Dr. Kilian Raschel
Daniel Altemeier
Department of Mathematics
Bielefeld University

SS 2011

Stochastik 2 - Exercises 12

The processing of this exercise-sheet is voluntary, you are not asked to hand it over and it won't be discussed in the tutorials, nevertheless you may see it as a chance for practice and a better insight.

Exercise 12.I: (Explosion-time criteria for general birth-death-processes)
Assume $\big(X_t\big)_{t\geq 0}$ to be the birth-death-process with birth-rates $\big(\lambda_i\big)_{i\in\mathbb{N}\cup\{0\}}$ and death-rates $\big(\mu_i\big)_{i\in\mathbb{N}\setminus\{0\}}$, all strictly positive.

1. Draw the transition graph and denote the Q-matrix.
2. Try to apply the 'detailed-balanced'-concept, define

$$b_j := \frac{\lambda_0 \cdot \ldots \cdot \lambda_{j-1}}{\mu_1 \cdot \ldots \mu_j},\ j \in \mathbb{N}$$

   and show that $\pi : \pi(j) = \pi_0 b_j, j \in \mathbb{N}$ is <u>the</u> invariant distribution, compute π_0.
3. Argue that the 'detailed balanced' concept is equivalent to the $\pi Q = 0$-condition.
4. Show that $\sum_{j=0}^{\infty} b_j = \infty$ implies that $\pi = 0$ and therefore is no invariant distribution and not Markovian.
5. For $\sum_{j=0}^{\infty} b_j < \infty$, show that

$$\pi_j = \lim_{t\to\infty} P_{ij}(t)\ \forall\, i, j \in \mathbb{N}_0.$$

Exercise 12.II: (repitition)
For a M/M/1-queue compute the expected time of return to 0 (from state 1), the lecture has given a hint how we can do that.
Advice: Show that $\mathbb{E}_2\big[s^{H^{(0)}}\big] = \mathbb{E}_1\big[s^{H^{(0)}}\big]^2$ and trivially $\mathbb{E}_1\big[s^{H^{(0)}}\big|X_1 = 0\big] = s$, then by

$$\varphi(s) = \mathbb{E}_1\big[s^{H^{(0)}}\big] = p\mathbb{E}_1\big[s^{H^{(0)}}\big|X_1 = 2\big] + q\mathbb{E}_1\big[s^{H^{(0)}}\big|X_1 = 0\big],$$

the problem is (almost) solved.

Exercise 12.III: (M/M/s - queue)

1. Give a sufficient condition for non-explosiveness of the queue
2. Characterize the invariant distribution in case of non-explosiveness (Try the BE/FE as well as the 'detailed-balance'-concept), verify

$$\pi_i = \begin{cases} \left(\frac{\lambda}{\mu}\right)^i \frac{1}{i!}\pi_0 & \text{if } i = 0, \ldots, s \\ \\ \left(\frac{\lambda}{\mu}\right)^i \frac{1}{s^{i-s}s!}\pi_0 & \text{if } i \geq s+1 \end{cases} , \ i \in \mathbb{N}$$

3. What happens in the case $s \nearrow \infty$?
4. In equilibrium, what the expected rate of departures?