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Stochastik 2 - Exercises 3

Handover date: **Friday, Apr 29th, 10:00**

Please put your solutions into the mailbox 200 which belongs to the head of the tutorials, Ms. Katharina von der Lühe. The mailbox can be found in the copy-room V3-128. Before the insertion of the solution please check that the sheets are ordered correctly and tacked. Write down your name in a legible handwriting on the the first sheet of your solution.

Exercise 3.I:
Consider $\big(X_n\big)_{n\in\mathbb{N}}$ as Markov Chain with transition matrix

$$P = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 1/8 & 0 & 3/4 & 1/8 & 0 \end{pmatrix},$$

let $X_0 = 0$. Decide for all states if they are recurrent or transient, make use of the theorems from the lecture, do NOT compute anything.

Exercise 3.II - 1st part:
Consider the symmetric 2-dimensional random walk $X = \big(X_n\big)_{n\in\mathbb{N}}$ on $\mathbb{Z}^2$ with $X_0 = 0 = (0,0)$. Use the following instructions to prove that every state $z \in \mathbb{Z}^2$ is recurrent.

(step 1) *Use:* The transition probability is an equal distribution on the all states that have distant 1 in '$p = 2$'-norm, i.e.

$$\forall\ i,j \in \mathbb{Z}^2: \qquad p_{ij} := \begin{cases} 1/4 & , \text{ if } \|i-j\|_2 = 1 \\ 0 & , \text{ else.} \end{cases}$$

together with $X_0 = 0$ the Markovian Chain is totally characterized.

(step 2) Denote $X_n = \big(X_n^{(1)}, X_n^{(2)}\big), n \in \mathbb{N}$ and show that the sum $S(n) := X_n^{(1)} + X_n^{(2)}, n \in \mathbb{N}$ and the difference $D(n) := X_n^{(1)} - X_n^{(2)}, n \in \mathbb{N}$ are each a *symmetric 1-dimensional random walk on $\mathbb{Z}$ with start in* 0.

(step 3) Show that S_n and D_n are independent for all $n \in \mathbb{N}$.

(step 4) Use that $\big\{X_n = 0 = (0,0)\big\}$ is equivalent to $\big\{S_n = 0\big\} \cap \big\{D_n = 0\big\}$, then use the results from the lecture (see Proposition A.III.9) to estimate $p_{00}^{(n)}$ of X.

(step 5) Deduce that X is recurrent.

Exercise 3.II 2nd part:
Consider now the symmetric 3-dimensional random walk $X = \big(X_n\big)_{n\in\mathbb{N}}$ on $\mathbb{Z}^3$ with $X_0 = 0 = (0,0,0)$. Again following the below instructions to show that in this case every $z \in \mathbb{Z}^3$ is transient.

(step 1) Give a precise definition of what *symmetric random walk on* $\mathbb{Z}^3$ *with* $X_0 = 0$ means.

(step 2) For arbitrary $i \in I$, choose w.l.o.g.[1] $i = 0$, the process can only return to 0 in an even number of steps and, the numbers of steps 'up' and 'down' must coincide (say i each), just like the ones 'north' and 'south' (say j each), and the ones 'west' and 'east' (say k each) must do. Furthermore $i + j + k = n$.
Show that the following holds for all $n \in \mathbb{N}$:

$$p_{00}^{(2n)} = \sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \frac{(2n)!}{\big(i!j!k!\big)^2} \left(\frac{1}{6}\right)^{2n}$$

Prove then:

$$= \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \binom{n}{i\ j\ k}^2 \cdot \left(\frac{1}{3}\right)^{2n}$$

(step 3) Now, argue that

$$\sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} \binom{n}{i\ j\ k} \left(\frac{1}{3}\right)^{n} = 1$$

(Think of putting n balls in 3 different boxes, compute the probability to find i, j, k balls in box $1, 2, 3$, then the claim is obvious)

(step 4) In the case[2] $n = 3m$ for some $m \in \mathbb{N}$, show that by an application of *Stirling's* inequality, we get

$$p_{00}^{(2n)} \leq \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \binom{n}{m\ m\ m} \left(\frac{1}{3}\right)^{n} \sim \frac{1}{2A^3} \left(\frac{6}{n}\right)^{3/2} \quad \text{when } n \text{ large.}$$

(step 5) Show the following inequalities:

$$p_{00}^{(6m)} \geq \frac{1}{6^2} p_{00}^{(6m-2)} \quad \text{and} \quad p_{00}^{(6n)} \geq \frac{1}{6^4} p_{00}^{6m-4} \quad \text{for all } m \geq 1.$$

(step 6) Deduce that

$$\sum_{n\in\mathbb{N}} p_{00}^{(n)} < \infty.$$

(step 7) Conclude that for all *symmetric random walks on* $\mathbb{Z}^d$ with $d \geq 4$, all states are transient.

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[1] w.l.o.g. $\hat{=}$ without loss of generality
[2] You may use the fact, that here: $\binom{n}{i\ j\ k} \leq \binom{n}{m\ m\ m}$ for all $i, j, k \geq 0 : i + j + k = n$.