LARGE DEVIATIONS FOR LOCALLY MONOTONE SPDES DRIVEN BY LÉVY NOISE

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ABSTRACT. We establish a Freidlin-Wentzell type large deviation principle (LDP) for a class of stochastic partial differential equations with locally monotone coefficients driven by Lévy noise. Our results essentially improve a recent work on this topic (Bernoulli, 2018) by the second named author of this paper and his collaborator, because we drop the compactness embedding assumptions, and we also make the conditions for the coefficient of the noise term more specific and weaker. To obtain our results, we utilize an improved sufficient criteria of Budhiraja, Chen, Dupuis, and Maroulas for functions of Poisson random measures, and the techniques introduced by the first and second named authors of this paper in [26] play important roles.

As an application, for the first time, the Freidlin-Wentzell type LDPs for many SPDEs driven by Lévy noise in unbounded domains of $\mathbb{R}^d$, which are generally lack of compactness embeddings properties, are achieved, like e.g., stochastic p-Laplace equation, stochastic Burgers-type equations, stochastic 2D Navier-Stokes equations, stochastic equations of non-Newtonian fluids, etc.

Keywords: Freidlin-Wentzell type large deviation principle; Lévy noise; locally monotone; stochastic partial differential equations

1. Introduction

The Freidlin-Wentzell type large deviation principle (LDP) describes how the solutions of equations behave when the noise in the equation approaches zero, see [16] for more details. The LDP theory provides a powerful mathematical framework for studying the behavior of rare events and extreme fluctuations in a wide variety of complex systems. For example, large deviation theory plays a crucial role in statistical mechanics by providing a framework for analyzing rare fluctuations in physical systems, shedding light on phenomena such as phase transitions and fluctuations in thermodynamic quantities (see e.g. [13] for more details); in finance and economics, large deviation theory is applied to model extreme events in financial markets, such as stock market crashes or large price movements (see e.g. [17] for more details).

Different from the Gaussian noise, which is commonly used in many standard engineering and statistical applications, where the underlying processes can be adequately described by normal distributions, Lévy noise provides a flexible and powerful framework for modeling complex systems that exhibit non-Gaussian behavior, heavy-tailed distributions, and long-range dependence. This makes it well-suited for capturing the dynamics of real-world phenomena that cannot be adequately described by traditional Gaussian processes, see e.g. [23] for more details about Lévy noise.

Although there have been lots of papers investigated the LDPs for stochastic evolution equations (SEEs) and stochastic partial differential equations (SPDEs) driven by Gaussian noise (see e.g. [5, 7, 8, 11, 14, 21, 27] and the references therein), there has not been very much results

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about the LDPs for SEEs and SPDEs driven by Lévy noise. The first paper about large deviations for SEEs driven by Lévy processes is [24], in which the authors obtained large deviation results for an Ornstein–Uhlenbeck type process driven by additive Lévy noise. By using the contraction principle in the theory of large deviations (see e.g. [9, Theorem 4.2]), [29] dealt with the LDPs for two-dimensional stochastic Navier–Stokes equations driven by additive Lévy noise. In [25], the authors used the theorem of Varadhan and Bryc (see e.g. [12, Theorem 1.3.8]) coupled with the techniques of Feng and Kurtz [15] to prove a LDP result for solutions of abstract SEEs perturbed by multiplicative Lévy noise, on a larger space (hence with a weaker topology) than the actual state space of the solution. Since Budhiraja, Dupuis and Maroulas introduced the weak convergence approach in [6], it has been applied to study the LDPs in various dynamical systems driven by Lévy noise, like e.g., the LDPs for (2.3) on some nuclear spaces with a monotone condition and Hilbert spaces with a locally monotone condition were established in [3] and [28], respectively; in [30], the authors proved the LDPs for a class of SEEs driven by multiplicative Lévy noise on Hilbert spaces, which is the actual state space of the solution; the LDPs for two-dimensional stochastic Navier–Stokes equations driven by multiplicative Lévy noises were investigated in [31] and [2] (we remark here that the strong solutions in [29, 31] are concerned in the probabilistic sense, while the strong solutions in [2] are concerned in the PDE sense). We stress that all the existing results on this topic (though we do not list all the works here), except for two papers [26] and [33] (cf. the next paragraph for further details), require the compactness of embeddings. However, it is very common to encounter situations where embeddings lack compactness, especially in some interesting state spaces such as the general σ-finite measurable spaces, and in particular, the unbounded domains in $\mathbb{R}^d$. Therefore, without the compactness conditions, whether the LDPs for SPDEs driven by Lévy noise can be established or not is significantly worthwhile to be investigated.

As far as we know, there are only two papers about the LDPs for SPDEs driven by Lévy noises without using the compactness assumptions. In [26], the first and second named authors of this paper proved the LDPs for a class of stochastic porous media equations driven by Lévy noise on general σ-finite measurable spaces. In [33], the second and third named authors of this paper and Liu established a LDP for stochastic nonlinear Schrödinger equation with either focusing or defocusing nonlinearity driven by nonlinear multiplicative Lévy noise in the Marcus canonical form on the whole $\mathbb{R}^d$. However, there are many other SPDEs, like e.g., stochastic p-Laplace equation, stochastic Burgers-type equations, stochastic 2D Navier-Stokes equations, stochastic equations of non-Newtonian fluids, etc., and the LDPs for these SPDEs driven by Lévy noises in the absence of compactness embeddings are open problems to be solved. Since these equations can be formulated within the framework of locally monotone SPDEs, this strongly motivates us to solve the problems by establishing the LDPs for a class of SPDEs with locally monotone coefficients driven by Lévy noises, without using the compactness embeddings hypothesis (cf. Theorem 3.7 for our main result). We would like to emphasize that, as an application of our current work, the LDPs for the aforementioned SPDE examples driven by Lévy noises in unbounded domains of $\mathbb{R}^d$, are achieved for the first time.

We will employ the same criteria as the ones used in [26, 33]. It is an improved sufficient criteria proposed in [20] by the second named author of this paper and his collaborators, and can be regarded as an adaption of the weak convergence approach introduced in [6] and [3] for the case of Poisson random measures. The improved sufficient criteria was first introduced by Matoussi, Sabbagh, and Zhang in [21] for the Wiener case, and has been proved to be more effective and suitable to deal with SPDEs with highly nonlinear terms, see e.g. [10, 27] for the Wiener case. The main procedures of our proofs are to establish the well-posedness of solutions to the skeleton equations, the convergence of the solutions to the skeleton equations, and some convergence between the solutions to the controlled SPDEs and the solutions to the skeleton equations.

As mentioned before, in [28], the second named author of this paper and Xiong also proved the LDPs for (2.3), but our current paper essentially improves the results in [28]. The main improvement is that we drop the compactness assumption for the Gelfand triples used in [28]. Another improvement is that we make the conditions for the coefficient of the noise term more specific and weaker. To be precise, we note that $L_2(\nu_T)\cap\mathcal{H}_2 \subset L_{\beta+2}(\nu_T)\cap\mathcal{H}_2$, $\beta \geqslant 0$, (cf. Section 3 for the definitions of the spaces and Remark 3.1 for the proof; see also [4]), which makes [28, page:2848, **(H5)**] more concrete. In our paper, we use the concrete assumption (cf. **(H6)** in Section 3). Also, we find that to obtain our results, it suffices to assume that $G_f \in \mathcal{H}^{\varpi_0}\cap L_2(\nu_T)$ for some $\varpi_0 \in (0,+\infty)$ (cf. **(H7)** in Section 3). Since $\mathcal{H}_2 \subset \mathcal{H}^\infty \subset \mathcal{H}^\varpi$, $\forall \varpi \in (0,+\infty)$ (cf. (3.1) below), it means our assumption for G_f is weaker than [28, page:2849, **(H6)**], where G_f is assumed to be in $\mathcal{H}_2 \cap L_2(\nu_T)$. To achieve these improvements, we can not follow the methods used in [28], but have to employ different approaches/new ideas, which surprisingly simplify the proofs than expected. We utilize the improved sufficient criteria proposed in [20], which has been proven to be an effective criteria in [20, 26, 33] and seems to be a more appropriate criteria compared to the one used in [28]. We adopt a series of technical methods, including time discretization, a cut-off argument, and relative entropy estimates of a sequence of probability measures as used in [26] to prove the convergence of the solutions to the skeleton equations. With all the discussions above, we can conclude that although both [28] and our present paper are concerned with the general framework, our work in fact has fundamental improvements compared to [28]. These consequently lead us to use different strategies and techniques, making the entire program a nontrivial endeavor.

The remainder of this paper is organized as follows: Section 2 provides a review of basic notations related to Poisson random measures and presents fundamental knowledge about Gelfand triples. Section 3 states the hypotheses and the main result: the large deviations for (2.3). Section 4 is dedicated to proving the existence and uniqueness of solutions to the skeleton equations. Sections 5 and 6 are focused on proving the improved sufficient criteria.

2. Preliminaries

Set $\mathbb{N} := \{1, 2, 3, \cdots\}$, $\mathbb{R} := (-\infty, +\infty)$ and $\mathbb{R}_+ := [0, +\infty)$. For a metric space S, the Borel σ-field on S will be written as $\mathcal{B}(S)$. We denote by $C_c(S)$ the space of real-valued continuous functions with compact supports. Let $C([0, T]; S)$ be the space of continuous functions $g : [0, T] \to S$ endowed with the uniform convergence topology. Let $D([0, T]; S)$ be the space of all càdlàg functions $g : [0, T] \to S$ endowed with the Skorokhod topology.

For a locally compact Polish space S, the space of all Borel measures on S is denoted by $M(S)$, and $M_{FC}(S)$ denotes the set of all $\mu \in M(S)$ with $\mu(O) < +\infty$ for each compact subset $O \subseteq S$. We endow $M_{FC}(S)$ with the weakest topology such that for each $g \in C_c(S)$ the mapping $\mu \in M_{FC}(S) \to \int_S g(s)\mu(ds)$ is continuous. This topology is metrizable such that $M_{FC}(S)$ is a Polish space, see [6] for more details.

We fix $T > 0$ throughout this paper. Assume that Z is a locally compact Polish space with a σ-finite measure $\nu \in M_{FC}(Z)$. The probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ is specified as follows.

$$\Omega := M_{FC}\big([0, T] \times Z \times \mathbb{R}_+\big), \qquad \mathcal{F} := \mathcal{B}(\Omega).$$

We introduce the coordinate mapping

$$\bar{N} : \Omega \to \Omega : \bar{N}(\omega) = \omega, \ \forall \omega \in \Omega.$$

Define for each $t \in [0, T]$ the σ-algebra

$$\mathcal{G}_t := \sigma\left\{\bar{N}((0, s] \times A) : \ 0 \leqslant s \leqslant t, \ A \in \mathcal{B}\big(Z \times \mathbb{R}_+\big)\right\}.$$

For the given ν, it follows from [18, Sec.I.8] that there exists a unique probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that $\bar{N}$ is a Poisson random measure (PRM) on $[0, T] \times Z \times \mathbb{R}_+$ with intensity measure $\mathrm{Leb}_T \otimes \nu \otimes \mathrm{Leb}_\infty$, where Leb_T and Leb_∞ stand for the Lebesgue measures on $[0, T]$ and $\mathbb{R}_+$ respectively.

We denote by $\mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$ the $\mathbb{P}$-completion of $\{\mathcal{G}_t\}_{t\in[0,T]}$ and $\mathcal{P}$ the $\mathbb{F}$-predictable σ-field on $[0,T]\times\Omega$. The PRM $\bar{N}$ is defined on the (filtered) probability space $(\Omega, \mathcal{F}, \mathbb{F} := \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$. The corresponding compensated PRM is denoted by $\widetilde{\bar{N}}$.

Denote

$$\mathcal{R}_+ := \{\varphi\colon [0,T]\times\Omega\times Z \to \mathbb{R}_+ : \varphi \text{ is } (\mathcal{P}\otimes\mathcal{B}(Z))/\mathcal{B}(\mathbb{R}_+)\text{-measurable}\}.$$

For any $\varphi \in \mathcal{R}_+$, $N^\varphi : \Omega \to M_{FC}([0,T]\times Z)$ is a counting process on $[0,T]\times Z$ defined by

$$N^\varphi((0,t]\times A) := \int_{(0,t]\times A\times\mathbb{R}_+} 1_{[0,\varphi(s,z)]}(r)\,\bar{N}(dr,dz,ds),\ 0\le t\le T,\ A\in\mathcal{B}(Z). \tag{2.1}$$

Here N^φ can be viewed as a controlled random measure, with φ selecting the intensity.

Analogously, $\widetilde{N}^\varphi$ is defined by replacing $\bar{N}$ with $\widetilde{\bar{N}}$ in (2.1). When $\varphi \equiv c > 0$, we write $N^\varphi = N^c$ and $\widetilde{N}^\varphi = \widetilde{N}^c$.

Let V be a reflexive and separable Banach space, which is densely and continuously injected in a separable Hilbert space $(H,\ \langle\cdot,\cdot\rangle_H)$. Identifying H with its dual we get

$$V \subset H \cong H^* \subset V^*,$$

where the star '*' denotes the dual spaces. Denote ${}_{V^*}\langle\cdot,\cdot\rangle_V$ the duality between V^* and V, then we have

$${}_{V^*}\langle u,v\rangle_V = \langle u,v\rangle_H,\quad \forall\, u\in H,\ v\in V. \tag{2.2}$$

Now we consider the following type of SPDEs driven by a Lévy process: for any $\epsilon\in(0,1)$,

$$dX_t^\epsilon = \mathcal{A}(t,X_t^\epsilon)dt + \epsilon\int_Z f(t,X_{t-}^\epsilon,z)\widetilde{N}^{\epsilon^{-1}}(dz,dt),\quad t\in[0,T], \tag{2.3}$$

with initial data $x\in H$, where $\mathcal{A} : [0,T]\times V \to V^*$ is a $\mathcal{B}([0,T])\otimes\mathcal{B}(V)/\mathcal{B}(V^*)$-measurable function and $f : [0,T]\times H\times Z\to H$ is a $\mathcal{B}([0,T])\otimes\mathcal{B}(H)\otimes\mathcal{B}(Z)/\mathcal{B}(H)$-measurable function.

Definition 2.1. *An H-valued càdlàg $\mathbb{F}$-adapted process $X^\epsilon = \{X_t^\epsilon\}_{t\in[0,T]}$ is called a solution of (2.3), if for its $dt\times\mathbb{P}$-equivalent class $\widehat{X}^\epsilon$ we have*

(1) $\widehat{X}^\epsilon \in L^\alpha([0,T];V)\cap L^2([0,T];H)$, *$\mathbb{P}$-a.s.;*

(2) *the following equality holds $\mathbb{P}$-a.s.:*

$$X_t^\epsilon = x + \int_0^t \mathcal{A}(s,\overline{X}_s^\epsilon)ds + \epsilon\int_0^t\int_Z f(s,X_{s-}^\epsilon,z)\widetilde{N}^{\epsilon^{-1}}(dz,ds),\ \forall t\in[0,T],$$

where $\overline{X}^\epsilon$ is any V-valued progressively measurable $dt\times\mathbb{P}$ version of $\widehat{X}^\epsilon$.

Remark 2.2. *It is a well-known and typical conclusion in probability theory that $dt\times\mathbb{P}$-equivalent processes/versions are regarded as the same stochastic process, and it is always impossible to find one version to satisfy all required properties. In the above definition, three $dt\times\mathbb{P}$-equivalent versions "X^ϵ, $\widehat{X}^\epsilon$, $\overline{X}^\epsilon$" are implicitly required to ensure that each version satisfies some required properties.*

In the rest of this paper, when there is no danger of causing ambiguity, we denote $dt\times\mathbb{P}$-equivalent processes/versions of a given process X by itself.

3. Hypotheses and main result

In this paper, we assume the following conditions. Suppose that there exist constants $\alpha > 1,\ \beta\geqslant 0,\ \theta>0,\ C>0$, a function $F\in L^1([0,T];\mathbb{R}^+)$ and a function $\rho : V\to[0,+\infty)$ which is Borel measurable and bounded on the balls, such that the following conditions hold for all $v, v_1, v_2\in V$ and $t\in[0,T]$:

(H1) (Hemicontinuity) The map $s\mapsto {}_{V^*}\langle\mathcal{A}(t,v_1+sv_2),v\rangle_V$ is continuous on $\mathbb{R}$.

(H2) (Local monotonicity)

$$2\ {}_{V^*}\langle \mathcal{A}(t,v_1)-\mathcal{A}(t,v_2), v_1-v_2\rangle_V \leqslant (F_t+\rho(v_2))\|v_1-v_2\|_H^2.$$

(H3) (Coercivity)

$$2\ {}_{V^*}\langle \mathcal{A}(t,v), v\rangle_V + \theta\|v\|_V^\alpha \leqslant F_t(1+\|v\|_H^2).$$

(H4) (Growth)

$$\|\mathcal{A}(t,v)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leqslant (F_t + C\|v\|_V^\alpha)(1+\|v\|_H^\beta).$$

(H5)

$$\rho(v) \leqslant C(1+\|v\|_V^\alpha)(1+\|v\|_H^\beta).$$

For simplicity we write ν_T for $\mathrm{Leb}_T \otimes \nu$. For $p \in (0,+\infty)$, define

$$\mathcal{H}_p := \Big\{ h : [0,T]\times Z \to \mathbb{R}^+ :\ \exists \delta > 0, s.t.\ \forall O \in \mathcal{B}([0,T])\otimes\mathcal{B}(Z) \text{ with } \nu_T(O) < +\infty,$$
$$\text{we have } \int_O \exp(\delta h^p(t,z))\nu(dz)dt < \infty \Big\},$$

and

$$L_p(\nu_T) := \Big\{ h : [0,T]\times Z \to \mathbb{R}^+ : \int_0^T \int_Z h^p(t,z)\nu(dz)dt < \infty \Big\}.$$

For $\varpi \in (0,+\infty)$, define

$$\mathcal{H}^\varpi :=\ \big\{ h : [0,T]\times Z \to \mathbb{R}^+ : \forall \Gamma \in \mathcal{B}([0,T])\otimes\mathcal{B}(Z) \text{ with } \nu_T(\Gamma) < +\infty,$$
$$\text{we have } \int_\Gamma \exp(\varpi h(s,z))\nu(dz)ds < \infty \big\},$$

and denote by $\mathcal{H}^\infty = \bigcap_{\varpi\in(0,+\infty)} \mathcal{H}^\varpi$. By [3, Remark 3.2], we have

$$\mathcal{H}_2 \subset \mathcal{H}^\infty. \tag{3.1}$$

To study the large deviation principle (LDP) of (2.3), besides the conditions **(H1)-(H5)**, we further need

(H6) There exists $L_f \in L_2(\nu_T)\cap\mathcal{H}_2$ such that

$$\|f(t,v,z)\|_H \leqslant L_f(t,z)(1+\|v\|_H), \quad \forall (t,v,z) \in [0,T]\times V\times Z.$$

(H7) There exists $G_f \in \mathcal{H}^{\varpi_0}\cap L_2(\nu_T)$ for some $\varpi_0 \in (0,+\infty)$, such that

$$\|f(t,v_1,z)-f(t,v_2,z)\|_H \leqslant G_f(t,z)\|v_1-v_2\|_H, \quad \forall (t,z)\in[0,T]\times Z, \quad v_1,v_2\in V.$$

Remark 3.1. *We claim that $L_2(\nu_T)\cap\mathcal{H}_2 \subset L_{\beta+2}(\nu_T)\cap\mathcal{H}_2$.*

Proof. Let $h \in \mathcal{H}_2\cap L_2(\nu_T)$. By the definition of $\mathcal{H}_2$, there exists a $\delta > 0$ such that $\forall O \in \mathcal{B}([0,T])\otimes\mathcal{B}(Z)$ with $\nu_T(O) < +\infty$, we have $\int_O \exp(\delta h^2(t,z))\nu(dz)dt < +\infty$. For this δ and $\beta \geqslant 0$, there exists $M > 0$, which depends on δ and β, such that

$$\exp(\delta y^2) \geqslant y^{\beta+2},\ \forall y \geqslant M. \tag{3.2}$$

Denote $E := \{(t,z)\in[0,T]\times Z : h(t,z)\geqslant M\}$ and $E^c := [0,T]\times Z \setminus E$. Since $h \in L_2(\nu_T)$, we have

$$\nu_T(E) \leqslant \frac{1}{M^2}\int_0^T\int_Z h^2(t,z)\nu(dz)dt < \infty,$$

hence by the definition of $\mathcal{H}_2$,

$$\int_E \exp(\delta h^2(t,z))\nu(dz)dt < \infty.$$

Then,

$$\int_0^T\int_Z h^{\beta+2}(t,z)\nu(dz)dt$$

$$
\begin{aligned}
&= \int_E h^{\beta+2}(t,z)\nu(dz)dt + \int_{E^c} h^{\beta+2}(t,z)\nu(dz)dt \\
&\leqslant \int_E \exp(\delta h^2(t,z))\nu(dz)dt + M^\beta \int_{E^c} h^2(t,z)\nu(dz)dt \\
&\leqslant \int_E \exp(\delta h^2(t,z))\nu(dz)dt + M^\beta \int_0^T \int_Z h^2(t,z)\nu(dz)dt \\
&< +\infty,
\end{aligned}
$$

where we used (3.2) in the first inequality. Therefore, $h \in L_{\beta+2}(\nu_T) \cap \mathcal{H}_2$. $\square$

With a minor modification of [1, Theorem 1.2] and using Remark 3.1, we have the following existence and uniqueness result for the solution of (2.3).

Proposition 3.2. *Suppose that conditions **(H1)-(H7)** hold. Then* (2.3) *has a unique solution* $X^\epsilon = \{X_t^\epsilon\}_{t\in[0,T]}$.

Remark 3.3. *The main differences between the proofs of Proposition 3.2 and* [1, Theorem 1.2] *lie in two aspects. There is no Wiener noise term in* (2.3)*, which makes the proof of Proposition 3.2 simpler. The mapping F appearing in Conditions (H2)-(H4) of this paper is a deterministic function, while in the conditions of* [1, Theorem 1.2]*, F could depend on ω. This leads to the following modifications: The local monotonicity and coercivity conditions in* [1, Theorem 1.2] *are satisfied for* (2.3) *with the constant C replaced by an $L^1([0,T];\mathbb{R}^+)$ function, e.g., local monotonicity holds with C replaced by $\int_Z G_f^2(t,z)\nu(dz) + F_t$ and coercivity holds with C replaced by F_t.*

Remark 3.4. *The conditions in Proposition 3.2 are satisfied by a very large class of SPDEs driven by a multiplicative pure jump Lévy noise, including the stochastic porous medium equation, stochastic p-Laplace equation, stochastic Burgers type equations, stochastic 2D Navier-Stokes equations and many other stochastic hydrodynamical systems.* [1, Section 2] *presents many concrete examples to illustrate the applications of this proposition. It is omitted in this paper.*

In the present paper, we aim to establish a LDP for the solution of (2.3) as $\epsilon \to 0$ on $D([0,T];H)$.

We first state the LDP we are concerned with. The theory of large deviations is concerned with events $A \in \mathcal{B}(D([0,T];H))$ for which probability $\mathbb{P}(X^\epsilon \in A)$ converges to zero exponentially fast as $\epsilon \to 0$. The exponential decay rate of such probabilities is typically expressed in terms of a "rate function" I defined as below.

Definition 3.5. (Rate function) *A function $I : D([0,T];H) \to [0,+\infty]$ is called a rate function on $D([0,T];H)$, if for each $M < +\infty$ the level set $\{y \in D([0,T];H) : I(y) \leqslant M\}$ is a compact subset of $D([0,T];H)$. For $A \in \mathcal{B}(D([0,T];H))$, we define $I(A) := \inf_{y\in A} I(y)$.*

Definition 3.6. (Large deviation principle) *Let I be a rate function on $D([0,T];H)$. The sequence $\{X^\epsilon\}_{\epsilon\in(0,1)}$ is said to satisfy a large deviation principle (LDP) on $D([0,T];H)$ with rate function I if the following two conditions hold.*

a. LDP upper bound. For each closed subset O of $D([0,T];H)$,

$$\limsup_{\epsilon\to 0} \epsilon \log \mathbb{P}(X^\epsilon \in O) \leqslant -I(O).$$

b. LDP lower bound. For each open subset G of $D([0,T];H)$,

$$\liminf_{\epsilon\to 0} \epsilon \log \mathbb{P}(X^\epsilon \in G) \geqslant -I(G).$$

Before stating our main result, we need to introduce the so-called skeleton equation. For each Borel measurable function $g\colon [0,T]\times Z \to [0,+\infty)$, define

$$Q(g) := \int_{[0,T]\times Z} \ell\big(g(s,z)\big)\,\nu(dz)ds,$$

where $\ell(x) = x\log x - x + 1$, $\ell(0) := 1$. For each $N \in \mathbb{N}$, define

$$S^N := \Big\{ g : [0,T]\times Z \to [0,\infty) : \, Q(g) \leqslant N \Big\}.$$

Any $g \in S^N$ can be identified with a measure $\hat{g} \in M_{FC}([0,T]\times Z)$, defined by

$$\hat{g}(A) := \int_A g(s,z)\,\nu(dz)ds, \ \forall A \in \mathcal{B}([0,T]\times Z).$$

This identification induces a topology on S^N under which S^N is a compact space (see the Appendix of [3]), which is used throughout the paper.

Denote

$$S := \bigcup_{N\in\mathbb{N}} S^N.$$

For any $g \in S$, consider the following deterministic PDE (called the skeleton equation): for any $t \in [0,T]$,

$$Y_t^g = x + \int_0^t \mathcal{A}(s, Y_s^g)ds + \int_0^t \int_Z f(s, Y_s^g, z)(g(s,z)-1)\nu(dz)ds, \text{ in } V^*. \tag{3.3}$$

By Proposition 4.1 below, this equation has a unique solution $Y^g \in C([0,T];H)\cap L^\alpha([0,T];V)$.

The following is the main result of this paper.

Theorem 3.7. *Assume that conditions **(H1)**-**(H7)** hold. Then the family $\{X^\epsilon\}_{\epsilon\in(0,1)}$ satisfies a LDP on $D([0,T];H)$ with the rate function $I : D([0,T];H) \to [0,+\infty]$ defined by*

$$I(\phi) := \inf\Big\{ Q(g) : \ g \in S \text{ such that } Y^g = \phi \Big\}, \qquad \phi \in D([0,T];H).$$

Here Y^g is the unique solution to (3.3). By convention, $\inf\emptyset = +\infty$.

Proof. Proposition 4.1 allows us to define a map

$$\Gamma : S \ni g \mapsto Y^g \in C([0,T];H)\cap L^\alpha([0,T];V), \tag{3.4}$$

where Y^g is the unique solution of (3.3).

Let $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of compact sets satisfying that $Z_n \subseteq Z$ and $Z_n \nearrow Z$. For each $n\in\mathbb{N}$, let

$$\mathcal{R}_{b,n} = \Big\{ \psi \in \mathcal{R}_+ : \psi(t,z,\omega) \in \begin{cases} [\frac{1}{n}, n], & \text{if } z \in Z_n \\ \{1\}, & \text{if } z \in Z_n^c \end{cases} \text{ for all } (t,\omega)\in[0,T]\times\Omega \Big\},$$

and $\mathcal{R}_b = \bigcup_{n=1}^\infty \mathcal{R}_{b,n}$. For any $N\in\mathbb{N}$, let $\mathcal{S}^N$ be a space of stochastic processes on Ω defined by

$$\mathcal{S}^N := \{\psi \in \mathcal{R}_b : \ \psi(\cdot,\cdot,\omega) \in S^N \text{ for } \mathbb{P}\text{-a.e. } \omega\in\Omega\}.$$

By Proposition 3.2, the Yamada-Watanabe theorem and Girsonov theorem for Poisson random measures, for $\epsilon\in(0,1)$, there exists a map

$$\Gamma^\epsilon : M_{FC}\big([0,T]\times Z\big) \to D([0,T];H)\cap L^\alpha([0,T];V)$$

such that $X^\epsilon = \Gamma^\epsilon(N^{\epsilon^{-1}})$ is the unique solution to (2.3). Moreover, for any $N\in(0,\infty)$ and $\psi_\epsilon \in \mathcal{S}^N$, let

$$X^{\psi_\epsilon} := \Gamma^\epsilon(N^{\epsilon^{-1}\psi_\epsilon}), \tag{3.5}$$

then X^{ψ_ϵ} is the unique solution of the integral equation (or generally called "controlled SPDE"):

$$\begin{aligned} X_t^{\psi_\epsilon} &= x + \int_0^t \mathcal{A}(s, X_s^{\psi_\epsilon})ds + \epsilon\int_0^t\int_Z f(s, X_{s-}^{\psi_\epsilon}, z)\Big(N^{\epsilon^{-1}\psi_\epsilon}(dz,ds) - \epsilon^{-1}\nu(dz)ds\Big) \\ &= x + \int_0^t \mathcal{A}(s, X_s^{\psi_\epsilon})ds + \epsilon\int_0^t\int_Z f(s, X_{s-}^{\psi_\epsilon}, z)\widetilde{N}^{\epsilon^{-1}\psi_\epsilon}(dz,ds) \end{aligned}$$

$$+\int_0^t\int_Z f(s,X_s^{\psi_\epsilon},z)\big(\psi_\epsilon(s,z)-1\big)\nu(dz)ds,\ \forall t\in[0,T]. \tag{3.6}$$

For the details of the proof of the above result, we refer [2, Lemma 7.1].

According to [20, Theorem 4.4], which is an adaption of the original results given in [6, Theoerm 4.2] and [3, Theorems 2.3 and 2.4], to complete the proof of the theorem, it is sufficient to verify the following two claims:

(**LDP1**) For any given $N\in\mathbb{N}$, let $\psi,\psi_n\in S^N$, $n\in\mathbb{N}$, be such that $\psi_n\to\psi$ in S^N as $n\to+\infty$. Then

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|\Gamma(\psi_n)(t)-\Gamma(\psi)(t)\|_H=0.$$

(**LDP2**) For any given $N\in\mathbb{N}$, let $\{\psi_\epsilon,\ \epsilon>0\}\subset\mathcal{S}^N$. Then, for any $\delta>0$,

$$\lim_{\epsilon\to 0}\mathbb{P}\left(\sup_{t\in[0,T]}\|\Gamma^\epsilon(N^{\epsilon^{-1}\psi_\epsilon})(t)-\Gamma(\psi_\epsilon)(t))\|_H\geqslant\delta\right)=0.$$

The verification of (**LDP1**) will be given in Proposition 5.1; see Section 5. (**LDP2**) will be established in Proposition 6.1; see Section 6. □

Finally, we introduce the following results, which will be used later.

Lemma 3.8. *(i) Let $\chi\in\mathcal{H}_2\cap L_2(\nu_T)$, we have*

$$\sup_{\hbar\in S^N}\int_0^T\int_Z|\chi(t,z)|^2(\hbar(t,z)+1)\nu(dz)dt<\infty,\ \forall N\in\mathbb{N}. \tag{3.7}$$

(ii) Let $\chi\in\mathcal{H}^\varpi\cap L_2(\nu_T)$ for some $\varpi\in(0,+\infty)$, we have

$$\sup_{\hbar\in S^N}\int_0^T\int_Z\chi(t,z)|\hbar(t,z)-1|\nu(dz)dt<\infty,\ \forall N\in\mathbb{N}. \tag{3.8}$$

(iii) Let $\chi\in\mathcal{H}^\infty\cap L_2(\nu_T)$ and $N\in\mathbb{N}$, we have

$$\lim_{\delta\to 0}\sup_{\hbar\in S^N}\sup_{\substack{O\in\mathcal{B}([0,T])\\ \mathrm{Leb}_T(O)\leqslant\delta}}\int_O\int_Z\chi(t,z)|\hbar(t,z)-1|\nu(dz)dt=0. \tag{3.9}$$

(iv) Let $\chi\in\mathcal{H}^\infty\cap L_2(\nu_T)$ and $N\in\mathbb{N}$, then for any $\varepsilon>0$ there exists a compact set $K_\varepsilon\subset Z$ such that

$$\sup_{\hbar\in S^N}\int_0^T\int_{K_\varepsilon^c}\chi(t,z)|\hbar(t,z)-1|\nu(dz)dt\leqslant\varepsilon, \tag{3.10}$$

where K_ε^c is the complement of K_ε.

(v) Let $\chi\in\mathcal{H}^\infty$, K be a compact subset of Z and $N\in\mathbb{N}$, we have

$$\lim_{j\to\infty}\sup_{\hbar\in S^N}\int_0^T\int_K\chi(t,z)\cdot 1_{\{\chi(t,z)\geqslant j\}}(t,z)\hbar(t,z)\nu(dz)dt=0. \tag{3.11}$$

Remark 3.9. *The proofs of (3.7), (3.8), (3.10) and (3.11) can be found in the proofs of* [3, (3.3), (3.4), (3.23), (3.26)] *respectively and use (3.1) and the fact that for any compact subset $K\subset Z$, $\nu(K)<+\infty$. (3.9) is quoted from* [30, Remark 2]*, which is a little more general than* [3, (3.5)]*, i.e.,*

$$\lim_{\delta\to 0}\sup_{\hbar\in S^N}\sup_{\substack{0\leqslant s\leqslant t\leqslant T\\ t-s\leqslant\delta}}\int_s^t\int_Z\chi(t,z)|\hbar(r,z)-1|\nu(dz)dr=0.$$

4. Well-posedness for the skeleton equation

Before verifying the **(LDP1)** and **(LDP2)**, we need to prove the following proposition, i.e., the existence and uniqueness of solutions to the skeleton equation (3.3).

Proposition 4.1. *Suppose that conditions* ***(H1)-(H7)*** *hold. For any $g \in S$, there exists a unique solution $Y^g \in C([0,T];H) \cap L^\alpha([0,T];V)$ satisfying (3.3). Moreover,*

$$\sup_{g\in S^N}\Big(\sup_{t\in[0,T]}\|Y_t^g\|_H^2+\int_0^T\|Y_t^g\|_V^\alpha dt\Big)\leqslant C_N<+\infty,\ \forall N\in\mathbb{N}, \tag{4.1}$$

where C_N is defined as in (4.9).

Proof. The proof is mainly divided into three steps.

(*Step 1*) For $J \in C([0,T];H)\cap L^\alpha([0,T];V)$, consider the following equation:

$$\begin{cases} dZ_t^J=\mathcal{A}(t,Z_t^J)dt+\displaystyle\int_Z f(t,J(t),z)(g(t,z)-1)\nu(dz)dt,\ t\in[0,T],\\ Z_0^J=x\in H. \end{cases} \tag{4.2}$$

In this step, we aim to prove the existence and uniqueness of solutions to (4.2) with $Z^J \in C([0,T];H)\cap L^\alpha([0,T];V)$ satisfying

$$Z_t^J=x+\int_0^t\mathcal{A}(s,Z_s^J)ds+\int_0^t\int_Z f(s,J(s),z)(g(s,z)-1)\nu(dz)ds \text{ in } V^*, \tag{4.3}$$

and

$$\sup_{t\in[0,T]}\|Z_t^J\|_H^2+\theta\int_0^T\|Z_s^J\|_V^\alpha ds\leqslant C_T, \tag{4.4}$$

where C_T is defined as in (4.6).

Assume that $\{e_1,e_2,...\}\subset V$ is an orthonormal basis of H such that $span\{e_1,e_2,...\}$ is dense in V. Denote $H_n:=span\{e_1,...,e_n\}$. Let $P_n: V^*\to H_n$ be defined by

$$P_ny:=\sum_{i=1}^n {}_{V^*}\langle y,e_i\rangle_V e_i,\ y\in V^*.$$

It is easy to see that $P_n|_H$ is just the orthogonal projection of H onto H_n, and, for $u_1,u_2\in V$, $v\in H_n$, we have

$$\begin{aligned} &{}_{V^*}\langle P_n\mathcal{A}(t,u_1)+\int_Z P_nf(t,u_2,z)(g(t,z)-1)\nu(dz),v\rangle_V\\ =&\ \langle P_n\mathcal{A}(t,u_1)+\int_Z P_nf(t,u_2,z)(g(t,z)-1)\nu(dz),v\rangle_H\\ =&\ {}_{V^*}\langle \mathcal{A}(t,u_1)+\int_Z f(t,u_2,z)(g(t,z)-1)\nu(dz),v\rangle_V. \end{aligned} \tag{4.5}$$

For each $n\in\mathbb{N}$, consider the following equation on H_n:

$$\begin{cases} dZ_t^{J,n}=P_n\mathcal{A}(t,Z_t^{J,n})dt+\displaystyle\int_Z P_nf(t,J(t),z)(g(t,z)-1)\nu(dz)dt,\\ Z_0^{J,n}=P_nx\in H. \end{cases}$$

From [19, Theorem 3.1.1], we know that there exists a unique solution $Z^{J,n}=\{Z_t^{J,n}\}_{t\in[0,T]}$ to the above equation satisfying the following integral equation:

$$Z_t^{J,n}=P_nx+\int_0^t P_n\mathcal{A}(s,Z_s^{J,n})ds+\int_0^t\int_Z P_nf(s,J(s),z)(g(s,z)-1)\nu(dz)ds,\ \forall t\in[0,T].$$

Applying the chain rule to $\|Z_t^{J,n}\|_H^2$, by **(H3)**, **(H6)** and (4.5), we have

$$\begin{aligned}\|Z_t^{J,n}\|_H^2 = &\ \|P_n x\|_H^2 + 2\int_0^t {}_{V^*}\langle P_n\mathcal{A}(s,Z_s^{J,n}), Z_s^{J,n}\rangle_V ds\\ &+2\int_0^t {}_{V^*}\langle \int_Z P_n f(s,J(s),z)\big(g(s,z)-1\big)\nu(dz), Z_s^{J,n}\rangle_V ds\\ \leqslant &\ \|x\|_H^2 + \int_0^t F_s ds + \int_0^t F_s\cdot\|Z_s^{J,n}\|_H^2 ds - \int_0^t \theta\|Z_s^{J,n}\|_V^\alpha ds\\ &+2\int_0^t\int_Z L_f(s,z)(1+\|J(s)\|_H)\cdot\|Z_s^{J,n}\|_H\cdot|g(s,z)-1|\nu(dz)ds\\ \leqslant &\ \|x\|_H^2 + \int_0^t F_s ds + \int_0^t F_s\cdot\|Z_s^{J,n}\|_H^2 ds - \int_0^t \theta\|Z_s^{J,n}\|_V^\alpha ds\\ &+2\big(1+\sup_{s\in[0,T]}\|J(s)\|_H\big)\int_0^t L_f(s)\cdot\big(\|Z_s^{J,n}\|_H^2+1)ds,\ \forall t\in[0,T],\end{aligned}$$

where

$$L_f(s) := \int_Z L_f(s,z)|g(s,z)-1|\nu(dz).$$

By (3.8) we know that $L_f\in L^1([0,T];\mathbb{R}^+)$. Since $F\in L^1([0,T];\mathbb{R}^+)$ and $J\in C([0,T];H)$, by Gronwall's inequality, we have

$$\sup_{t\in[0,T]}\|Z_t^{J,n}\|_H^2 + \theta\int_0^T \|Z_s^{J,n}\|_V^\alpha ds \leqslant C_T,$$

where

$$\begin{aligned}C_l = &\ \Big(\|x\|_H^2 + \int_0^l F_s ds + 2\big(1+\sup_{s\in[0,l]}\|J(s)\|_H\big)\int_0^l L_f(s)ds\Big)\\ &\cdot e^{\int_0^l F_s ds + 2(1+\sup_{s\in[0,l]}\|J(s)\|_H)\int_0^l L_f(s)ds},\ l\in[0,T].\end{aligned}\tag{4.6}$$

Following the similar ideas as [1, Theorem 4.1], we obtain the existence and uniqueness of solutions to (4.2) satisfying (4.3) and (4.4). Besides, (4.4) still holds with T replaced by any $l\in[0,T]$.

(*Step 2*) Choosing $M\geqslant 4(1+\|x\|_H^2)$, by (4.6) there exists $0<l_0<T$ such that

$$\Big(\|x\|_H^2 + \int_0^{l_0} F_s ds + 2(1+M)\int_0^{l_0} L_f(s)ds\Big)\cdot e^{\int_0^{l_0} F_s ds + 2(1+M)\int_0^{l_0} L_f(s)ds} \leqslant M.$$

Define

$$\Lambda_{l_0,M} := \{\chi\in C([0,l_0];H)\cap L^\alpha([0,l_0];V) : \sup_{s\in[0,l_0]}\|\chi(s)\|_H + \theta\int_0^{l_0}\|\chi(s)\|_V^\alpha ds\leqslant M\}.$$

From (4.4) we see that for any $J\in\Lambda_{l_0,M}$, one has $Z^J\in\Lambda_{l_0,M}$. Define a function $d:\Lambda_{l_0,M}\times\Lambda_{l_0,M}\to[0,+\infty)$ by

$$d(\chi^1,\chi^2) = \sup_{s\in[0,l_0]}\|\chi^1(s)-\chi^2(s)\|_H,\ \forall\chi^1,\chi^2\in\Lambda_{l_0,M}.$$

It is easy to see that d is a metric on $\Lambda_{l_0,M}$. In fact, $(\Lambda_{l_0,M},d)$ is a complete metric space. Before clarifying this, we remark that in this paper $\Lambda_{l_0,M}$ is not equipped with the following usual metric

$$\tilde{d}(\chi^1,\chi^2) := \sup_{s\in[0,l_0]}\|\chi^1(s)-\chi^2(s)\|_H + \Big(\int_0^{l_0}\|\chi^1(s)-\chi^2(s)\|_V^\alpha ds\Big)^{\frac{1}{\alpha}},\ \forall\chi^1,\chi^2\in\Lambda_{l_0,M}.$$

Because as seen later in this step (see the arguments under (4.7)), we only prove that the unique solution to (4.2) on $[0,l_1]$ (cf. (4.8) for l_1) is a contraction from $(\Lambda_{l_1,M},d)$ to $(\Lambda_{l_1,M},d)$, which is sufficient to use the Banach fixed-point theorem. To clarify that $(\Lambda_{l_0,M},d)$ is complete, we need to prove that every Cauchy sequence in $(\Lambda_{l_0,M},d)$ converges and its limit is in $\Lambda_{l_0,M}$. Suppose that $\{\chi_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\Lambda_{l_0,M},d)$, hence also a Cauchy sequence in $(C([0,l_0];H),\sup_{s\in[0,l_0]}\|\cdot\|_H)$. Then there exists some $\chi\in C([0,l_0];H)$ such that as $n\to+\infty$,

$$\chi_n\longrightarrow\chi,\ \text{strongly in } C([0,l_0];H)\subset L^\alpha([0,l_0];H).$$

Since $\theta\int_0^{l_0}\|\chi_n(s)\|_V^\alpha ds\leqslant M$, $\forall n\in\mathbb{N}$, there exists some $\bar{\chi}\in L^\alpha([0,l_0];V)$ such that as $n\to+\infty$,

$$\chi_n\longrightarrow\bar{\chi},\ \text{weakly in } L^\alpha([0,l_0];V)\subset L^\alpha([0,l_0];H).$$

Therefore, $\chi=\bar{\chi}$, ds-a.s.. Since

$$\sup_{s\in[0,l_0]}\|\chi_n(s)\|_H+\theta\int_0^{l_0}\|\chi_n(s)\|_V^\alpha ds\leqslant M,$$

by weak lower semicontinuity of the norms, we may pass to the limit and get

$$\sup_{s\in[0,l_0]}\|\chi(s)\|_H+\theta\int_0^{l_0}\|\chi(s)\|_V^\alpha ds\leqslant M.$$

From Step 1, we know that for any $J_1,J_2\in\Lambda_{l_0,M}$, there exist unique solutions to (4.2) with J replaced by J_1 and J_2, for simplicity, denoted as $Z^1:=Z^{J_1}$ and $Z^2:=Z^{J_2}$, respectively. We have for $t\in[0,l_0]$,

$$\begin{aligned}&Z_t^1-Z_t^2\\ =&\int_0^t\mathcal{A}(s,Z_s^1)-\mathcal{A}(s,Z_s^2)ds+\int_0^t\int_Z\big(f(s,J_1(s),z)-f(s,J_2(s),z)\big)(g(s,z)-1)\nu(dz)ds.\end{aligned}$$

Applying the chain rule to $\|Z_t^1-Z_t^2\|_H^2$, by **(H2)** and **(H7)**,

$$\begin{aligned}&\|Z_t^1-Z_t^2\|_H^2\\ =&\ 2\int_0^t {}_{V^*}\langle\mathcal{A}(s,Z_s^1)-\mathcal{A}(s,Z_s^2),Z_s^1-Z_s^2\rangle_V ds\\ &+2\int_0^t\Big\langle\int_Z\big(f(s,J_1(s),z)-f(s,J_2(s),z)\big)(g(s,z)-1)\nu(dz),Z_s^1-Z_s^2\Big\rangle_H ds\\ \leqslant&\int_0^t(F_s+\rho(Z_s^2))\|Z_s^1-Z_s^2\|_H^2ds\\ &+2\int_0^t\int_Z G_f(s,z)\|J_1(s)-J_2(s)\|_H\cdot|g(s,z)-1|\cdot\|Z_s^1-Z_s^2\|_H\nu(dz)ds\\ \leqslant&\int_0^t\big(F_s+\rho(Z_s^2)\big)\|Z_s^1-Z_s^2\|_H^2ds\\ &+\int_0^t\int_Z G_f(s,z)\|J_1(s)-J_2(s)\|_H^2\cdot|g(s,z)-1|\nu(dz)ds\\ &+\int_0^t\int_Z G_f(s,z)\|Z_s^1-Z_s^2\|_H^2\cdot|g(s,z)-1|\nu(dz)ds,\ \forall t\in[0,l_0].\end{aligned}\tag{4.7}$$

By Gronwall's inequality, we know that for $l\in[0,l_0]$,

$$\begin{aligned}&\sup_{t\in[0,l]}\|Z_t^1-Z_t^2\|_H^2\\ \leqslant&\sup_{t\in[0,l]}\|J_1(t)-J_2(t)\|_H^2\cdot\int_0^l G_f(s)ds\cdot\exp\Big\{\int_0^l F_s+\rho(Z_s^2)+G_f(s)ds\Big\},\end{aligned}$$

where

$$G_f(s):=\int_Z G_f(s,z)|g(s,z)-1|\nu(dz),$$

which by (3.8) one has $G_f \in L^1([0,T];\mathbb{R}^+)$. Since $F \in L^1([0,T];\mathbb{R}^+)$, by **(H5)** and (4.4),

$$\begin{aligned}
&\exp\Big\{\int_0^{l_0} F_s + \rho(Z_s^2) + G_f(s)ds\Big\} \\
\leqslant\ & \exp\Big\{\int_0^{l_0} F_s + C(1+\|Z_s^2\|_V^\alpha)(1+\|Z_s^2\|_H^\beta) + G_f(s)ds\Big\} \\
:=\ & C_{M,l_0,\theta,\int_0^T F_s ds,\int_0^T G_f(s)ds} < +\infty.
\end{aligned}$$

Therefore, choosing $l_1 \in (0, l_0]$ such that

$$\int_0^{l_1} G_f(s)ds \cdot C_{M,l_0,\theta,\int_0^T F_s ds,\int_0^T G_f(s)ds} \leqslant \frac{1}{2}, \tag{4.8}$$

which means the unique solution to (4.2) on $[0, l_1]$ can be regarded as a map from $\Lambda_{l_1,M}$ to $\Lambda_{l_1,M}$ and it is a contraction. Hence, by the Banach fixed-point theorem, there exists a unique solution to (3.3) on $[0, l_1]$ in the space $C([0,l_1];H) \cap L^\alpha([0,l_1];V)$.

(*Step 3*) Assume that there exists a unique solution to (3.3) on $[0,T]$, denote it as Y^g. Applying the chain rule to $\|Y_t^g\|_H^2$, by (2.2), **(H3)** and **(H6)**,

$$\begin{aligned}
\|Y_t^g\|_H^2 =\ & \|x\|_H^2 + 2\int_0^t {}_{V^*}\langle \mathcal{A}(s,Y_s^g), Y_s^g\rangle_V ds \\
& +2\int_0^t {}_{V^*}\langle \int_Z f(s,Y_s^g,z)\big(g(s,z)-1\big)\nu(dz), Y_s^g\rangle_V ds \\
\leqslant\ & \|x\|_H^2 + \int_0^t F_s ds + \int_0^t F_s\cdot\|Y_s^g\|_H^2 ds - \int_0^t \theta\|Y_s^g\|_V^\alpha ds \\
& +4\int_0^t L_f(s)ds + 4\int_0^t L_f(s)\cdot\|Y_s^g\|_H^2 ds,\ \forall t\in[0,T].
\end{aligned}$$

Then,

$$\begin{aligned}
&\sup_{t\in[0,T]}\|Y_t^g\|_H^2 + \theta\int_0^T \|Y_s^g\|_V^\alpha ds \\
\leqslant\ & \|x\|_H^2 + \int_0^T F_s + 4L_f(s)ds + \int_0^T (F_s + 4L_f(s))\cdot\|Y_s^g\|_H^2 ds.
\end{aligned}$$

By Gronwall's inequality and (3.8),

$$\begin{aligned}
&\sup_{t\in[0,T]}\|Y_t^g\|_H^2 + \int_0^T \|Y_s^g\|_V^\alpha ds \\
\leqslant\ & \big(\|x\|_H^2 + \int_0^T F_s + 4L_f(s)ds\big)\cdot e^{\int_0^T F_s + 4L_f(s)ds} \\
:=\ & C_N,
\end{aligned} \tag{4.9}$$

where C_N is dependent on $\int_0^T F_s ds$, $\int_0^T L_f(s)ds$, N and $\|x\|_H$, but independent of g.

Combing the results in Step 2 and Step 3, using the standard arguments, we know that there exists a unique solution $Y^g \in C([0,T];H)\cap L^\alpha([0,T];V)$ satisfying (3.3) on $[0,T]$. (4.1) follows from (4.9). ☐

5. The verification of **(LDP1)**

Recall from (3.4) that

$$\Gamma : S \ni g \mapsto Y^g \in C([0,T];H)\cap L^\alpha([0,T];V),$$

where Y^g is the unique solution of the skeleton equation (3.3). In this section, we aim to prove the following proposition.

Proposition 5.1. *For any given $N \in \mathbb{N}$, let $\psi, \psi_n \in S^N$, $n \in \mathbb{N}$, be such that $\psi_n \to \psi$ in S^N as $n \to +\infty$. Then*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|\Gamma(\psi_n)(t) - \Gamma(\psi)(t)\|_H = 0.$$

Proof. From Proposition 4.1, we know that $X^\psi := \Gamma(\psi)$ is the unique solution to the following deterministic PDE

$$\begin{cases} dX_t^\psi = \mathcal{A}(t, X_t^\psi)dt + \int_Z f(t, X_t^\psi, z)(\psi(t,z) - 1)\nu(dz)dt, \\ X_0^\psi = x \in H, \end{cases}$$

while $X^{\psi_n} := \Gamma(\psi_n)$ is the unique solution to the following deterministic PDE

$$\begin{cases} dX_t^{\psi_n} = \mathcal{A}(t, X_t^{\psi_n})dt + \int_Z f(t, X_t^{\psi_n}, z)(\psi_n(t,z) - 1)\nu(dz)dt, \\ X_0^{\psi_n} = x \in H. \end{cases}$$

Applying the chain rule to $\|X_t^{\psi_n} - X_t^\psi\|_H^2$, by (2.2),

$$\begin{aligned} &\|X_t^{\psi_n} - X_t^\psi\|_H^2 \\ = \ & 2\int_0^t {}_{V^*}\langle \mathcal{A}(s, X_s^{\psi_n}) - \mathcal{A}(s, X_s^\psi), X_s^{\psi_n} - X_s^\psi\rangle_V ds \\ &+2\int_0^t \Big\langle \int_Z f(s, X_s^{\psi_n}, z)\big(\psi_n(s,z) - 1\big) - f(s, X_s^\psi, z)\big(\psi(s,z) - 1\big)\nu(dz), \\ &\qquad\qquad X_s^{\psi_n} - X_s^\psi\Big\rangle_H ds, \ \forall t \in [0,T]. \end{aligned} \tag{5.1}$$

By **(H2)**, **(H5)** and (4.1),

$$\begin{aligned} &2\int_0^t {}_{V^*}\langle \mathcal{A}(s, X_s^{\psi_n}) - \mathcal{A}(s, X_s^\psi), X_s^{\psi_n} - X_s^\psi\rangle_V ds \\ \leqslant \ & 2\int_0^t \big(F_s + C(1 + \|X_s^\psi\|_V^\alpha)(1 + \|X_s^\psi\|_H^\beta)\big) \cdot \|X_s^{\psi_n} - X_s^\psi\|_H^2 ds \\ = \ & 2\int_0^t (F_s + C) \cdot \|X_s^{\psi_n} - X_s^\psi\|_H^2 ds + 2C\int_0^t \|X_s^\psi\|_V^\alpha \cdot \|X_s^{\psi_n} - X_s^\psi\|_H^2 ds \\ &+2C\int_0^t \|X_s^\psi\|_H^\beta \cdot \|X_s^{\psi_n} - X_s^\psi\|_H^2 ds \\ &+2C\int_0^t \|X_s^\psi\|_V^\alpha \cdot \|X_s^\psi\|_H^\beta \cdot \|X_s^{\psi_n} - X_s^\psi\|_H^2 ds \\ \leqslant \ & 2\int_0^t (F_s + C) \cdot \|X_s^{\psi_n} - X_s^\psi\|_H^2 ds + 2C\int_0^t \|X_s^\psi\|_V^\alpha \cdot \|X_s^{\psi_n} - X_s^\psi\|_H^2 ds \\ &+2C \sup_{\hbar\in S^N} \sup_{t\in[0,T]} \|X_t^\hbar\|_H^\beta \cdot \int_0^t \|X_s^{\psi_n} - X_s^\psi\|_H^2 ds \\ &+2C \sup_{\hbar\in S^N} \sup_{t\in[0,T]} \|X_t^\hbar\|_H^\beta \cdot \int_0^t \|X_s^\psi\|_V^\alpha \cdot \|X_s^{\psi_n} - X_s^\psi\|_H^2 ds \\ := &\int_0^t 2(F_s + C_{N,\beta}\|X_s^\psi\|_V^\alpha + C_{N,\beta}) \cdot \|X_s^{\psi_n} - X_s^\psi\|_H^2 ds, \quad \forall t \in [0,T]. \end{aligned} \tag{5.2}$$

Here $C_{N,\beta}$ only depends on C_N appearing in (4.1) and β. For simplicity, denote the second term in the right hand-side of (5.1) as $I_n(t)$ and rewrite it as following:

$$I_n(t)$$

$$
\begin{aligned}
&:= 2\int_0^t \Big\langle \int_Z f(s,X_s^\psi,z)\big((\psi_n(s,z)-1)-(\psi(s,z)-1)\big)\nu(dz), X_s^{\psi_n}-X_s^\psi\Big\rangle_H ds\\
&\quad +2\int_0^t \Big\langle \int_Z \big(f(s,X_s^{\psi_n},z)-f(s,X_s^\psi,z)\big)(\psi_n(s,z)-1)\nu(dz), X_s^{\psi_n}-X_s^\psi\Big\rangle_H ds\\
&:= Q_{n,1}(t)+Q_{n,2}(t).
\end{aligned}\tag{5.3}
$$

Denote

$$
G_n(s):=\int_Z G_f(s,z)|\psi_n(s,z)-1|\nu(dz).
$$

From (3.8) we know that

$$
\sup_{n\in\mathbb{N}}\int_0^T G_n(s)ds<\infty.\tag{5.4}
$$

For $Q_{n,2}(t)$, from **(H7)** we have

$$
|Q_{n,2}(t)|\leqslant 2\int_0^t G_n(s)\|X_s^{\psi_n}-X_s^\psi\|_H^2 ds.\tag{5.5}
$$

Substituting (5.2)-(5.5) to (5.1), we get

$$
\begin{aligned}
&\|X_t^{\psi_n}-X_t^\psi\|_H^2\\
\leqslant\ &\int_0^t 2\big(F_s+C_{N,\beta}\|X_s^\psi\|_V^\alpha+C_{N,\beta}+G_n(s)\big)\cdot\|X_s^{\psi_n}-X_s^\psi\|_H^2 ds+Q_{n,1}(t),\ \ \forall t\in[0,T].
\end{aligned}
$$

From (4.1) we know that $\|X^\psi\|_V^\alpha\in L^1([0,T];\mathbb{R}^+)$, hence by Gronwall's inequality and (5.4), we obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|X_t^{\psi_n}-X_t^\psi\|_H^2\\
\leqslant\ &\sup_{t\in[0,T]}|Q_{n,1}(t)|\cdot e^{2\int_0^T F_s+C_{N,\beta}\|X_s^\psi\|_V^\alpha+C_{N,\beta}+G_n(s)ds}\\
:=\ &C_{G_f,N,\beta,\alpha,T}\cdot\sup_{t\in[0,T]}|Q_{n,1}(t)|.
\end{aligned}\tag{5.6}
$$

Here $C_{G_f,N,\beta,\alpha,T}$ is independent of n.

Now let us estimate $|Q_{n,1}(t)|$. By Lemma 3.8, we know that for any $\varepsilon>0$, there exists a compact set $K_\varepsilon\subset Z$ such that (3.10) holds. We rewrite

$$
\begin{aligned}
&Q_{n,1}(t)\\
=\ &2\int_0^t\int_{K_\varepsilon}\Big\langle f(s,X_s^\psi,z)\big(\psi_n(s,z)-\psi(s,z)\big), X_s^{\psi_n}-X_s^\psi\Big\rangle_H\nu(dz)ds\\
&+2\int_0^t\int_{K_\varepsilon^c}\Big\langle f(s,X_s^\psi,z)\big(\psi_n(s,z)-\psi(s,z)\big), X_s^{\psi_n}-X_s^\psi\Big\rangle_H\nu(dz)ds\\
:=\ &I_{n,1}(t)+I_{n,2}(t).
\end{aligned}\tag{5.7}
$$

By **(H6)** and (4.1),

$$
\begin{aligned}
&\sup_{t\in[0,T]}|I_{n,2}(t)|\\
\leqslant\ &2\int_0^T\int_{K_\varepsilon^c}L_f(s,z)\cdot(\|X_s^\psi\|_H+1)\cdot|\psi_n(s,z)-\psi(s,z)|\cdot\|X_s^{\psi_n}-X_s^\psi\|_H\nu(dz)ds\\
\leqslant\ &2\sup_{s\in[0,T]}\Big[(\|X_s^\psi\|_H+1)\cdot\|X_s^{\psi_n}-X_s^\psi\|_H\Big]\cdot\\
&\qquad\Big(\int_0^T\int_{K_\varepsilon^c}L_f(s,z)|\psi_n(s,z)-1|\nu(dz)ds+\int_0^T\int_{K_\varepsilon^c}L_f(s,z)|\psi(s,z)-1|\nu(dz)ds\Big)
\end{aligned}
$$

$$\leqslant \varepsilon C_N. \tag{5.8}$$

To estimate $I_{n,1}(t)$, define

$$A_{L_f,J} = \{(s,z) \in [0,T] \times Z : L_f(s,z) \geqslant J\}.$$

In the following, for any $M \subset [0,T] \times Z$, denote $M^c := [0,T] \times Z \setminus M$. Denote

$$\begin{aligned}
& I_{n,1}(t) \\
= \ & 2\int_0^t \int_{K_\varepsilon} \Big\langle f(s, X_s^\psi, z)\big(\psi_n(s,z) - \psi(s,z)\big), X_s^{\psi_n} - X_s^\psi \Big\rangle_H 1_{A_{L_f,J}}(s,z)\nu(dz)ds \\
& +2\int_0^t \int_{K_\varepsilon} \Big\langle f(s, X_s^\psi, z)\big(\psi_n(s,z) - \psi(s,z)\big), X_s^{\psi_n} - X_s^\psi \Big\rangle_H 1_{A^c_{L_f,J}}(s,z)\nu(dz)ds \\
:= \ & I_{n,1,J}(t) + I_{n,1,J^c}(t).
\end{aligned} \tag{5.9}$$

By **(H6)** and (4.1),

$$\begin{aligned}
& \sup_{t\in[0,T]} |I_{n,1,J}(t)| \\
\leqslant \ & 2\int_0^T \int_{K_\varepsilon} L_f(s,z)(\|X_s^\psi\|_H + 1)\big(\psi_n(s,z) + \psi(s,z)\big) \cdot \\
& \qquad\qquad (\|X_s^{\psi_n}\|_H + \|X_s^\psi\|_H) 1_{A_{L_f,J}}(s,z)\nu(dz)ds \\
\leqslant \ & 2\sup_{s\in[0,T]} \big[(\|X_s^\psi\|_H + 1)(\|X_s^{\psi_n}\|_H + \|X_s^\psi\|_H)\big] \cdot \\
& \qquad\qquad \int_0^T \int_{K_\varepsilon} L_f(s,z)\big(\psi_n(s,z) + \psi(s,z)\big) 1_{A_{L_f,J}}(s,z)\nu(dz)ds \\
\leqslant \ & C_N \sup_{\hbar \in S^N} \int_0^T \int_{K_\varepsilon} L_f(s,z)\hbar(s,z) 1_{A_{L_f,J}}(s,z)\nu(dz)ds,
\end{aligned} \tag{5.10}$$

by (3.11), we know that for $\varepsilon > 0$, there exists $J_\varepsilon > 0$ such that

$$\sup_{\hbar \in S^N} \int_0^T \int_{K_\varepsilon} L_f(s,z)\hbar(s,z) 1_{A_{L_f,J_\varepsilon}}(s,z)\nu(dz)ds \leqslant \frac{\varepsilon}{C_N},$$

so choose J in (5.10) to be J_ε, then (5.10) yields

$$\sup_{t\in[0,T]} |I_{n,1,J_\varepsilon}(t)| \leqslant \varepsilon. \tag{5.11}$$

Substituting (5.7)-(5.11) to (5.6), we get

$$\begin{aligned}
& \sup_{t\in[0,T]} \|X_t^{\psi_n} - X_t^\psi\|_H^2 \\
\leqslant \ & C_{G_f,N,\beta,\alpha,T} \cdot \Big(\varepsilon + \sup_{t\in[0,T]} |I_{n,1,J^c_\varepsilon}(t)| + \varepsilon C_N\Big).
\end{aligned} \tag{5.12}$$

To estimate $|I_{n,1,J^c_\varepsilon}(t)|$, denote

$$U^n(s) = X_s^{\psi_n} - X_s^\psi, \ U^n(\bar{s}_m) = X_{\bar{s}_m}^{\psi_n} - X_{\bar{s}_m}^\psi,$$

where

$$\bar{s}_m = t_{k+1} \equiv (k+1)T \cdot 2^{-m}, \text{ for } s \in [kT2^{-m}, (k+1)T2^{-m}).$$

Then

$$\sup_{t\in[0,T]} |I_{n,1,J^c_\varepsilon}(t)| \leqslant \sum_{i=1}^4 \tilde{I}_i, \tag{5.13}$$

where

$$\tilde{I}_1 = \sup_{t\in[0,T]} \Big|\int_0^t\int_{K_\varepsilon} \big\langle f(s,X^\psi_s,z)\big(\psi_n(s,z)-\psi(s,z)\big), U^n(s)-U^n(\bar{s}_m)\big\rangle_H 1_{A^c_{L_f,J_\varepsilon}}(s,z)\nu(dz)ds\Big|,$$

$$\tilde{I}_2 = \sup_{t\in[0,T]} \Big|\int_0^t\int_{K_\varepsilon} \big\langle\big(f(s,X^\psi_s,z)-f(s,X^\psi_{\bar{s}_m},z)\big)\big(\psi_n(s,z)-\psi(s,z)\big), U^n(\bar{s}_m)\big\rangle_H 1_{A^c_{L_f,J_\varepsilon}}(s,z)\nu(dz)ds\Big|,$$

$$\tilde{I}_3 = \sup_{1\leqslant k\leqslant 2^m}\sup_{t_{k-1}\leqslant t\leqslant t_k} \Big|\int_{t_{k-1}}^t\int_{K_\varepsilon} \big\langle f(s,X^\psi_{\bar{s}_m},z)\big(\psi_n(s,z)-\psi(s,z)\big), U^n(\bar{s}_m)\big\rangle_H 1_{A^c_{L_f,J_\varepsilon}}(s,z)\nu(dz)ds\Big|,$$

$$\tilde{I}_4 = \sum_{k=1}^{2^m} \Big|\int_{t_{k-1}}^{t_k}\int_{K_\varepsilon} \big\langle f(s,X^\psi_{\bar{s}_m},z)\big(\psi_n(s,z)-\psi(s,z)\big), U^n(\bar{s}_m)\big\rangle_H 1_{A^c_{L_f,J_\varepsilon}}(s,z)\nu(dz)ds\Big|.$$

Note that $\tilde{I}_i, i=1,...,4$, are all dependent on n,m,ε. For simplicity, we omit these parameters.

Now, let us estimate $\tilde{I}_i, i=1,2,3,4$. From [3, Remark 3.3] we know that for any $a,b\in(0,+\infty)$ and $\sigma\in[1,+\infty)$,

$$ab \leqslant e^{\sigma a} + \frac{1}{\sigma}(b\log b - b + 1) = e^{\sigma a} + \frac{1}{\sigma}\ell(b). \tag{5.14}$$

Choosing $a=1$ and $b=\psi_n(s,z)$ or $\psi(s,z)$ in (5.14), using (4.1), similarly as to get [26, (5.23)],

$$\begin{aligned}\tilde{I}_1 \leqslant\ & C_{N,T}J_\varepsilon\cdot e^\sigma\nu(K_\varepsilon)\Big[\Big(\int_0^T \|X^{\psi_n}_s - X^{\psi_n}_{\bar{s}_m}\|^2_H ds\Big)^{\frac{1}{2}} + \Big(\int_0^T \|X^{\psi}_s - X^{\psi}_{\bar{s}_m}\|^2_H ds\Big)^{\frac{1}{2}}\Big] \\ & + J_\varepsilon\cdot\frac{C_{N,T}}{\sigma}.\end{aligned} \tag{5.15}$$

Here $C_{N,T}$ only depends on C_N appearing in (4.1) and T.

To estimate $\int_0^T \|X^{\psi_n}_s - X^{\psi_n}_{\bar{s}_m}\|^2_H ds$, note that for any $s\in[0,T]$,

$$X^{\psi_n}_{\bar{s}_m} - X^{\psi_n}_s = \int_s^{\bar{s}_m} \mathcal{A}(t,X^{\psi_n}_t)dt + \int_s^{\bar{s}_m}\int_Z f(t,X^{\psi_n}_t,z)\big(\psi_n(t,z)-1\big)\nu(dz)dt.$$

Applying the chain rule to $\|X^{\psi_n}_{\bar{s}_m} - X^{\psi_n}_s\|^2_H$, by (2.2),

$$\begin{aligned}&\|X^{\psi_n}_{\bar{s}_m} - X^{\psi_n}_s\|^2_H \\ =\ & 2\int_s^{\bar{s}_m} {}_{V^*}\big\langle \mathcal{A}(t,X^{\psi_n}_t), X^{\psi_n}_t - X^{\psi_n}_s\big\rangle_V dt \\ & + 2\int_s^{\bar{s}_m}\Big\langle \int_Z f(t,X^{\psi_n}_t,z)\big(\psi_n(t,z)-1\big)\nu(dz), X^{\psi_n}_t - X^{\psi_n}_s\Big\rangle_H dt,\ \forall s\in[0,T].\end{aligned}$$

Integrating the above equality over $[0,T]$ with respect to s, we get

$$\begin{aligned}&\int_0^T \|X^{\psi_n}_{\bar{s}_m} - X^{\psi_n}_s\|^2_H ds \\ =\ & 2\int_0^T\int_s^{\bar{s}_m} {}_{V^*}\big\langle \mathcal{A}(t,X^{\psi_n}_t), X^{\psi_n}_t - X^{\psi_n}_s\big\rangle_V dtds \\ & + 2\int_0^T\int_s^{\bar{s}_m}\Big\langle \int_Z f(t,X^{\psi_n}_t,z)\big(\psi_n(t,z)-1\big)\nu(dz), X^{\psi_n}_t - X^{\psi_n}_s\Big\rangle_H dtds.\end{aligned}$$

By Young's inequality, **(H4)** and Fubini's theorem, there exists a positive constant $C_\alpha\in(0,+\infty)$, which only depends on α and may change from line to line, such that

$$\begin{aligned}& 2\int_0^T\int_s^{\bar{s}_m} {}_{V^*}\big\langle \mathcal{A}(t,X^{\psi_n}_t), X^{\psi_n}_t - X^{\psi_n}_s\big\rangle_V dtds \\ \leqslant\ & 2\int_0^T\int_s^{\bar{s}_m} \frac{\|\mathcal{A}(t,X^{\psi_n}_t)\|^{\frac{\alpha}{\alpha-1}}_{V^*}}{\frac{\alpha}{\alpha-1}} + \frac{\|X^{\psi_n}_t - X^{\psi_n}_s\|^\alpha_V}{\alpha}dtds\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{2(\alpha-1)}{\alpha}\int_0^T\int_s^{\bar{s}_m}\big(F_t+C\|X_t^{\psi_n}\|_V^\alpha\big)\big(1+\|X_t^{\psi_n}\|_H^\beta\big)dtds\\
&\quad +C_\alpha\int_0^T\int_s^{\bar{s}_m}\|X_t^{\psi_n}\|_V^\alpha+\|X_s^{\psi_n}\|_V^\alpha\ dtds\\
&= \frac{2(\alpha-1)}{\alpha}\int_0^T\int_s^{\bar{s}_m}F_t+F_t\|X_t^{\psi_n}\|_H^\beta+C\|X_t^{\psi_n}\|_V^\alpha+C\|X_t^{\psi_n}\|_V^\alpha\|X_t^{\psi_n}\|_H^\beta\ dtds\\
&\quad +C_\alpha\int_0^T\int_s^{\bar{s}_m}\|X_t^{\psi_n}\|_V^\alpha\ dtds+C_\alpha\int_0^T\int_s^{\bar{s}_m}\|X_s^{\psi_n}\|_V^\alpha\ dtds\\
&\leqslant C_\alpha\int_0^T\int_s^{\bar{s}_m}F_t+\|X_t^{\psi_n}\|_V^\alpha\ dtds+C_\alpha\cdot\Big[\sup_{\hbar\in S^N}\sup_{t\in[0,T]}\|X_t^\hbar\|_H^\beta\Big]\cdot\int_0^T\int_s^{\bar{s}_m}F_t+\|X_t^{\psi_n}\|_V^\alpha\ dtds\\
&\quad +\frac{T}{2^m}C_\alpha\cdot\sup_{\hbar\in S^N}\int_0^T\|X_s^\hbar\|_V^\alpha ds\\
&= C_\alpha\int_0^T\int_{\bar{s}_{m-1}}^t F_t+\|X_t^{\psi_n}\|_V^\alpha\ dsdt+C_\alpha\cdot\Big[\sup_{\hbar\in S^N}\sup_{t\in[0,T]}\|X_t^\hbar\|_H^\beta\Big]\cdot\int_0^T\int_{\bar{s}_{m-1}}^t F_t+\|X_t^{\psi_n}\|_V^\alpha\ dsdt\\
&\quad +\frac{T}{2^m}C_\alpha\cdot\sup_{\hbar\in S^N}\int_0^T\|X_s^\hbar\|_V^\alpha ds\\
&\leqslant \frac{T}{2^m}C_\alpha\cdot\Big[\int_0^T F_tdt+\sup_{\hbar\in S^N}\int_0^T\|X_t^\hbar\|_V^\alpha dt\Big]\\
&\quad +\frac{T}{2^m}C_\alpha\cdot\Big[\sup_{\hbar\in S^N}\sup_{t\in[0,T]}\|X_t^\hbar\|_H^\beta\Big]\cdot\Big[\int_0^T F_tdt+\sup_{\hbar\in S^N}\int_0^T\|X_t^\hbar\|_V^\alpha dt\Big]\\
&\quad +\frac{T}{2^m}C_\alpha\cdot\sup_{\hbar\in S^N}\int_0^T\|X_s^\hbar\|_V^\alpha ds.
\end{aligned}
$$

Since $F\in L^1([0,T];\mathbb{R}^+)$ and (4.1) holds, the above inequality can be dominated by $\frac{1}{2^m}C_{\alpha,T,N}$, where $C_{\alpha,T,N}$ only depends on C_N appearing in (4.1), T and α.

By **(H6)** and (4.1),

$$
\begin{aligned}
&\int_0^T\int_s^{\bar{s}_m}\Big\langle\int_Z f(t,X_t^{\psi_n},z)\big(\psi_n(t,z)-1\big)\nu(dz),X_t^{\psi_n}-X_s^{\psi_n}\Big\rangle_H dtds\\
&\leqslant \int_0^T\int_s^{\bar{s}_m}\int_Z L_f(t,z)\big(1+\|X_t^{\psi_n}\|_H\big)\|X_t^{\psi_n}-X_s^{\psi_n}\|_H\cdot|\psi_n(t,z)-1|\nu(dz)dtds\\
&\leqslant 4\Big[\sup_{\hbar\in S^N}\sup_{t\in[0,T]}\|X_t^\hbar\|_H^2+1\Big]\int_0^T\int_s^{\bar{s}_m}\int_Z L_f(t,z)\cdot|\psi_n(t,z)-1|\nu(dz)dtds\\
&\leqslant C_{N,T}\sup_{\hbar\in S^N}\sup_{s\in[0,T]}\int_s^{\bar{s}_m}\int_Z L_f(t,z)\cdot|\hbar(t,z)-1|\nu(dz)dt.
\end{aligned}
$$

Since by (3.9),

$$
\lim_{m\to+\infty}\sup_{\hbar\in S^N}\sup_{s\in[0,T]}\int_s^{\bar{s}_m}\int_Z L_f(t,z)\cdot|\hbar(t,z)-1|\nu(dz)dt=0,
$$

we then have

$$
\lim_{m\to+\infty}\sup_{n\in\mathbb{N}}\int_0^T\|X_{\bar{s}_m}^{\psi_n}-X_s^{\psi_n}\|_H^2ds=0. \tag{5.16}
$$

Using similar arguments, we also have

$$
\lim_{m\to+\infty}\int_0^T\|X_{\bar{s}_m}^{\psi}-X_s^{\psi}\|_H^2ds=0. \tag{5.17}
$$

Substituting (5.16) and (5.17) into (5.15), we get

$$\limsup_{m\to+\infty}\sup_{n\in\mathbb{N}}\tilde{I}_1\leqslant J_\varepsilon\cdot\frac{C_{N,T}}{\sigma}.$$

Since σ is arbitrary in $[1,+\infty)$, we obtain

$$\lim_{m\to+\infty}\sup_{n\in\mathbb{N}}\tilde{I}_1=0.$$

Similarly as to get [26, (5.37), (5.39)], we have

$$\lim_{m\to+\infty}\sup_{n\in\mathbb{N}}\tilde{I}_2=0,$$

and

$$\limsup_{m\to+\infty}\sup_{n\in\mathbb{N}}\tilde{I}_3=0.$$

Hence, for any $\gamma>0$, there exists $m_\gamma>0$ such that for all $m\geqslant m_\gamma$,

$$\sum_{i=1}^{3}\sup_{n\in\mathbb{N}}\tilde{I}_i\leqslant\gamma. \tag{5.18}$$

Similarly as to get [26, (5.46)], we have

$$\lim_{n\to+\infty}\left|\int_{t_{k-1}}^{t_k}\int_{K_\varepsilon}\langle f(s,X^\psi_{\bar{s}_m},z),U^n(\bar{s}_m)\rangle_H\big(\psi_n(s,z)-\psi(s,z)\big)1_{A^c_{L_f,J_\varepsilon}}(s,z)\nu(dz)ds\right|=0. \tag{5.19}$$

For the fixed γ and m_γ as above, (5.19) implies that

$$\lim_{n\to+\infty}\tilde{I}_4=0. \tag{5.20}$$

Then, by (5.13), (5.18) and (5.20),

$$\limsup_{n\to+\infty}\sup_{t\in[0,T]}|I_{n,1,J^c_\varepsilon}(t)|\leqslant\gamma,$$

which implies

$$\lim_{n\to+\infty}\sup_{t\in[0,T]}|I_{n,1,J^c_\varepsilon}(t)|=0. \tag{5.21}$$

since γ is arbitrary in $(0,+\infty)$.

Now, taking (5.21) into (5.12), we finally obtain

$$\limsup_{n\to+\infty}\sup_{t\in[0,T]}\|X^{\psi_n}_t-X^\psi_t\|^2_H\leqslant C_{G_f,N,\beta,\alpha,T}\cdot(\varepsilon+C_N\cdot\varepsilon),$$

since ε is arbitrary in $(0,+\infty)$, it follows that

$$\lim_{n\to+\infty}\sup_{t\in[0,T]}\|X^{\psi_n}_t-X^\psi_t\|^2_H=0,$$

which completes the proof of Proposition 5.1. □

6. The verification of **(LDP2)**

Recall from (3.5) that $X^{\psi_\epsilon}:=\Gamma^\epsilon(N^{\epsilon^{-1}\psi_\epsilon})$ is the unique solution of (3.6). Hence **(LDP2)** is verified once the following proposition is proved.

Proposition 6.1. *For any given $N\in\mathbb{N}$, let $\{\psi_\epsilon,\ \epsilon>0\}\subset\mathcal{S}^N$. Then, for any $\delta>0$,*

$$\lim_{\epsilon\to0}\mathbb{P}\left(\sup_{t\in[0,T]}\|X^{\psi_\epsilon}_t-\Gamma(\psi_\epsilon)(t)\|_H\geqslant\delta\right)=0. \tag{6.1}$$

Proof. For any given $N \in \mathbb{N}$, let $\{\psi_\epsilon,\ \epsilon > 0\} \subset \mathcal{S}^N$. Recall from (3.4) that $Y^{\psi_\epsilon} := \Gamma(\psi_\epsilon) \in C([0,T],H) \cap L^\alpha([0,T],V)$ is the unique solution of the following equation

$$Y_t^{\psi_\epsilon} = x + \int_0^t \mathcal{A}(s, Y_s^{\psi_\epsilon})\, ds + \int_0^t \int_Z f(s, Y_s^{\psi_\epsilon}, z)(\psi_\epsilon(s,z) - 1)\nu(dz)ds.$$

Taking (3.6) into account, we get

$$\begin{aligned} X_t^{\psi_\epsilon} - Y_t^{\psi_\epsilon} = &\int_0^t \mathcal{A}(s, X_s^{\psi_\epsilon}) - \mathcal{A}(s, Y_s^{\psi_\epsilon})\, ds + \epsilon \int_0^t \int_Z f(s, X_{s-}^{\psi_\epsilon}, z) \widetilde{N}^{\epsilon^{-1}\psi_\epsilon}(dz, ds) \\ &+ \int_0^t \int_Z \big(f(s, X_s^{\psi_\epsilon}, z) - f(s, Y_s^{\psi_\epsilon}, z)\big)\big(\psi_\epsilon(s,z) - 1\big)\nu(dz)ds, \quad \forall t \in [0,T]. \end{aligned}$$

By applying the Itô formula, we infer

$$\begin{aligned} \|X_t^{\psi_\epsilon} - Y_t^{\psi_\epsilon}\|_H^2 = &\ 2\int_0^t {}_{V^*}\langle \mathcal{A}(s, X_s^{\psi_\epsilon}) - \mathcal{A}(s, Y_s^{\psi_\epsilon}), X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon}\rangle_V \ ds \\ &+2\epsilon \int_0^t \int_Z \langle f(s, X_{s-}^{\psi_\epsilon}, z), X_{s-}^{\psi_\epsilon} - Y_{s-}^{\psi_\epsilon}\rangle_H \ \widetilde{N}^{\epsilon^{-1}\psi_\epsilon}(dz, ds) \\ &+2\int_0^t \int_Z \langle \big(f(s, X_s^{\psi_\epsilon}, z) - f(s, Y_s^{\psi_\epsilon}, z)\big)\big(\psi_\epsilon(s,z) - 1\big), X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon}\rangle_H \ \nu(dz)ds \\ &+\epsilon^2 \int_0^t \int_Z \|f(s, X_{s-}^{\psi_\epsilon}, z)\|_H^2 \ N^{\epsilon^{-1}\psi_\epsilon}(dz, ds), \quad \forall t \in [0,T]. \end{aligned}$$

Owing to **(H2)** and **(H7)**, we deduce

$$\begin{aligned} \|X_t^{\psi_\epsilon} - Y_t^{\psi_\epsilon}\|_H^2 \leqslant &\int_0^t (F_s + \rho(Y_s^{\psi_\epsilon}))\|X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon}\|_H^2 \ ds \\ &+2\epsilon \int_0^t \int_Z \langle f(s, X_{s-}^{\psi_\epsilon}, z), X_{s-}^{\psi_\epsilon} - Y_{s-}^{\psi_\epsilon}\rangle_H \ \widetilde{N}^{\epsilon^{-1}\psi_\epsilon}(dz, ds) \\ &+2\int_0^t \int_Z G_f(s,z)|\psi_\epsilon(s,z) - 1| \cdot \|X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon}\|_H^2 \nu(dz)ds \\ &+\epsilon^2 \int_0^t \int_Z \|f(s, X_{s-}^{\psi_\epsilon}, z)\|_H^2 \ N^{\epsilon^{-1}\psi_\epsilon}(dz, ds) \\ \leqslant &\ 2\int_0^t (G_\epsilon(s) + F_s + \rho(Y_s^{\psi_\epsilon}))\|X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon}\|_H^2 \ ds \\ &+2\epsilon \int_0^t \int_Z \langle f(s, X_{s-}^{\psi_\epsilon}, z), X_{s-}^{\psi_\epsilon} - Y_{s-}^{\psi_\epsilon}\rangle_H \ \widetilde{N}^{\epsilon^{-1}\psi_\epsilon}(dz, ds) \\ &+\epsilon^2 \int_0^t \int_Z \|f(s, X_{s-}^{\psi_\epsilon}, z)\|_H^2 \ N^{\epsilon^{-1}\psi_\epsilon}(dz, ds), \quad \forall t \in [0,T], \end{aligned} \tag{6.2}$$

where $G_\epsilon(s) := \int_Z G_f(s,z)|\psi_\epsilon(s,z) - 1|\nu(dz)$. From (3.8) there exists a constant C_{G_f} such that, for any $\epsilon \in (0,1]$,

$$\int_0^T G_\epsilon(s)ds \leqslant C_{G_f} < +\infty, \ \mathbb{P}\text{-a.s.}. \tag{6.3}$$

On applying Gronwall's inequality, we find

$$\begin{aligned} &\sup_{t\in[0,T]} \|X_t^{\psi_\epsilon} - Y_t^{\psi_\epsilon}\|_H^2 \\ \leqslant &\ \Big(2\epsilon \sup_{t\in[0,T]} \Big| \int_0^t \int_Z \langle f(s, X_{s-}^{\psi_\epsilon}, z), X_{s-}^{\psi_\epsilon} - Y_{s-}^{\psi_\epsilon}\rangle_H \ \widetilde{N}^{\epsilon^{-1}\psi_\epsilon}(dz, ds)\Big| \\ &\quad +\epsilon^2 \int_0^T \int_Z \|f(s, X_{s-}^{\psi_\epsilon}, z)\|_H^2 \ N^{\epsilon^{-1}\psi_\epsilon}(dz, ds)\Big) \cdot e^{2\int_0^T G_\epsilon(s) + F_s + \rho(Y_s^{\psi_\epsilon}) ds} \end{aligned}$$

$$\leqslant C_{N,\int_0^T F_s ds, C_{G_f}} \Big(2\epsilon \sup_{t\in[0,T]} \Big| \int_0^t \int_Z \langle f(s, X_{s-}^{\psi_\epsilon}, z), X_{s-}^{\psi_\epsilon} - Y_{s-}^{\psi_\epsilon} \rangle_H \, \widetilde{N}^{\epsilon^{-1}\psi_\epsilon}(dz, ds) \Big|$$
$$+\epsilon^2 \int_0^T \int_Z \|f(s, X_{s-}^{\psi_\epsilon}, z)\|_H^2 \, N^{\epsilon^{-1}\psi_\epsilon}(dz, ds) \Big), \ \mathbb{P}\text{-a.s..} \tag{6.4}$$

in which (6.3), (4.1) and **(H5)** were used to get

$$2 \int_0^T G_\epsilon(s) + F_s + \rho(Y_s^{\psi_\epsilon}) ds \leqslant C_{N,\int_0^T F_s ds, C_{G_f}} < +\infty, \ \mathbb{P}\text{-a.s.},$$

where $C_{N,\int_0^T F_s ds, C_{G_f}}$ is a constant, and only depends on C_N appearing in (4.1), $\int_0^T F_s ds$ and C_{G_f}.

Applying Doob's inequality for $p = 1$ (c.f. [22, Theorem 1] or [32, Proposition 2.2]) in the second term on the right-hand side of (6.2) gives

$$\begin{aligned}
& 2\epsilon \, \mathbb{E} \sup_{t\in[0,T]} \Big| \int_0^t \int_Z \langle f(s, X_{s-}^{\psi_\epsilon}, z), X_{s-}^{\psi_\epsilon} - Y_{s-}^{\psi_\epsilon} \rangle_H \, \widetilde{N}^{\epsilon^{-1}\psi_\epsilon}(dz, ds) \Big| \\
\leqslant\ & 2\epsilon \, \mathbb{E} \Big[\int_0^T \int_Z \langle f(s, X_s^{\psi_\epsilon}, z), X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon} \rangle_H^2 \, \epsilon^{-1} \psi_\epsilon(s,z) \nu(dz) ds \Big]^{\frac{1}{2}} \\
\leqslant\ & 2\epsilon^{\frac{1}{2}} \, \mathbb{E} \Big[\int_0^T \int_Z 2|L_f(s,z)|^2 (1 + \|X_s^{\psi_\epsilon}\|_H^2) \|X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon}\|_H^2 \psi_\epsilon(s,z) \nu(dz) ds \Big]^{\frac{1}{2}} \\
\leqslant\ & 2\epsilon^{\frac{1}{2}} \, \mathbb{E} \Big[\sup_{s\in[0,T]} \|X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon}\|_H^2 \int_0^T \int_Z 2|L_f(s,z)|^2 (1 + \|X_s^{\psi_\epsilon}\|_H^2) \psi_\epsilon(s,z) \nu(dz) ds \Big]^{\frac{1}{2}} \\
\leqslant\ & \epsilon^{\frac{1}{2}} \, \mathbb{E} \Big[\sup_{s\in[0,T]} \|X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon}\|_H^2 \Big] + \epsilon^{\frac{1}{2}} \, \mathbb{E} \Big[\int_0^T \int_Z 2|L_f(s,z)|^2 (1 + \|X_s^{\psi_\epsilon}\|_H^2) \psi_\epsilon(s,z) \nu(dz) ds \Big] \\
\leqslant\ & \epsilon^{\frac{1}{2}} \, \mathbb{E} \Big[\sup_{s\in[0,T]} \|X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon}\|_H^2 \Big] + 2\epsilon^{\frac{1}{2}} \mathbb{E} \Big[\sup_{s\in[0,T]} (1 + \|X_s^{\psi_\epsilon}\|_H^2) \int_0^T \int_Z |L_f(s,z)|^2 \psi_\epsilon(s,z) \, \nu(dz) ds \Big] \\
\leqslant\ & \epsilon^{\frac{1}{2}} \, \mathbb{E} \Big[\sup_{s\in[0,T]} \|X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon}\|_H^2 \Big] + 2\epsilon^{\frac{1}{2}} C_{L_f,N} \, \mathbb{E} \Big[\sup_{s\in[0,T]} (1 + \|X_s^{\psi_\epsilon}\|_H^2) \Big],
\end{aligned} \tag{6.5}$$

where $C_{L_f,N} := \sup_{\hbar\in S^N} \int_0^T \int_Z |L_f(t,z)|^2 (\hbar(t,z) + 1) \nu(\mathrm{d}z) \mathrm{d}t < +\infty$ by (3.7) and we also used Assumption **(H6)** in the second inequality and Young's inequality in the fourth inequality.

For the last term on the right-hand side of (6.2), using again Assumption **(H6)**, we find

$$\begin{aligned}
& \epsilon^2 \mathbb{E} \Big[\int_0^T \int_Z \|f(s, X_{s-}^{\psi_\epsilon}, z)\|_H^2 \, N^{\epsilon^{-1}\psi_\epsilon}(dz, ds) \Big] \\
=\ & \epsilon \, \mathbb{E} \Big[\int_0^T \int_Z \|f(s, X_s^{\psi_\epsilon}, z)\|_H^2 \, \psi_\epsilon(s,z) \, \nu(dz) ds \Big] \\
\leqslant\ & \epsilon \, \mathbb{E} \Big[\int_0^T \int_Z 2|L_f(s,z)|^2 (1 + \|X_s^{\psi_\epsilon}\|_H^2) \, \psi_\epsilon(s,z) \, \nu(dz) ds \Big] \\
\leqslant\ & 2\epsilon \, C_{L_f,N} \, \mathbb{E} \Big[\sup_{s\in[0,T]} (1 + \|X_s^{\psi_\epsilon}\|_H^2) \Big].
\end{aligned} \tag{6.6}$$

Combining (6.5), (6.6) with (6.4), we obtain for any $\epsilon \in (0, 1]$,

$$\begin{aligned}
& \mathbb{E} \Big[\sup_{s\in[0,T]} \|X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon}\|_H^2 \Big] \\
\leqslant\ & \epsilon^{\frac{1}{2}} C_{N,\int_0^T F_s ds, C_{G_f}} \, \mathbb{E} \Big[\sup_{s\in[0,T]} \|X_s^{\psi_\epsilon} - Y_s^{\psi_\epsilon}\|_H^2 \Big] \\
& + 2(\epsilon^{\frac{1}{2}} + \epsilon) \, C_{L_f,\int_0^T F_s ds, N, C_{G_f}} \, \mathbb{E} \Big[\sup_{s\in[0,T]} (1 + \|X_s^{\psi_\epsilon}\|_H^2) \Big],
\end{aligned} \tag{6.7}$$

where $C_{L_f,\int_0^T F_s ds,N,C_{G_f}}$ is a constant, and only depends on C_N appearing in (4.1), L_f, $\int_0^T F_s ds$ and C_{G_f}.

By applying the Itô formula to $\|X_t^{\psi_\epsilon}\|_H^2$, and using Assumptions **(H3)** and **(H6)**, we infer

$$
\begin{aligned}
&\|X_t^{\psi_\epsilon}\|_H^2\\
=\;& \|x\|_H^2 + 2\int_0^t {}_{V^*}\langle \mathcal{A}(s,X_s^{\psi_\epsilon}), X_s^{\psi_\epsilon}\rangle_V\, ds + 2\epsilon\int_0^t\int_Z \langle f(s,X_{s-}^{\psi_\epsilon},z), X_{s-}^{\psi_\epsilon}\rangle_H\, \widetilde{N}^{\epsilon^{-1}\psi_\epsilon}(dz,ds)\\
&+2\int_0^t\int_Z \langle f(s,X_s^{\psi_\epsilon},z)\big(\psi_\epsilon(s,z)-1\big), X_s^{\psi_\epsilon}\rangle_H\, \nu(dz)ds\\
&+\epsilon^2\int_0^t\int_Z \|f(s,X_{s-}^{\psi_\epsilon},z)\|_H^2\, N^{\epsilon^{-1}\psi_\epsilon}(dz,ds)\\
\leqslant\;& \|x\|_H^2 + \int_0^t \big(F_s+4L_\epsilon(s)\big)(1+\|X_s^{\psi_\epsilon}\|_H^2)ds + \epsilon^2\int_0^t\int_Z \|f(s,X_{s-}^{\psi_\epsilon},z)\|_H^2\, N^{\epsilon^{-1}\psi_\epsilon}(dz,ds)\\
&+2\epsilon\int_0^t\int_Z \langle f(s,X_{s-}^{\psi_\epsilon},z), X_{s-}^{\psi_\epsilon}\rangle_H\, \widetilde{N}^{\epsilon^{-1}\psi_\epsilon}(dz,ds),,\quad \forall t\in[0,T],
\end{aligned}
$$

where $L_\epsilon(t) := \int_Z L_f(t,z)\cdot|\psi_\epsilon(s,z)-1|\nu(dz)$. From (3.8) there exists a constant C_{L_f} such that, for any $\epsilon\in(0,1]$,

$$\int_0^T L_\epsilon(s)ds \leqslant C_{L_f} < +\infty,\ \mathbb{P}\text{-a.s..}$$

Gronwall's inequality and the above inequality now provide

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|X_t^{\psi_\epsilon}\|_H^2\\
\leqslant\;& \Big(\|x\|_H^2 + \int_0^T F_s ds + 4C_{L_f} + 2\epsilon\sup_{t\in[0,T]}\Big|\int_0^t\int_Z \langle f(s,X_{s-}^{\psi_\epsilon},z), X_{s-}^{\psi_\epsilon}\rangle_H\, \widetilde{N}^{\epsilon^{-1}\psi_\epsilon}(dz,ds)\Big|\\
&\qquad +\epsilon^2\int_0^T\int_Z \|f(s,X_{s-}^{\psi_\epsilon},z)\|_H^2\, N^{\epsilon^{-1}\psi_\epsilon}(dz,ds)\Big)\cdot e^{\int_0^T F_s ds+4C_{L_f}},\ \mathbb{P}\text{-a.s..}
\end{aligned}
\tag{6.8}
$$

Similar to (6.5), in view of Doob's inequality for $p=1$ and Assumption **(H6)**, we obtain

$$
\begin{aligned}
&2\epsilon\,\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\int_0^t\int_Z \langle f(s,X_{s-}^{\psi_\epsilon},z), X_{s-}^{\psi_\epsilon}\rangle_H\, \widetilde{N}^{\epsilon^{-1}\psi_\epsilon}(dz,ds)\Big|\Big]\\
\leqslant\;& 2\epsilon^{\frac12}\,\mathbb{E}\Big[\int_0^T\int_Z 2|L_f(s,z)|^2(1+\|X_s^{\psi_\epsilon}\|_H^2)\|X_s^{\psi_\epsilon}\|_H^2\psi_\epsilon(s,z)\nu(dz)ds\Big]^{\frac12}\\
\leqslant\;& 4\epsilon^{\frac12}\,C_{L_f,N}^{\frac12}\mathbb{E}\Big[\sup_{s\in[0,T]}(1+\|X_s^{\psi_\epsilon}\|_H^2)\Big].
\end{aligned}
\tag{6.9}
$$

Taking the estimates (6.6) and (6.9) into account, (6.8) yields

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[0,T]}\|X_s^{\psi_\epsilon}\|_H^2\Big] \leqslant\;& \Big(\|x\|_H^2 + \int_0^T F_s ds + 4C_{L_f} + 4\epsilon^{\frac12}\,C_{L_f,N}^{\frac12}\mathbb{E}\Big[\sup_{s\in[0,T]}(1+\|X_s^{\psi_\epsilon}\|_H^2)\Big]\\
&+2\epsilon\,C_{L_f,N}\,\mathbb{E}\Big[\sup_{s\in[0,T]}(1+\|X_s^{\psi_\epsilon}\|_H^2)\Big]\Big)\cdot e^{\int_0^T F_s ds+4C_{L_f}}.
\end{aligned}
$$

Let $\epsilon_0\in(0,1)$ be such that $(4\epsilon_0^{\frac12}\,C_{L_f,N}^{\frac12}+2\epsilon_0\,C_{L_f,N})e^{\int_0^T F_s ds+4C_{L_f}} \leqslant \frac12$. Then for any $\epsilon\in(0,\epsilon_0]$, we conclude

$$
\begin{aligned}
\frac12\mathbb{E}\Big[\sup_{s\in[0,T]}\|X_s^{\psi_\epsilon}\|_H^2\Big] \leqslant\;& \Big(\|x\|_H^2 + \int_0^T F_s ds + 4C_{L_f}\Big)\cdot e^{\int_0^T F_s ds+4C_{L_f}} + \frac12\\
:=\;& C_{\|x\|_H,\int_0^T F_s ds, C_{L_f}} < +\infty.
\end{aligned}
\tag{6.10}
$$

Taking (6.10) into (6.7), we get that for any $\epsilon \in (0, \epsilon_0]$,

$$\mathbb{E}\Big[\sup_{s\in[0,T]}\|X_s^{\psi_\epsilon}-Y_s^{\psi_\epsilon}\|_H^2\Big] \leqslant \epsilon^{\frac{1}{2}}C_{N,\int_0^T F_s ds, C_{G_f}}\,\mathbb{E}\Big[\sup_{s\in[0,T]}\|X_s^{\psi_\epsilon}-Y_s^{\psi_\epsilon}\|_H^2\Big]$$
$$+2(\epsilon^{\frac{1}{2}}+\epsilon)(2C_{\|x\|_H,\int_0^T F_s ds, C_{L_f}}+1)\,C_{L_f,\int_0^T F_s ds, N, C_{G_f}}.$$

Let $\epsilon_1 \in (0, \epsilon_0]$ be such that $\epsilon_1^{\frac{1}{2}}C_{N,\int_0^T F_s ds, C_{G_f}} < \frac{1}{2}$. Then for any $\epsilon \in (0, \epsilon_1]$,

$$\mathbb{E}\Big[\sup_{s\in[0,T]}\|X_s^{\psi_\epsilon}-Y_s^{\psi_\epsilon}\|_H^2\Big] \leqslant 4(\epsilon^{\frac{1}{2}}+\epsilon)(2C_{\|x\|_H,\int_0^T F_s ds, C_{L_f}}+1)\,C_{L_f,\int_0^T F_s ds, N, C_{G_f}},$$

which implies (6.1).

The proof of Proposition 6.1 is complete. □

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