Homework
Waves in Evolution Equations
Summer term 2017

Wolf-Jürgen Beyn
Christian Döding

Due: Wed. May 17, 12:00, V3-128, mailbox 128 (Christian Döding)

Tutorial: Tue. 23.05. 2017, 14-16, V5-148

Exercise 6: [Equivariance for a general elliptic principal part]
Consider a reaction diffusion equation with a general principal part for a real-valued function $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$,

$$u_t = \sum_{i,j=1}^{d} A_{ij} D_i D_j u + f(u), \quad x \in \mathbb{R}^d, t \geq 0. \tag{RD}$$

The matrix $A = (A_{ij}) \in \mathbb{R}^{d,d}$ is assumed to be symmetric positive definite. Determine a Lie group that is isomorphic to the special orthogonal group $\mathrm{SO}(d)$ such that the right-hand side of (RD) is equivariant with respect to its action. How would you then define a rotating wave for equation (RD)?
Hint: Set up a linear coordinate transformation in $\mathbb{R}^d$ which transforms the principal part of (RD) into Δu and then transfer the well-known action and the notion of a rotating wave.

(8 points)

Exercise 7: [Equivariance of a complex-valued evolution equation]
Consider an evolution equation for a complex-valued function $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{C}$, $d \geq 2$ given by

$$u_t = A\Delta u + g(|u|)u, \quad x \in \mathbb{R}^d, t \geq 0, \tag{CE}$$

where $A \in \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ is smooth. Determine an action of the (Lie) group $\mathrm{SO}(d) \times S^1$ (recall the unit circle $S^1 = \mathbb{R}/_{2\pi\mathbb{Z}}$) on functions $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{C}$ with respect to which the right hand side of (CE) is equivariant. Which equation is satisfied by wave profiles that rotate about the origin and whose images simultaneously rotate in $\mathbb{C}$?

(6 points)