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SS 2011

Stochastik 2 - Exercises 4

Handover date: **Friday, May 6th, 10:00**

Please put your solutions into the mailbox 200 which belongs to the head of the tutorials, Ms. Katharina von der Lühe. The mailbox can be found in the copy-room V3-128. Before the insertion of the solution please check that the sheets are ordered correctly and tacked. Write down your name in a legible handwriting on the the first sheet of your solution.

Exercise 4.I:(Stopping times)

Consider $X = (X_n)_{n\in\mathbb{N}}$ a Markov Chain with countable state space I on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F} = \bigcup_{i\in\mathbb{N}} \mathcal{F}_i$ is the natural filtration of the Markov Chain X.

(a) Show that for all $i \in I$, the *first hitting time*,

$$\tau_i(\omega) := \inf\{\, n \in \mathbb{N} \mid X_n(\omega) = i\} \quad \forall\ \omega \in \Omega,$$

with the convention $\inf\{\ \emptyset\ \} = +\infty$, is a *stopping time*[1].

(b) Show that for any $A \subseteq I$ the first hitting

$$\tau_A := \inf\{\ n \in \mathbb{N} \mid X_n \in A\ \},$$

is a stopping time, where we again use the convention $\inf\{\emptyset\} = +\infty$.

(c) Show that the random variable

$$\iota_i := \sup\{\ n \in \mathbb{N} \mid X_n(\omega) = i\ \}$$

(set $\sup(\emptyset) = -\infty$)in general is not a stopping time.

Exercise 4.II:

Again let $X = (X_n)_{n\in\mathbb{N}}$ be a Markov Chain with countable state space I on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is again the natural filtration of X. Consider two $\mathcal{F}$-stopping times S and T. Show that the following claims hold:

1. The random variable $S + T$ is again a stopping time, does the same hold true for $S - T$ if $S > T$?
2. If $S \leq T$, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.
3. For which $\alpha \in \mathbb{R}_+$ is αT again a stopping time?

[1] see A.IV.1 b)

Exercise 4.III:
Consider $X = \big(X_n\big)_{n\in\mathbb{N}}$ a Markov Chain on $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ with countable state space I.

(a) Show that the random variable

$$X_T(\omega) := X_{T(\omega)}(\omega) \quad \text{for all } \omega \in \Omega,$$

is measurable w.r.t. [2] $\mathcal{F}_T$ if T is a stopping time.

(b) Find an example for $T : \Omega \to \mathbb{N}$ such that X_T is not $\mathcal{F}_T$-measurable.

[2] An I-valued random variable $Y : \Omega \to I$ is called measurable with respect to some set $\mathcal{A}$ of subsets of Ω, if for all $i \in I$, the set $\{\ \omega \in \Omega \mid Y = i\ \}$ is included in $\mathcal{A}$