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Stochastik 2 - Exercises 8 *3rd edition*

Handover date: **Friday, June 10th, 10:00**

Please put your solutions into the mailbox 200 which belongs to the head of the tutorials, Ms. Katharina von der Lühe. The mailbox can be found in the copy-room V3-128. Before the insertion of the solution please check that the sheets are ordered correctly and tacked. Write down your name in a legible handwriting on the the first sheet of your solution.

Exercise 8.I: (repetition)

Let $\big(A(i,j)\big)_{1\leq i,j\leq n}, n \in \mathbb{N}\cup\{\infty\} = I$, a matrix with the following properties:

1. $A(i,j) \geq 0$ for $i \neq j$,
2. $-a(i) := A(i,i) < 0$ and $\sum_{j\in I} A(i,j) = 0$ for all $j \in I$,
3. $a := \sup_{i\in I} a(i) < \infty$.

Now, we construct a Markovian process $\big(M_t\big)_{t\geq 0}$ that "'jumps from i to j with rate $A(i,j)$"'. Therefore, study the following matrix Π:

$$\Pi(i,j) = \delta_{i,j} + \frac{A(i,j)}{a}, \quad i,j \in I.$$

(i) As a starter...

[a] Show, that Π satisfies all the properties of a transition matrix

(ii) Assume now, that for all $i \in I$ on a properly defined probability space $\big(\Omega, \mathcal{F}, \mathbb{P}_i\big)$, the following objects are defined:

I First, a Markov-chain $\big(Z_j\big)_{j\in I}$ with transition probability Π and start in i,

II Second, an independent Poisson-process $N(t)_{t\geq 0}$ with intensity a.

Now, define $M_t := \big(Z_{N(t)}\big)_{t\geq 0}$ and prove **[b]**-**[d]**:

[b] For all $i,j \in I$:

$$\mathbb{P}_i\big[M_t = j\big] = \exp\big(tA\big)(i,j) := \sum_{n\geq 0}\left(\frac{t^n A^n}{n!}\right)(i,j)$$

Show that $A^n(i,j) \leq (2a)^n$ and deduce that the right-hand side is well defined. Furthermore, show that

$$\frac{d}{dt}\mathbb{P}_i\big[M_t = j\big]\Big|_{t=0} = A(i,j)\forall\, i,j \in I.$$

[c] $\big(M_t\big)_{t\geq 0}$ is a Markov-process with transition semigroup $\Pi_t := \exp(At), t \geq 0$, That means, for all $n \geq 1, 0 = t_0 < t_1 < \ldots < t_n$ and $x_0, x_1, \ldots, x_n \in I$:

$$\mathbb{P}_{x_0}\big[M_{t_1} = x_1, \ldots, M_{t_n} = x_n\big] = \prod_{k=1}^{n} \Pi_{t_k - t_{k-1}}\big(x_{k-1}, x_k\big).$$

[d] Now, let $\tau_0^\star = 0$ and $Z_0^\star = i$ and define recursively for $n \geq 1$:

$$\tau_n^\star := \inf_{t > \tau_{n-1}^\star} : M_t \neq Z_{n-1}^\star,$$

the *time* of the n-th jump and

$$Z_n^\star = M_{\tau_n^\star}$$

the *target* of the n-th jump. Then:

[d.1] $\big(Z_n^\star\big)_{n\geq 0}$ is a Markov-chain with state-space I, transition matrix

$$\Pi^\star(i,j) = \begin{cases} \dfrac{\Pi(i,j)}{1 - \Pi(i,i)} & \text{if } i \neq j \\ 0 & \text{else} \end{cases}.$$

[d.2] Depending on the 'current' state $\big(Z_n^\star\big)_{n\geq 0}$, the waiting times $\tau_{n+1}^\star - \tau_n^\star$ in state $Z_n^\star, n \geq 1$, are independent and exponentially distributed each with parameter $a(Z_n^\star)$.

Exercise 8.II:(finite explosion time)

In general weakening the assumption of bounded jump rates, we face the problem of a finite explosion time. In the above setting assume the following: Let $I = \mathbb{N}$ and assume

$$A(i, i+1) = -A(i,i) = i^2, i \in I = \mathbb{N}.$$

1. Determine the discrete jump-chain $\big(Z_n^\star\big)_{n\in\mathbb{N}}$ with start in $i_0 = 1$.
2. Compute the distribution of the jump-times $\big(\tau_n^\star\big)_{n\geq 1}$
3. Show that the CTMC $\big(M_t\big)_{t\geq 0}$ $\mathbb{P}_{i_0}$-almost surely explodes in finite time $\sup_{n\geq 1} \tau_n^\star$, i.e.
$$\mathbb{E}^{i_0}\left[\sup_{n\geq 1} \tau_n^\star\right] < \infty.$$