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SS 2011

Stochastik 2 - Presence-Exercises 4:

Presence-Exercise 4.I:

Gas molecules move about randomly in a box which is divided into two halves symmetrically by a partition. A hole is made in the partition. Suppose there are N molecules in the box of which exactly one passes the partition each time-period. Show that the number of molecules on one side of the partition just after a molecule has passed through the hole evolves as a Markov Chain. What are the transition probabilities? What is the invariant distribution of this chain? Explain in detail the construction of your Markov Chain.

Presence-Exercise 4.II:

Assume $(X_n)_{n\in\mathbb{N}}$ to be Markov Chain on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with transition matrix P on (countable) finite state-space I, then the set of invariant distributions of the stochastic matrix P is *convex*, i.e. for all ν,μ invariant measures on I under $\mathbb{P}$, for all $\lambda \in [0,1]$, the measure $\lambda\nu + (1-\lambda)\mu =: \eta$ is again an invariant measure on I under $\mathbb{P}$.

(Note, that there are two claims to show: First, the property that a convex combination of probability measures is again a probability measure and second the invariance property.)

Presence-Exercise 4.III:

Find an example of

(...) a full-rank $n \times n$-matrix P such that there is exactly one invariant distribution.

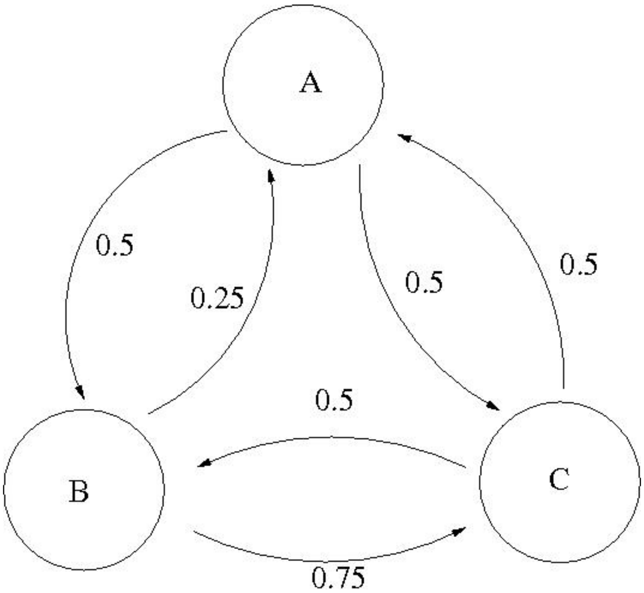
(...) a full-rank $n \times n$-matrix P such that there are infinitely many invariant distributions.

(...) a Markov Chain on a finite countable state space I, such that for some $i \in I$ the probabilities $p_{ij}^{(n)}$ do not converge (in n) for every j.

Presence-Exercise 4.IV:

Remember presence-exercise 1.III (1) observe the distribution of X at multiple points in time with start in $(1,0,0)$ and (2) compute all invariant measures of P. (3) Discribe your observations . The distribution of $X_{10}, X_{100}, X_{1000}$ under $\mathbb{P}_{(1,0,0)}$ is given by

Figure 1:



$\mathbb{P}_{X_{10},(1,0,0)}, \dots$

$$\mathbb{P}_{X_1,(1,0,0)} = \begin{pmatrix} 0 & 0.500 & 0.500 \end{pmatrix}$$
$$\mathbb{P}_{X_2,(1,0,0)} = \begin{pmatrix} 0.375 & 0.25 & 0.375 \end{pmatrix}$$
$$\mathbb{P}_{X_3,(1,0,0)} = \begin{pmatrix} 0.250 & 0.375 & 0.375 \end{pmatrix}$$
$$\mathbb{P}_{X_4,(1,0,0)} = \begin{pmatrix} 0.281 & 0.313 & 0.406 \end{pmatrix}$$
$$\mathbb{P}_{X_5,(1,0,0)} = \begin{pmatrix} 0.281 & 0.344 & 0.375 \end{pmatrix}$$
$$\mathbb{P}_{X_{10},(1,0,0)} = \begin{pmatrix} 0.277 & 0.3333 & 0.3901 \end{pmatrix}$$
$$\mathbb{P}_{X_{10},(1,0,0)} = \begin{pmatrix} 0.277 & 0.3333 & 0.3901 \end{pmatrix}$$
$$\mathbb{P}_{X_{100},(1,0,0)} = \begin{pmatrix} 0.2778 & 0.3333 & 0.3889 \end{pmatrix}$$