Homework
Waves in Evolution Equations
Summer term 2017

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Due: Wed. July 5, 12:00, V3-128, mailbox 128 (Christian Döding)

Tutorial: Tue. 11.07.2017, 14-16, V5-148

Exercise 20: [Exponential dichotomy of the adjoint operator]
Let $S(t,s)$, $t,s \in J$ be the solution operator of a linear differential operator $\mathcal{L}v = \dot{v} - A(\cdot)v$, $A \in C(J,\mathbb{R}^{m,m})$ on an interval $J \subseteq \mathbb{R}$ (nonempty interior, bounded or unbounded).

a) Show that $\frac{d}{dt}S(t,s) = A(t)S(t,s)$ and $\frac{d}{ds}S(t,s) = -S(t,s)A(s)$ for $t,s \in J$. Conclude that the (adjoint) differential operator $\mathcal{L}^\star v = \dot{v} + A(\cdot)^\top v$ has the solution operator $S^\star(t,s) = S(s,t)^\top, t,s \in J$. (Hint: use a fundamental matrix)

b) If $\mathcal{L}$ has an exponential dichotomy on J with projectors $P_s(t), t \in J$ and constants $\alpha, K > 0$ (w.r.t. the Euclidean norm) then the adjoint operator also has an exponential dichotomy on J with projectors $P_s^\star(t) := (I_m - P_s(t))^\top, t \in J$ and the same constants.

(6 points)

Exercise 21: [Special solutions of inhomogeneous equations]
Let $\mathcal{L}$ be a linear differential operator on an interval J as in Exercise 20 which has an exponential dichotomy with projectors $P_s(t), t \in J$ and constants $K, \alpha > 0$. Define Green's matrix for $t,s \in J$ via

$$G(t,s) = \begin{cases} S(t,s)P_s(s), & t \geq s, \\ S(t,s)(P_s(s) - I_m), & t < s, \end{cases}$$

and for $r \in C_\mathrm{b}(J,\mathbb{R}^m)$ (continuous, bounded) let

$$v(t) = \int_J G(t,s)r(s)ds, \quad t \in J.$$

a) Prove that $v \in C_\mathrm{b}(J,\mathbb{R}^m) \cap C^1(J,\mathbb{R}^m)$ solves $\mathcal{L}v = r$ in J and derive an estimate $\|v\|_\infty \leq C\|r\|_\infty$. If, in addition, $A(\cdot)$ is bounded show that $\dot{v}$ is bounded as well and one has an estimate $\|\dot{v}\|_\infty \leq \tilde{C}\|r\|_\infty$.

b) If $J = [t_0,\infty)$ for some $t_0 \in \mathbb{R}$ and $\|r\|_\eta = \sup_{t\in J} e^{\eta t}|r(t)| < \infty$ for some $0 < \eta < \alpha$, then $\|v\|_\eta \leq C\|r\|_\eta$ for some constant C.

c) If $J = [t_0,\infty)$ for some $t_0 \in \mathbb{R}$ and the limits $\lim_{t\to\infty} A(t) = A_\infty$, $\lim_{t\to\infty} r(t) = r_\infty$ exist, then A_∞ is invertible and $\lim_{t\to\infty} v(t) = -A_\infty^{-1}r_\infty$, $\lim_{t\to\infty}\dot{v}(t) = 0$.
Hint: Show that $\lim_{t\to\infty}\int_J G(t,s)dsA_\infty = \lim_{t\to\infty}\int_J G(t,s)A(s)ds = -I_m$ and use Exercise 20 a).

(8 points)