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WS 2012/13

Probability Theory III - Exercises 1

Handover date: **Wednesday, Oct 24th, 12:00**

Please put your solutions into the mailbox 161 which belongs to the head of the tutorials, Daniel Altemeier. The mailbox can be found in the copy-room V3-128. Before the insertion of the solution please check that the sheets are ordered correctly and tacked. Write down your name in a legible handwriting on the the first sheet of your solution.

Let $B = (B_s)_{s\geq 0}$ be a continuous Brownian Motion on a filtered probability-space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\geq 0}, \mathbb{P}\big)$.

Exercise 1.I

Prove directly from the definition of the stochastic integral:

(i) $\displaystyle\int_0^t s dB(s) = tB_t - \int_0^t B_s ds$ für alle $t \geq 0$.

Start with a elementary function $f(t,\omega) = \sum_j e_j(\omega)\mathbb{1}_{[t_j,t_{j+1})}(t)$, where $(t_1,\dots,t_n)$ is a partition of $[0,t]$ and e_j always $\mathcal{F}_{t_j}$-measurable for all $j = 1,\dots,n$. Use the following advice:

$$\sum_j \big(s_{j+1}B_{j+1} - s_j B_j\big) = \sum_j s_j\big(B_{j+1} - B_j\big) + \sum_j B_{j+1}\big(s_{j+1} - s_j\big).$$

(ii) $\displaystyle\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$

Exercise 1.II

Consider $f : [0,\infty) \to \mathbb{R}$, i.e. deterministic, with $\int_0^t f^2(s)ds < \infty$ for all $t \in [0,\infty)$. Show that the stochastic integral $\int_0^t f(s)dB(s)$ is always a Gaussian Process.

Exercise 1.III

A result of Kiyoshi Itô yields the below formula for an iteration of stochastic integrals: For $n \in \mathbb{N}$

$$n! \int_{u_{n-1} \leq u_n \leq t} \int_{u_n \leq u_{n-1} \leq u_{n-2}} \cdots \int_{u_1 \leq u_2 \leq u_3} \int_{0 \leq u_1 \leq u_2} dB_{u_1} dB_{u_2} \ldots dB_{u_n} = t^{\frac{n}{2}} h_n\left(\frac{B_t}{\sqrt{t}}\right) \tag{1}$$

where h_n is the *Hermite polynomial of degree* n defined by

$$h_n(x) = (-1)^n \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2}\right); \quad n \in \mathbb{N}.$$

(i) Show that all stochastic integrals in (1) are well-defined.

(ii) Show (1) for $n = 1, 2, 3$. You may use the results of exercise I.1 and from the lecture.

(iii) Prove with the help of (ii) that the process defined by

$$N_t := B_t^3 - 3tB_t \text{ für } t \geq 0$$

is a $\{\mathcal{F}_s\}_{s \geq 0}$-martingale.

Exercise I.IV (*Preparation for a mini-talk*)

Prepare a short talk on the below stated topic for the tutorial on tuesday, october, 30th. For that you should present the material in your own words. The form of the presentation is left to you.

Explain the construction of the stochastic integral.