Dr. Kilian Raschel
Daniel Altemeier
Department of Mathematics
Bielefeld University

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Stochastik 2 - Exercises 9

Handover date: **Friday, June 17th, 10:00**

Please put your solutions into the mailbox 200 which belongs to the head of the tutorials, Ms. Katharina von der Lühe. The mailbox can be found in the copy-room V3-128. Before the insertion of the solution please check that the sheets are ordered correctly and tacked. Write down your name in a legible handwriting on the the first sheet of your solution.

Exercise 9.I:

From the lecture...

Example B.V.5 (Yule-Process). *Consider a bacterial culture and assume that cell divisions occur i.i.d. exponentially with rate* λ. *Every bacterium is immortal. Let* $X(t) \in \mathbb{N}$ *denote the number of bacteria in time* t. *Then* $\big(X(t)\big)_{t\in[0,\infty)}$ *is a CTMC on* $\mathbb{N}$, *the so-called* Yule-process, *a pure birth-process. Given the Q-matrix, the backward equation is solved by:*

$$P_{i,j}(t) = \begin{cases} \binom{j-1}{j-i}\big(\exp(\lambda t)-1\big)^{j-i}\exp(-j\lambda t), & \textit{if} \quad i \le j \\ 0, & \textit{else.} \end{cases}$$

1. Give the Q-matrix that belongs to the Yule-process.
2. Show that the backward equation is actually solved by the semi-group defined above.

Exercise 9.II:

In the lecture, the proof of the forward equation (Theorem B.V.6) was only briefly sketched, give a full and in-depth commentated proof.

Exercise 9.III:

Let X be some CTMC with Q-matrix

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 4 & -4 & 0 \\ 2 & 1 & -3 \end{pmatrix}.$$

Compute the probability $p_{23}(t)$ to reach state 3 in time t starting from state 2. You may solve this by using the below list of advice:

step 1: Compute the eigenvalues of the matrix

step 2: The matrix Q can be diagonalized, i.e. there is an invertible U such that:

$$Q = U \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix} U^{-1},$$

where y_1, y_2, y_3 are the eigenvalues of Q. Deduce that $\exp(Q)$ is diagonalizable with the same matrix U.

step 3: Deduce an explicit functional term for $p_{23}(t)$ depending on the entries of U and terms of the form $\alpha_i \exp(\beta_i t), \alpha_i, \beta_i \in \mathbb{R}, i = 1, 2, 3$.

step 4: Now, we know $P(0)$, $P'(0)$ and by the BE we know what $P''(0)$ looks like.

step 5: Denote a system of linear equations and solve it for the parameters α_1, α_2 and α_3.