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SS 2011

Stochastik 2 - Exercises 5

Handover date: **Friday, May 13th, 10:00**

Please put your solutions into the mailbox 200 which belongs to the head of the tutorials, Ms. Katharina von der Lühe. The mailbox can be found in the copy-room V3-128. Before the insertion of the solution please check that the sheets are ordered correctly and tacked. Write down your name in a legible handwriting on the the first sheet of your solution.

Exercise 5.I:

Observe once more the stochastic matrix

$$P = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 \end{pmatrix}$$

Compute $\lim_{n\to\infty} p_{ii}^n$ and show that there are at least 2 invariant probability distributions for P.

Exercise 5.II:

Assume once more that I is (countable) finite and a Markov Chain $X = \big(X_n\big)_{n\in\mathbb{N}}$ on some probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, where $\mathcal{F}$ is the natural filtration of the process X and the distribution $\mathbb{P}$ representable in a stochastic matrix P. Suppose for some $i \in I$ that

$$p_{ij}^{(n)} \xrightarrow[n\to\infty]{} \pi_j \text{ for all } j \in I. \tag{1}$$

(a) Show that $\pi = \big(\ \pi_j\ :\ j \in I\ \big)$ is an invariant distribution.

Advice:

(i) Show that π is a probability distribution on I, you need that for finite sums the '$\sum$' and the limit operator '$\lim_{\dots}$' are interchangable .

(ii) Use the Chapman-Kolmogorov equation to show, that π_j is indeed invariant under P.

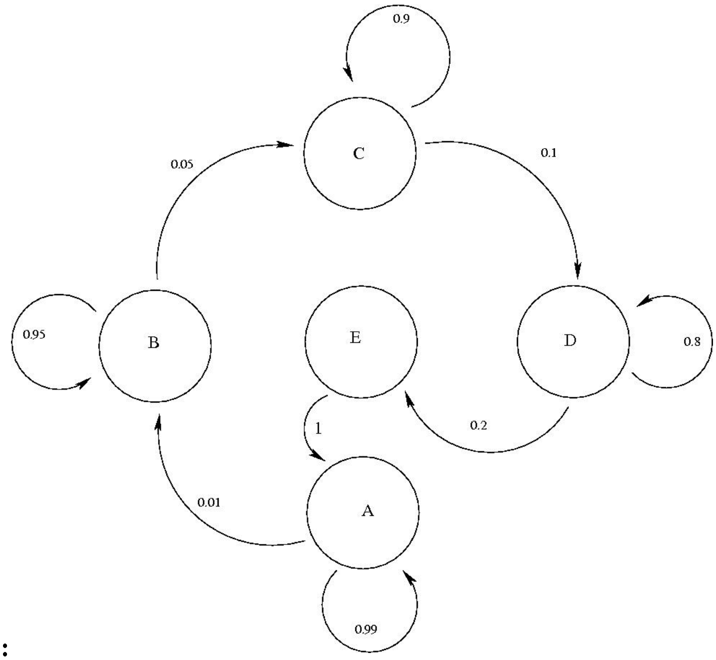
(b) Find an example such that equation 1 is valid, but π is not an invariant measure under P.

(One of the two steps above does in general not work within an infinite state-space.)

Exercise 5.III:

Remember presence-exercise I.III (1) gather the distribution of X_{100}, X_{1000} for all $i \in \{A, B, C, D, E\}$ as starting points X_0 and (2) compute all invariant distributions[1] of P. (3) Discribe your observations .

Figure 1: *Transition graph Presence-Exercise 1.III:*



Exercise 5.IV:

Observe the following transition P:

$$P := \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/10 & 0 & 0 & 1/2 & 0 & 2/5 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/4 & 3/4 \end{pmatrix}$$

(a) Draw the transition graph

(b) Let $X = \big(X_n\big)_{n\in\mathbb{N}}$ be a Markov Chain on $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ with transition matrix P. Give for all $e_1 = (1, 0, 0, 0, 0, 0), e_2 = (0, 1, 0, 0, 0), \ldots, e_6 = (0, 0, 0, 0, 0, 1)$ the distribution of the random variable X_{10}, X_{100} under $\mathbb{P}_{e_i}, i = 1, \ldots, 6$, which means to compute a 'long-time'-state distribution depending on the initial state of X.

(c) Compute all the invariant distributions of X.

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[1] We propose the use of a CAS, the explicit Gauss-elimination is NOT thought to be part of your solution.