm} \in B_\kappa(\xi)$ as in Theorem 3.3 or they are periodic as in Theorem 3.6.

4 Approximation of homoclinic chaos

In this section we show how the previous results may be used to approximate the conjugacy in Smale's Theorem.

For two nonempty subsets $A, B \subset X$ of some metric space (X, d) the *Hausdorff distance of* A, B is defined as

$$\mathrm{d_H}(A,B) = \max\{\mathrm{dist}(A,B), \mathrm{dist}(B,A)\} \tag{43}$$

where $\mathrm{dist}(A,B) = \sup_{a\in A} \inf_{b\in B} d(a,b)$. The map $\mathrm{d_H}$ is a metric on the set of nonempty closed subsets of X. Moreover, the triangle inequality

$$\mathrm{d_H}(A,B) \leq \mathrm{d_H}(A,C) + \mathrm{d_H}(C,B) \tag{44}$$

holds for all nonempty subsets $A, B, C \subset X$ (see for example [3, §1]).

We consider the compact space $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ where $\{0,1\}$ is equipped with the discrete and Σ_2 with the product topology. It is easy to see that this topology is also generated by the metric

$$d(a_{\mathbb{Z}}, b_{\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} \frac{|a_n - b_n|}{2^{|n|}}, \quad a_{\mathbb{Z}}, b_{\mathbb{Z}} \in \Sigma_2.$$

The *shift map* is given by

$$\sigma : \begin{array}{ccc} \Sigma_2 & \to & \Sigma_2 \\ (a_n)_{n\in\mathbb{Z}} & \mapsto & (a_{n+1})_{n\in\mathbb{Z}} \end{array}$$

and it is a homeomorphism with certain chaotic features.

According to [7, Theorem 4.8] we have the following result.

Theorem 4.1 (Smale's Homoclinic Theorem) *Let $x_{\mathbb{Z}}$ be a transversal homoclinic orbit of f with respect to the hyperbolic fixed point ξ. There exists an integer $p \in \mathbb{N}$ and a compact subset $Q \subset \mathbb{R}^k$ which is invariant under f^p and a homeomorphism $\Psi : \Sigma_2 \to Q$ such that*

$$f^p \circ \Psi = \Psi \circ \sigma. \tag{45}$$

For the purpose of numerical approximation we slightly modify the construction of Ψ in Palmer [7]. Moreover we assume $f \in C^2(\mathbb{R}^k, \mathbb{R}^k)$ and $b \in C^2(\mathbb{R}^{2k}, \mathbb{R}^k)$. The set $\Lambda = \Lambda_{THO}$ (see (16)) is compact hyperbolic and $\hat{\Lambda}$ is a bounded hyperbolic set by Corollary 2.8. Let the statements of Theorem 3.3, Corollary 3.5, Theorem 3.6 and Corollary 3.5 hold for $M = \hat{\Lambda}$ and some $\kappa, \sigma, \tilde{\sigma} > 0$, $N \in \mathbb{N}$. Taking $\varepsilon_0, \delta(\cdot)$ from the Shadowing Lemma applied to Λ and ρ from Proposition 3.1 applied to ξ we choose ε with

$$0 < \varepsilon < \min\{\frac{1}{2}\|x_0 - \xi\|, \kappa, \rho, \tilde{\sigma}, \varepsilon_0\}. \tag{46}$$

Notice the difference to Palmer's choice $0 < \varepsilon < \min\{\frac{1}{2}\|x_0 - \xi\|, \varepsilon_0\}$. Select $L \in \mathbb{N}$ with

$$\|x_n - \xi\| < \frac{\delta(\varepsilon)}{2} \quad \text{for } n = -L, L+1 \tag{47}$$

and let

$$C_0 = (\xi, \ldots, \xi),\ C_1 = (x_{-L}, \ldots, x_L) \in (\mathbb{R}^k)^p, \quad p = 2L+1. \tag{48}$$

For $a_{\mathbb{Z}} \in \Sigma_2$ string the segments C_0, C_1 together and get $(C_{a_n})_{n\in\mathbb{N}}$ which is a $\delta(\varepsilon)$ pseudo orbit by (47). This orbit has a unique ε shadowing orbit $y_{\mathbb{Z}}$ by the Shadowing Lemma. We set $\Psi(a_{\mathbb{Z}}) = y_0$ as the midpoint of this shadowing orbit. Thus we have the following construction of Ψ:

$$\begin{array}{ccccccc} \Sigma_2 & \longrightarrow & PO_{\delta(\varepsilon)}(\Lambda_{THO}) & \longrightarrow & S_{\mathbb{Z}} & \longrightarrow & R^k \\ a_{\mathbb{Z}} & \overset{\text{string}}{\longmapsto} & (C_{a_n})_{n\in\mathbb{Z}} = c_{\mathbb{Z}}(a_{\mathbb{Z}}) & \overset{\varepsilon\,\text{shadow}}{\longmapsto} & y_{\mathbb{Z}} = y_{\mathbb{Z}}(c_{\mathbb{Z}}(a_{\mathbb{Z}})) & \overset{\text{project}}{\longmapsto} & y_0 = \Psi(a_{\mathbb{Z}}) \end{array}. \tag{49}$$

By the proof of Palmer [7, Proof of Theorem 3.5] the statement of Theorem 4.1 holds for p, Ψ and $Q = \Psi(\Sigma_2)$.

By our construction (46) the orbits with midpoints $\Psi(a_{\mathbb{Z}})$, $a_{\mathbb{Z}} \in \Sigma_2$ lie in the hyperbolic set $\hat{\Lambda}$. Moreover, by the conjugacy (45) the union of all points of these orbits satisfies

$$\bigcup_{a_{\mathbb{Z}}\in\Sigma_2} \{x \in \mathbb{R}^k \,:\, \text{there exists } n \in \mathbb{N} \text{ such that } f^n(x) = \Psi(a_{\mathbb{Z}})\} = \bigcup_{i=0}^{p-1} f^i(Q) \subset \hat{\Lambda}$$

This set is compact because f is continuous and Q is compact.

A sequence $a_{\mathbb{Z}} \in \Sigma_2$ is called *homoclinic* if $\lim_{n\to\pm\infty} \sigma^n(a_{\mathbb{Z}}) = 0$. By the discrete topology on the set $\{0,1\}$ this is equivalent to the property that there exists an $m \in \mathbb{N}$ such that $a_n = 0$ for all $|n| \geq m$. Let

$$\Sigma_2^H(m) = \{a_{\mathbb{Z}} \in \Sigma_2 : a_n = 0 \text{ for all } |n| \geq m\}$$

be the set of *homoclinic sequences of stage* $m \in \mathbb{N}\backslash\{0\}$. This set is finite for every $m \geq 1$ and contains 2^{2m-1} elements. The topology on Σ_2 implies that the homoclinic sequences are dense in Σ_2. From $\Sigma_2^H(m) \subset \Sigma_2$ we get $\text{dist}(\Sigma_2^H(m), \Sigma_2) = 0$. Moreover, for every $a_{\mathbb{Z}} \in \Sigma_2$ there is $b_{\mathbb{Z}} \in \Sigma_2^H(m)$ defined by $b_n = a_n$ for $|n| < m$ such that $d(a_{\mathbb{Z}}, b_{\mathbb{Z}}) \leq 4 \cdot 2^{-m}$. One easily sees that $\text{dist}(\Sigma_2, \Sigma_2^H(m)) = 4 \cdot 2^{-m}$ and thus

$$\mathrm{d_H}(\Sigma_2, \Sigma_2^H(m)) = 4 \cdot 2^{-m} \quad \text{for all } m \geq 1 \tag{50}$$

holds.

The sequence $a_{\mathbb{Z}} \in \Sigma_2$ is called *periodic of period* $q \in \mathbb{N}$ if $a_{n+q} = a_n$ for all $n \in \mathbb{Z}$ and q is minimal with this property. Similarly to the homoclinic codes we define

$$\Sigma_2^P(m) = \{a_{\mathbb{Z}} \in \Sigma_2 : a_{n+2m-1} = a_n \text{ for all } n \in \mathbb{Z}\}$$

for $m \in \mathbb{N}\backslash\{0\}$. This set has as many elements as $\Sigma_2^H(m)$ and

$$\mathrm{d_H}(\Sigma_2, \Sigma_2^P(m)) \leq 4 \cdot 2^{-m} \quad \text{for all } m \geq 1 \tag{51}$$

holds.

We notice that the conjugacy (45) implies that

$$\bar{x}_{\mathbb{Z}}(a_{\mathbb{Z}}) := (f^n(\Psi(a_{\mathbb{Z}})))_{n\in\mathbb{Z}}, \quad a_{\mathbb{Z}} \in \Sigma_2. \tag{52}$$

is an f-orbit of period $(2m-1)p$ if $a_{\mathbb{Z}}$ is a periodic code of period $m \geq 1$.

One way to approximate Q is to use Theorem 3.6 and calculate $\Psi(\Sigma_2^P(m))$ for large $m \in \mathbb{N}$. This will be done first. Then we will approximate Q by computing $\Psi(\Sigma_2^H(m))$ for large $m \in \mathbb{N}$.

Since Σ_2 is compact and Ψ is continuous, we can define the modulus of continuity

$$\omega(\varepsilon) = \sup\{\|\Psi(a_\mathbb{Z}) - \Psi(b_\mathbb{Z})\| : a_\mathbb{Z}, b_\mathbb{Z} \in \Sigma_2, d(a_\mathbb{Z}, b_\mathbb{Z}) \leq \varepsilon\}, \quad \varepsilon \geq 0 \tag{53}$$

and $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. From (43) we obtain

$$\mathrm{d_H}(\Psi(A), \Psi(B)) \leq \omega(\mathrm{d_H}(A, B)) \quad \text{for all } A, B \subset \Sigma_2. \tag{54}$$

Using (51) this implies

$$\mathrm{d_H}(Q, \Psi(\Sigma_2^P(m))) \leq \omega(4 \cdot 2^{-m}) \quad \text{for } m \geq 1.$$

Notice that we do not need $\varepsilon \leq \min\{\kappa, \rho\}$ in (46) here. By Theorem 3.6 and Corollary 3.7 we have

Proposition 4.2 *Consider the conjugacy Ψ from Theorem 4.1 constructed as above. Assume $f \in C^2(\mathbb{R}^k, \mathbb{R}^k)$ to be a C^1 diffeomorphism and use periodic boundary conditions b_{per} to define the operator Γ_J (see (15) and (35)). Then there exist $\delta > 0$, $N \in \mathbb{N}$ such that*

$$\mathrm{d_H}(Q, Q^P(m)) \leq \omega(4 \cdot 2^{-m})$$

holds for $(2m-1)p \geq N$, $Q^P(m) = \{[x_J(a_\mathbb{Z})]_0 : a_\mathbb{Z} \in \Sigma_2^P(m)\}$, $J = [n_-, n_+]$ and $n_+ = n_- + (2m-1)p$. Here $x_J(a_\mathbb{Z})$ is the limit (uniformly in m) of the Newton iterates (42) when started at the pseudo orbit

$$(C_{a_{-(m-1)}}, \dots, C_{a_0}, \dots C_{a_{m-1}}, \eta)$$

where C_0, C_1 are given as in (48) and η is the first entry in the segment $C_{a_{-(m-1)}}$. $x_J(a_\mathbb{Z})$ coincides on J with the true shadowing orbit $\bar{x}_\mathbb{Z}(a_\mathbb{Z})$ from (52) and is the unique zero of Γ_J in $B_\delta(x_J(a_\mathbb{Z}))$. For $\omega(\cdot)$ see (53).

For homoclinic codes we get a weaker result. Let

$$N(m) = (m-1)(2L+1) + L + 1 \quad \text{for } m \in \mathbb{N}. \tag{55}$$

Proposition 4.3 *Consider the conjugacy Ψ and f, b as in the construction following Theorem 4.1. Then there exist $\delta, \beta, C > 0$ and $N \in \mathbb{N}$ such that for every $m \geq 1$, $\pm n_\pm \geq N(m)$, $n_+ - n_- \geq N$, $J = [n_-, n_+]$*

$$\mathrm{d_H}(Q, Q^H(J, m)) \leq \omega(4 \cdot 2^{-m}) + Ce^{-\beta(\min\{-n_-, n_+\} - N(m))}.$$

Here $Q^H(J, m) = \{[x_J(a_\mathbb{Z})]_0 : a_\mathbb{Z} \in \Sigma_2^H(m)\}$ and $x_J(a_\mathbb{Z})$ denotes the unique zero of Γ_J in $B_\delta(\bar{x}_{|J}(a_\mathbb{Z}))$, $a_\mathbb{Z} \in \Sigma_2^H(m)$ with the true shadowing orbit $\bar{x}_\mathbb{Z}(a_\mathbb{Z})$ given by (52). Newton's method converges to this zero (with a rate uniform in L and m) when started at

$$c_{|J}(a_\mathbb{Z}) = (\xi, \dots, \xi, C_{a_{-(m-1)}}, \dots C_{a_0}, \dots, C_{a_{m-1}}, \xi, \dots, \xi).$$

Here the number of trailing and leading ξ's must be chosen so that $n_+ - n_- \geq N$ holds for the total length of $c_{|J}(a_\mathbb{Z})$.

Proof. We apply Corollary 3.5 to the bounded hyperbolic set $\hat{\Lambda}$. By construction of Ψ this set contains all orbits $\bar{x}_{\mathbb{Z}}(a_{\mathbb{Z}})$, $a_{\mathbb{Z}} \in \Sigma_2$ (see (52)). Take $C, \sigma > 0$ and $N \in \mathbb{N}$ from Corollary 3.5 as in the construction of Ψ. The choice of ε in (46) and Proposition 3.1 imply that $\bar{x}_{\mathbb{Z}}(a_{\mathbb{Z}})$ is homoclinic with respect to ξ for all $a_{\mathbb{Z}} \in \Sigma_2^H(m)$, $m \in \mathbb{N}$. Moreover, $\bar{x}_n(a_{\mathbb{Z}}) \in B_\varepsilon(\xi)$ for all $|n| \geq N(m)$. By Corollary 3.5 there exists a unique zero $x_J(a_{\mathbb{Z}})$ in $B_\sigma(\bar{x}_{|J}(a_{\mathbb{Z}}))$ for every $J = [n_-, n_+]$ with $\pm n_\pm \geq N(m)$ and $n_+ - n_- \geq N$. This solution satisfies

$$\|\bar{x}_{|J}(a_{\mathbb{Z}}) - x_J(a_{\mathbb{Z}})\|_\infty \leq Ce^{-\beta \min\{-N(m)-n_-, n_+-N(m)\}}.$$

In terms of the distance $\mathrm{d_H}$ this means

$$\mathrm{d_H}(\Psi(\Sigma_2^H(m)), Q^H(J,m)) \leq Ce^{-\beta(\min\{n_-,n_-\}-N(m))}$$

for all $m \in \mathbb{N}$, $\pm n_\pm \geq N(m)$, $n_+ - n_- \geq N$ and $J = [n_-, n_+]$. By (54) and (50)

$$\mathrm{d_H}(Q, \Psi(\Sigma_2^H(m))) \leq \omega(4 \cdot 2^{-m})$$

holds for all $m \geq 1$ and our assertions follows from the triangle inequality (44) and Corollary 3.7. ∎

5 Numerical implementation and applications

In this section we illustrate Propositions 4.2 and 4.3 by two examples and we conclude with a branch following of a double humped homoclinic orbit.

In the following examples we compute zeros of the nonlinear operator Γ_J (see (15)) by Newton's method. This is justified by Theorem 3.3. The calculations are done with a machine precision of about 10^{-16} and as in [2, Section4] we stop the Newton iteration (42) if

$$\|\Gamma_J'(y_J^n)^{-1}\Gamma_J(y_J^n)\|_\infty \leq 10^{-15}(1 + \|y_J^n\|_\infty).$$

In all of our examples we neither know an exact transversal homoclinic orbit nor the quantities used in the construction of Ψ. Therefore, we take a zero of $\Gamma_{[\bar{n}_-, \bar{n}_+]}$ with large $-\bar{n}_-$, $\bar{n}_+$ as an exact orbit $x_{[\bar{n}_-, \bar{n}_+]}$ as in [2]. Then we choose some $L \in \mathbb{N}$ and set C_0, C_1 as in (48).

Example 1 (Hénon's map)

Consider the map

$$f(x, y) = (1 + y - ax^2, bx)$$

which is a C^∞ diffeomorphism for $b \neq 0$, and for

$$a = 1.3, \quad b = 0.3 \tag{56}$$

it has a hyperbolic fixed point at $\xi = (0.631\ldots, 0.189\ldots)$. We set $J = [-50, 50]$ and define Γ_J by periodic boundary conditions. Then Newton's method is started at the rather crude guess v_J given by

$$v_n = \begin{cases} 0 & \text{if } n = 0 \\ \xi & \text{otherwise} \end{cases} \tag{57}$$

and it converges to a peridic orbit x_J with

$$x_0 = (0.338\ldots, -0.255\ldots).$$

For more details see [2].

We start the approximation of Smale's set with the values $L = 3$, $m = 2$ in Proposition 4.2. For a binary sequence

$$(a_{-(m-1)}, \ldots, a_0, \ldots, a_{m-1}) \tag{58}$$

we take the stringed vector

$$(C_{a_{-(m-1)}}, \ldots, C_{a_0}, \ldots, C_{a_{m-1}}, C_{a_{-(m-1)}}) =: (c_{-(m-1)(2L+1)-L}, \ldots, c_{m(2L+1)+L}).$$

To get an approximate zero $y_{\tilde{J}}$ of $\Gamma_{\tilde{J}}$ on $\tilde{J} = [-(m-1)(2L+1), m(2L+1)]$ ignore L elements on both sides and take

$$v_{\tilde{J}} = (c_{-(m-1)(2L+1)}, \ldots, c_{m(2L+1)})$$

as starting vector for the Newton iteration. The resulting point y_0 is then an element of $Q^P(m)$. In doing so for every binary sequence (58) we get the whole set $Q^P(m)$.

In Figure 2 we display the results of this computation in the xy–plane. Notice that the set is very thin and the points cluster in two regions around ξ and x_0.

In the Figures 3 and 4 we zoom into these clusters in the xy–coodinates and we introduce as a third coordinate the subscript i of the bit a_i. If $a_i = 0$ we draw a little box and a bar if $a_i = 1$.

In the figures we have added the bit $a_m = a_{-(m-1)}$ to indicate how the codes in $\Sigma_2^P(m)$ continue. In this way we can decode the points in $Q^P(m)$ and relate it to the clustering. In Figure 3, which shows the area around x_0, the bit a_0 is 1 for all points, while in the area near ξ given in Figure 4 the bit a_0 is always 0. In both figures the clustering continues. The next steps of separation depend on the bits a_1 and a_{-1}.

Figure 5 shows the whole set $Q^P(m)$. Notice that there are eight points coded but it is not possible to separate all of them in this scaling. At some 'double points' a box and a bar as a common symbol for both bits occur.

It is also possible to approximate Smale's set through homoclinic points as in Proposition 4.3. The resulting set $Q^H(J, m)$ differs from $Q^P(m)$ only in higher decimal places, so the figure is visually the same. We notice that for Figure 1 we took a trajectory through an element of $Q^H([n_-, n_+], m)$ with $-n_- = n_+ = N(m) + 39$ (see (55)).

We have continued this procedure to higher code and segment lengths, but the pictures remain almost the same exept that now the bit lines are longer and more code sequences

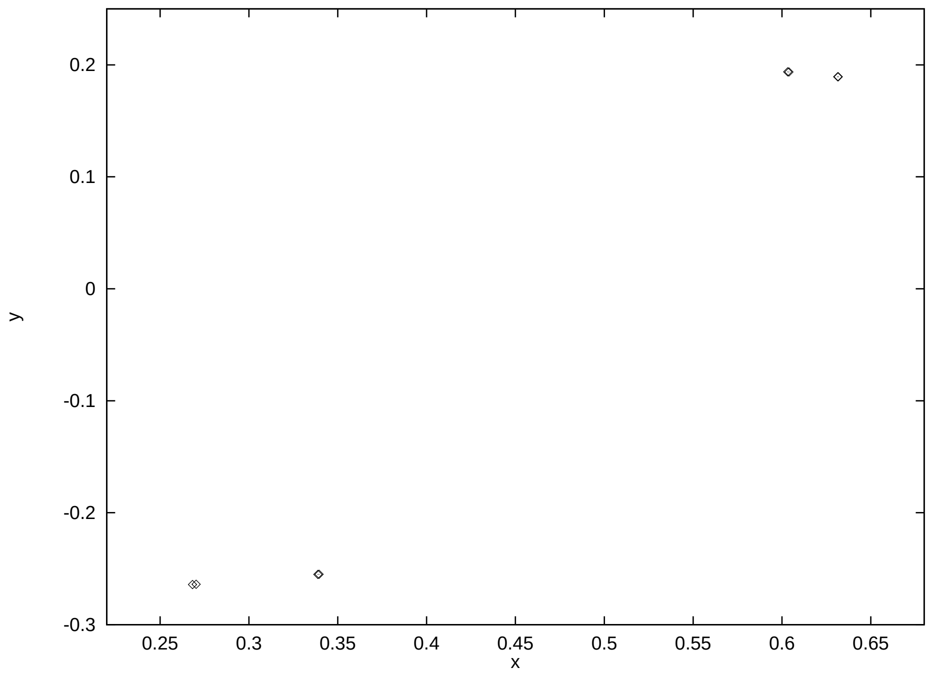

Figure 2: The set $Q^P(2)$ for Hénon's map.

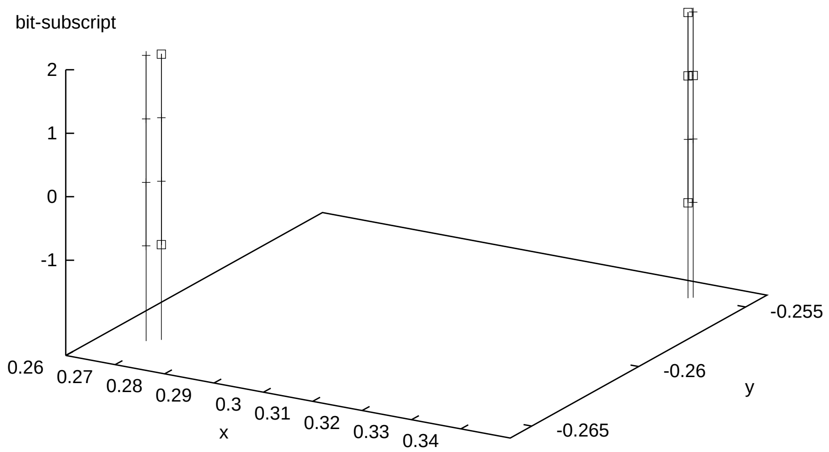

Figure 3: The points near x_0 in $Q^P(2)$ together with their codes for Hénon's map.

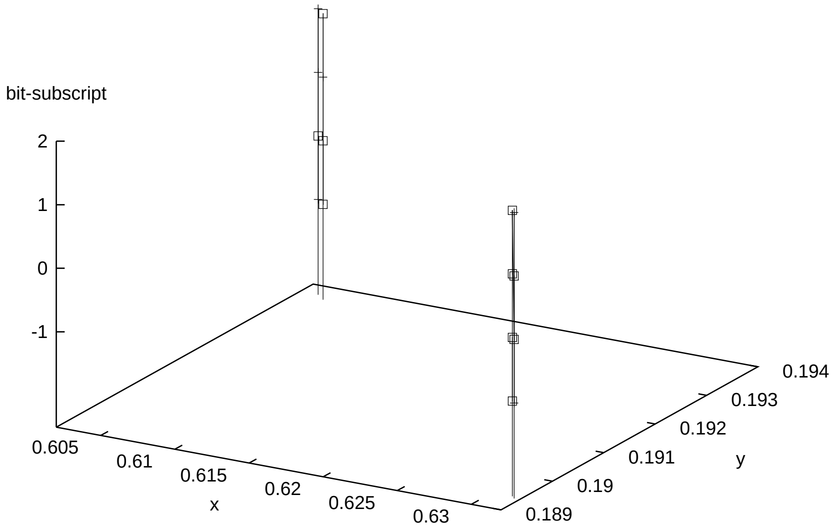

Figure 4: The points near ξ in $Q^P(2)$ together with their codes for Hénon's map.

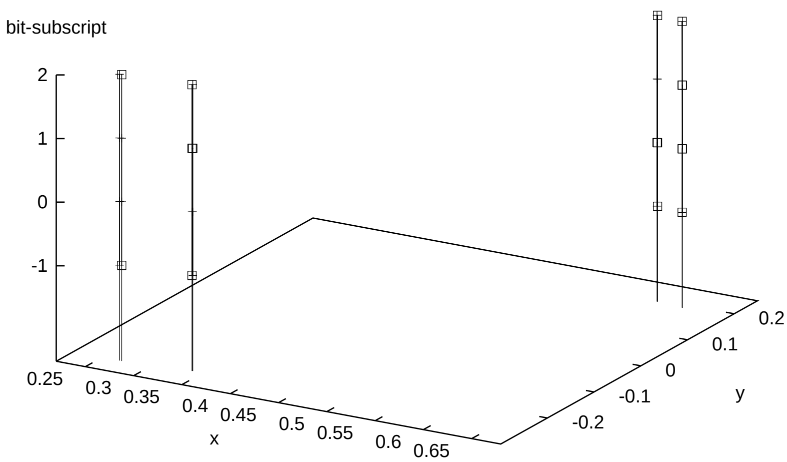

Figure 5: All points in $Q^P(2)$ together with their codes for Hénon's map.

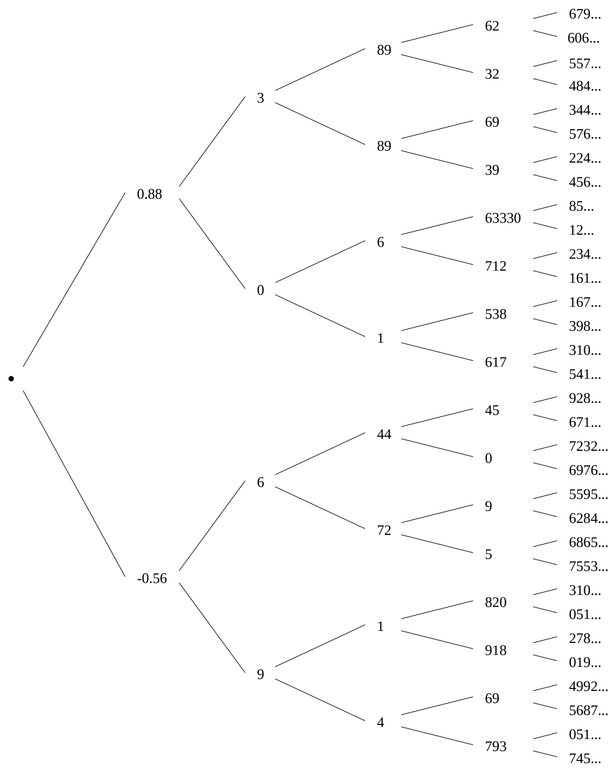

Figure 6: A binary tree for the decimals of the first component of the points in $Q^P(J, m)$ for the 3D-Hénon map (for further explanation see the text).

are indistinguishable in the graphics. Taking $m = 6$, $L = 15$, for example, we can separate just four points of the set $Q^P(m)$ within the precision used.

We certainly lack here a good idea of visualization which allows to plot the Cantor set and simultaneously its coding to sufficient depth.

Example 2 (3D–Hénon map)

Consider the map

$$f(x, y, z) = (a + bz - x^2, x, y)$$

with the parameter values (56). It is a C^∞ diffeomorphism with a hyperbolic fixed point at

$$\xi = (0.883\ldots, 0.883\ldots, 0.883\ldots).$$

We define the operator Γ_J by projection boundary conditions as in (36). On $J = [-50, 50]$ we find a solution $x_{[-50,50]}$ with

$$x_0 = (-0.566\ldots, 1.524\ldots, 0.276\ldots)$$

by starting the Newton iteration with the vector (57).

a	$\Psi(a)$
...0100010...	(0.8838932484... , 0.8838980990... , 0.8838951418...)
...0000010...	(0.8838932557... , 0.8838980839... , 0.8838951732...)
...0100000...	(0.8838962606... , 0.8838962830... , 0.8838962365...)
...0000000...	(0.8838962679... , 0.8838962679... , 0.8838962679...)
...0010010...	(0.8839839224... , 0.8836979742... , 0.8842758722...)
...0110010...	(0.8839839456... , 0.8836979217... , 0.8842759686...)
...0010000...	(0.8839869344... , 0.8836961579... , 0.8842769666...)
...0110000...	(0.8839869576... , 0.8836961053... , 0.8842770631...)
...0100100...	(0.8860633012... , 0.8825888469... , 0.8846840688...)
...0000100...	(0.8860633085... , 0.8825888318... , 0.8846841002...)
...0100110...	(0.8860712161... , 0.8825840679... , 0.8846869472...)
...0000110...	(0.8860712234... , 0.8825840528... , 0.8846869786...)
...0010100...	(0.8861538167... , 0.8823884299... , 0.8850645792...)
...0110100...	(0.8861538398... , 0.8823883773... , 0.8850646756...)
...0010110...	(0.8861617310... , 0.8823836499... , 0.8850674568...)
...0110110...	(0.8861617541... , 0.8823835972... , 0.8850675532...)
...0101010...	(-0.5664406976... , 1.5246581818... , 0.2761513895...)
...0001010...	(-0.5664407232... , 1.5246581823... , 0.2761514998...)
...0101000...	(-0.5664445671... , 1.5246595885... , 0.2761482265...)
...0001000...	(-0.5664445928... , 1.5246595890... , 0.2761483368...)
...0011010...	(-0.5667256865... , 1.5246564748... , 0.2773741015...)
...0111010...	(-0.5667257553... , 1.5246564735... , 0.2773743965...)
...0011000...	(-0.5667295595... , 1.5246578820... , 0.2773709571...)
...0111000...	(-0.5667296284... , 1.5246578807... , 0.2773712521...)
...0101100...	(-0.5691820051... , 1.5256548948... , 0.2738987662...)
...0001100...	(-0.5691820310... , 1.5256548954... , 0.2738988777...)
...0101110...	(-0.5691918019... , 1.5256584574... , 0.2738906728...)
...0001110...	(-0.5691918278... , 1.5256584579... , 0.2738907843...)
...0011100...	(-0.5694694992... , 1.5256535153... , 0.2751348767...)
...0111100...	(-0.5694695687... , 1.5256535141... , 0.2751351749...)
...0011110...	(-0.5694793051... , 1.5256570791... , 0.2751268319...)
...0111110...	(-0.5694793745... , 1.5256570779... , 0.2751271302...)

Table 1: Approximations of the 2^5 points $\Psi(a)$ in the set $Q^H(J,m)$ for the 3D-Hénon map together with their codes $a \in \Sigma_2^H(m)$.

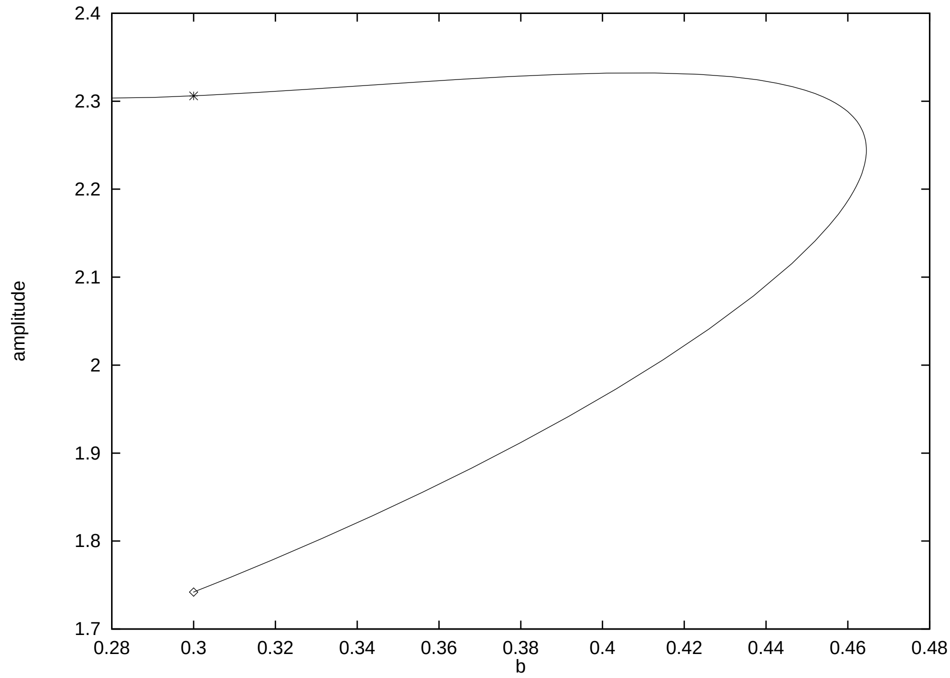

Figure 7: Following the primary homoclinic for Hénon's map.

We set $L = 6$, $m = 3$, $J = [n_-, n_+]$, $-n_- = n_+ = N(m) + 39$ (see (55)) and compute the set $Q^H(J, m)$ according to Proposition 4.3. In Table 1 the calculated elements of this set are given sorted by the first component of $\Psi(a)$, $a \in \Sigma_2^H(m)$. Notice the clustering in the components according to code bits. The first two clusters differ by the bit a_0 and the subsequent ones by a_1, a_{-1}, a_2 and a_{-2}.

This suggests to display the same information in a binary tree and this is done in Figure 6. It shows the decimal places of the first component of $\Psi(a)$, $a \in \Sigma_2^H(m)$. When passing from left to right in the tree we add one bit at each node (in the order a_0, a_1, a_{-1}, a_2, a_{-2}) and we go up if the bit has value 0 and go down otherwise.

Branch following of a double humped solution

As in [2] we can free a parameter in the nonlinear system (15) which defines Γ_J and apply a branch following procedure (cf. [1]). In the following we do this for Hénon's map with varying the parameter b.

We start with the primary homoclinic from Example 1 ($a = 1.3$, $b = 0.3$, $n_\pm = \pm 50$) and follow it through one turning point which correponds to a homoclinic tangency. In Figure 7 we show the resulting branch from [2] by plotting the *amplitude*

$$\sqrt{\sum_{n=n_-}^{n_+} \|y_n - \xi(b)\|^2}, \quad y_J \in (\mathbb{R}^k)^J, J = [n_-, n_+]$$

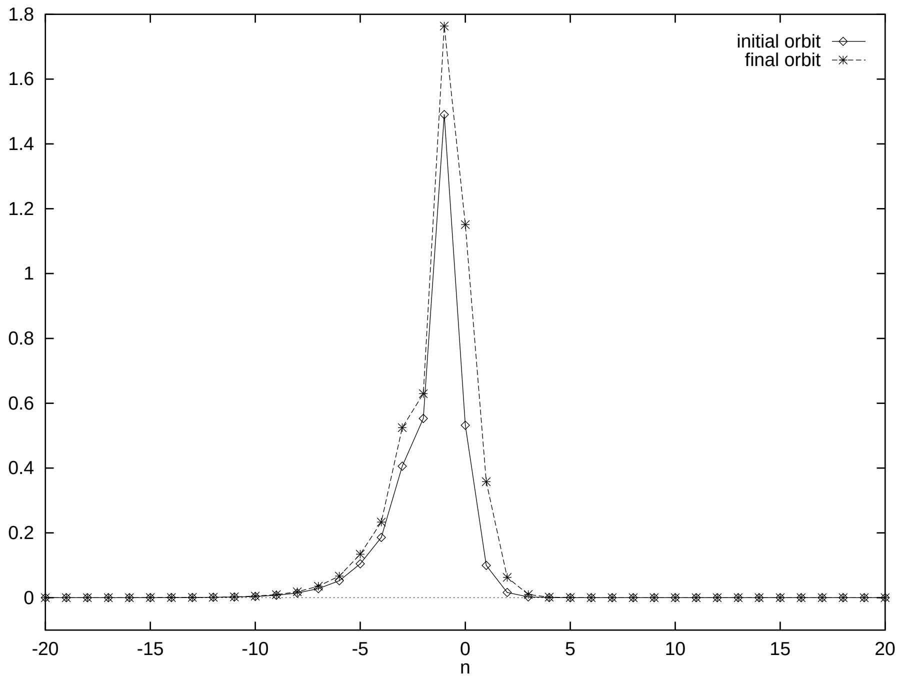

Figure 8: $\|y_n - \xi\|$ versus n for the initial and final primary homoclinic following the path for Hénon's map.

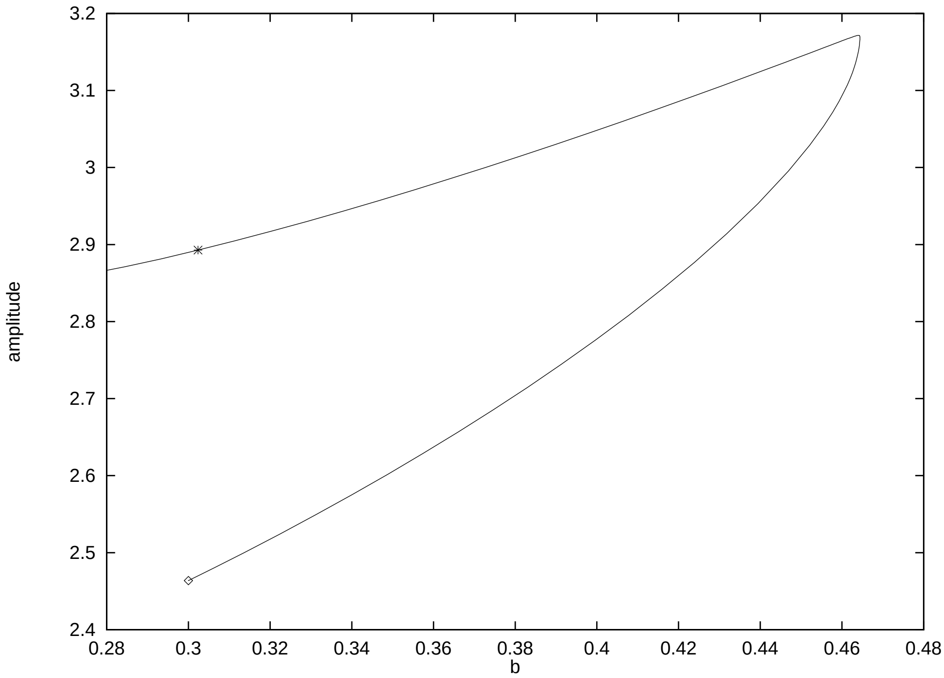

Figure 9: Following a double humped solution for Hénon's map.

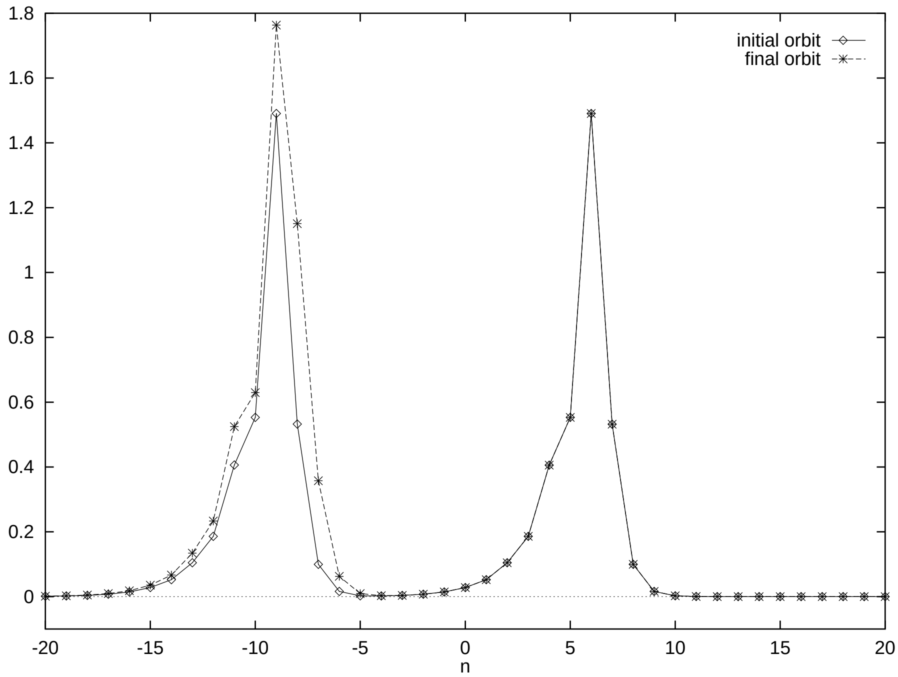

Figure 10: $\|y_n - \xi\|$ versus n for the initial and final homoclinic following the path of the double humped solution for Hénon's map.

versus the parameter. At $b = 0.3$ there are two different transversal homoclinics $x_{\mathbb{Z}}$, $\tilde{x}_{\mathbb{Z}}$ with numerical approximations y_J, $\tilde{y}_J$ depicted in Figure 8.

With y_J (marked by a little rhomb in Figure 7) we proceed as in Example 1 above and form the segment

$$C_1 = (y_{-L}, \dots, y_0, \dots, y_L), \quad L = 7.$$

The continuation of double humped homoclinics is then started at the shadowing orbit of

$$(\xi, \dots, \xi, C_1, C_1, \xi, \dots, \xi)$$

where we use 40 ξ's on each side.

Figure 9 shows the resulting branch and Figure 10 the initial and the final double humped homoclinic orbit at $b = 0.3$. We notice that there is a rather sharp turning point in Figure 9 which again corresponds to the homoclinic tangency. Second, looking at the shape of the humps in Figure 8 and Figure 10 we see that at the final point (marked by a little asterisk) the first hump now has the shape of $\tilde{y}_J$ whereas the second one remains almost the same. Hence we have found the shadowing orbit of

$$(\xi, \dots, \xi, \tilde{C}_1, C_1, \xi, \dots, \xi) \quad \text{where } \tilde{C}_1 = (\tilde{y}_{-L}, \dots, \tilde{y}_0, \dots, \tilde{y}_L).$$

As mentioned in Section 3 our approximation Theorem 3.3 still applies in this situation if we use the hyperbolic set

$$M = \{\xi\} \cup \{x_n : n \in \mathbb{Z}\} \cup \{\tilde{x}_n : n \in \mathbb{Z}\}.$$

In a similar way one finds that the branch started at the shadowing orbit of

$$(\xi, \dots, \xi, \tilde{C}_1, \tilde{C}_1, \xi, \dots, \xi)$$

ends at that of

$$(\xi, \dots, \xi, C_1, \tilde{C}_1, \xi, \dots, \xi).$$

As for example the monograph [6] shows there is a bewildering variety of phenomena created near homoclinic tangencies. We think that the methods provided in this paper may help in further exploring at least some of these phenomena from a different point of view. This will be the topic of further investigations.

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