Homework
Waves in Evolution Equations
Summer term 2017

Wolf-Jürgen Beyn
Christian Döding

Due: Wed. July 12, 12:00, V3-128, mailbox 128 (Christian Döding)

Tutorial: Tue. 18.07.2017, 14-16, V5-148

Exercise 22: [Solution of an inhomogeneous system in a Sobolev space]
Repeat the analysis of Exercise 21 for the function spaces $L^2(J, \mathbb{R}^m)$ and $H^1(J, \mathbb{R}^m)$. More precisely, let $\mathcal{L} = \frac{d}{dt} - A(\cdot)$ with $A \in C(J, \mathbb{R}^{m,m})$ have an exponential dichotomy on the interval J with projectors $P_s(t), t \in J$ and constants $K, \alpha > 0$. For $r \in L^2(J, \mathbb{R}^m)$ and with Green's function G from Exercise 21 define

$$v(t) = \int_J G(t,s)r(s)ds, \quad t \in J.$$

a) Show that $v \in L^2(J, \mathbb{R}^m)$ and derive an estimate $\|v\|_{L^2} \leq C_0 \|r\|_{L^2}$ for some $C_0 > 0$.

b) Under the additional assumption $A \in C_{\mathrm{b}}(J, \mathbb{R}^{m,m})$ show that $v \in H^1(J, \mathbb{R}^m)$ solves $\mathcal{L}v = r$ in J with the weak derivative and satisfies an estimate $\|v\|_{H^1} \leq C_1 \|r\|_{L^2}$ for some $C_1 > 0$.

Hints: In a) use Cauchy Schwarz and Fubini. In b) show that $A(\cdot)v + r$ is the weak derivative of v by applying test functions $\varphi \in C_0^\infty(J, \mathbb{R})$. Use integration by parts and Fubini.

(7 points)

Exercise 23: [Eigenvalues of the linearization about a travelling wave]
Let $v_\star \in C_b^2(\mathbb{R}, \mathbb{R}^m)$ and $\mu_\star \in \mathbb{R}$ satisfy the travelling wave equation

$$0 = v'' + \mu v' + f(v) \quad \text{in } \mathbb{R}, \tag{1}$$

where $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ and $\lim_{x \to \pm\infty} v_\star(x) = v_\pm$, $f(v_\pm) = 0$. Consider the linearized differential operator

$$\mathcal{L}v = v'' + \mu_\star v' + Df(v_\star(\cdot))v, \quad v \in C_b^2(\mathbb{R}, \mathbb{R}^m).$$

a) Show that $v_\star'$ is in the kernel of $\mathcal{L}$ by differentiating (1).

b) Compute a numerical approximation of the spectrum of $\mathcal{L}$ in the following way. Discretize the boundary eigenvalue problem

$$\mathcal{L}v = \lambda v \quad \text{in} \quad J = [x_-, x_+], \quad v(x_-) = v(x_+) = 0$$

with stepsize $h = \frac{x_+ - x_-}{N}$ and classical (centered) finite differences (Numerik II) to obtain a matrix eigenvalue problem $L_h v_h = \lambda v_h, v_h \in \mathbb{R}^{m(N-1)}$. Compute the spectrum of L_h for the wave of the Nagumo equation (Exercise 1, $b = \frac{1}{4}$) with the data $x_\pm = \pm j, N = 20j$, $j = 1, \ldots, 10$. For each j determine the maximum resp. minimum real part $\lambda_{\max}$ resp. $\lambda_{\min}$ of all eigenvalues and plot $|\lambda_{\max}|$ resp. $|\lambda_{\min}|$ in logarithmic scale versus $|x_\pm| = j$.

(7 points)