Dr. Kilian Raschel
Daniel Altemeier
Department of Mathematics
Bielefeld University

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Stochastik 2 - Exercises 2 *2nd version*

Handover date: **Thursday, Apr 21st, 12:00**

Please put your solutions into the mailbox 200 which belongs to the head of the tutorials, Ms. Katharina von der Lühe. The mailbox can be found in the copy-room V3-128. Before the insertion of the solution please check that the sheets are ordered correctly and tacked. Write down your name in a legible handwriting on the the first sheet of your solution.

Exercise 2.I:
Consider a Markov Chain $\big(X_n\big)_{n\in\mathbb{N}}$ with finite countable state-space I and specified by some transition matrix $P = (p_{i,j})_{i,j\in I}$.

(a) Show that: If $f_{ii} > 0 \; \forall\, i \in I$, then '$\leftrightsquigarrow$' is an equivalence-relation on I

(b) Prove Proposition A.III.8: If $i \rightsquigarrow j$, then

$$\text{State } i \text{ is recurrent} \quad \Rightarrow \quad \text{state } j \text{ is recurrent} .$$

Exercise 2.II:
Consider the following transition matrix P that corresponds to some Markov-Chain $\big(X_n\big)_{n\in\mathbb{N}}$, let $\{S_1, \ldots, S_5\}$ be the state-space.

$$P := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Draw the corresponding transition graph and note, that one can interpret the chain as *asymmetric random walk with absorbing boundary.*

(b) Find the probability, that the process X is absorbed in state S_1, if we assume to start in S_2.

Exercise 2.III:
Consider some diploid plant genome containing the alleles a and A. A typical mean of breeding plants is the autogamic reproduction. We consider our initial plant to exhibit the allele-combination Aa. Assume that inheritance chooses randomly (twice) one single allele from the current combination and there is only one offspring per generation. Let $\left(X_n\right)_{n\in\mathbb{N}}$ denote the current allele-combination (within the allele-combination the order of alleles does not matter).

(a) Draw the transition graph, give all *closed classes* of the process.

(b) Compute explicitely all the n-step transition probabilities $p_{i,i}^{(n)}$ for all $i, j \in \{AA, Aa, aa\}$ and all $n \in \mathbb{N}$. Which states are recurrent, which are transient?

Exercise 2.IV:
Consider the symmetric 2-dimensional random walk $X = \left(X_n\right)_{n\in\mathbb{N}}$ on $\mathbb{Z}^2$ with $X_0 = 0 = (0,0)$. Use the following instructions to prove that every state $z \in \mathbb{Z}$ is recurrent.

(step 1) Give a precise definition of what *symmetric 2-dimensional random walk on* $\mathbb{Z}^2$ *with* $X_0 = 0$ means.

(step 2) Denote $X_n = \left(X_n^{(1)}, X_n^{(2)}\right), n \in \mathbb{N}$ and show that the sum $S(n) := X_n^{(1)} + X_n^{(2)}, n \in \mathbb{N}$ and the difference $D(n) := X_n^{(1)} - X_n^{(2)}, n \in \mathbb{N}$ are each a *symmetric 1-dimensional random walk on* $\mathbb{Z}$ *with start in* 0.

(step 3) Show that S_n and D_n are independent for all $n \in \mathbb{N}$.

(step 4) Use that $\left\{X_n = 0 = (0,0)\right\}$ is equivalent to $\left\{S_n = 0\right\} \cap \left\{D_n = 0\right\}$, then use the results from the lecture (see Proposition A.III.) to estimate $p_{00}^{(n)}$ of X.

(step 5) Deduce that X is recurrent