Homework
Waves in Evolution Equations
Summer term 2017

Wolf-Jürgen Beyn
Christian Döding

Due: Wed. June 28, 12:00, V3-128, mailbox 128 (Christian Döding)

Tutorial: Tue. 05.07.2017, 14-16, V5-148

Exercise 18: [Invariance condition for a stable manifold]
Consider a two-dimensional system

$$\dot{u} = f(u), \; u = \begin{pmatrix} x \\ y \end{pmatrix}, \; f(u) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}, \tag{1}$$

where $f \in C^2(\mathbb{R}^2, \mathbb{R}^2)$. Assume that 0 is a hyperbolic saddle, i.e.

$$f(0) = 0, \quad Df(0) = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}, \quad \lambda_- < 0 < \lambda_+. \tag{2}$$

Let $H = \{(x, h(x)) : x \in \mathbb{R}\}$ be the graph of some function $h \in C^3(\mathbb{R}, \mathbb{R})$ satisfying $h(0) = 0$. Show the following

a) The manifold H is invariant under the flow of (1) if and only if

$$h'(x) f_1(x, h(x)) = f_2(x, h(x)) \quad \text{for all } x \in \mathbb{R}. \tag{3}$$

b) Let one of the conditions from a) hold. Show $h'(0) = 0$ and derive a formula for $h''(0)$.

(7 points)

Exercise 19: [Exponential behavior on the stable manifold]

a) Show that the following weighted spaces are Banach spaces for all $\eta > 0, k \geq 0$:

$$\begin{aligned} C^k_\eta([0,\infty), \mathbb{R}^m) = &\{v \in C^k([0,\infty), \mathbb{R}^m) : \quad \|v\|_{k,\eta} < \infty\}, \\ \|v\|_{k,\eta} := &\sum_{j=0}^{k} \sup_{t \geq 0} |e^{\eta t} v^{(j)}(t)|. \end{aligned} \tag{4}$$

b) Let $\dot{u} = f(u)$ with $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ be a dynamical system which has 0 as a hyperbolic saddle. Let $\mathbb{R}^m = X_s \oplus X_u$ be the decomposition into stable and unstable subspaces of $Df(0)$ and let P_s be the corresponding projector onto X_s. Show that there exist constants $\delta_0, \delta, \eta > 0$ such that the boundary value problem

$$\dot{v} = f(v) \text{ on } [0,\infty), \quad P_s v(0) = v_s \in X_s, \; |v_s| \leq \delta_0 \tag{5}$$

has a unique solution $v \in C^1_\eta([0,\infty), \mathbb{R}^m)$ with $\|v\|_{1,\eta} \leq \delta$.

Hint: Apply the Lipschitz inverse mapping theorem with parameter $\lambda = v_s$ (Appendix 8.4 of the manuscript) by using the weighted spaces from a). The main work is in the linear estimate.

(7 points)