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WS 2012/13

Probability Theory III - Homework 10

Due date: **Wednesday, January 9, 12:00 h**

Solutions to the assigned homework problems must be deposited in Daniel Altemeier's drop box 161 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Unless stated otherwise let $B = (B_u)_{u \geq 0}$ be a continuous standard Brownian Motion on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_u\}_{u \geq 0}, \mathbb{P}\big)$.

Exercise 10.I

Let $\phi \in C^2$, $\big(X_t\big)_{t \geq 0}$ an Itô diffusion with generator A_X and $Y_t := \phi(X_t), t \geq 0$. Prove that $\big(Y_t\big)_{t \geq 0}$ coincides in law with an Itô diffusion $\big(Z_t\big)_{t \geq 0}$ with generator A_Z if and only if

$$A_X\big(f \circ \phi\big) = \big(A_Z f\big) \circ \phi$$

for all polynomials $f(x_1, \ldots, x_n) = \sum_{i=1}^n a_i x_i + \sum_{i,j=1}^n c_{ij} x_i x_j$ where a_i, c_{ij} are real coefficients for all $i, j \in \mathbb{N}$.

Exercise 10.II

In connection with the deduction of the Black–Scholes formula for the price of an option, the following partial differential equation appears:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\rho u + \alpha x \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\beta^2 x^2 \dfrac{\partial^2 u}{\partial x^2} & \text{for } t > 0, x \in \mathbb{R} \\ \\ u(0, x) = (x - K)^+ & \text{for } x \in \mathbb{R}, \end{cases}$$

where for simplicity $\rho > 0, K > 0$ positive constants and $\alpha \geq 0$ and $\beta \geq 0$ nonnegative constants. Let $(x - K)^+ = \max\{x - K, 0\}$. Use the Feynman–Kac formula to prove that the solution u of this equation is given by

$$u(t, x) = \frac{\exp\big(-\rho t\big)}{\sqrt{2\pi t}} \int_{\mathbb{R}} \left(x \exp\left(\left(\alpha - \frac{1}{2}\beta^2 \right) t + \beta y \right) - K \right)^+ \exp\left(-\frac{y^2}{2t} \right) dy; \quad t > 0.$$

(This expression can be simplified further.)

Exercise 10.III

a) Define $\alpha(t) = \frac{1}{2}\ln\left(1 + \frac{2}{3}t^3\right)$. If $(B_t)_{t\geq 0}$ is a Brownian motion prove that there is another Brownian motion $(\tilde{B}_t)_{t\geq 0}$ such that

$$\int_0^{\alpha_t} \exp(s)dB_s = \int_0^t sd\tilde{B}_s.$$

b) Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}$. Show that $X_t = B_t^2, t \geq 0$, is a weak solution of the stochastic differential equation

$$dX_t = dt + 2\sqrt{|X_t|}d\tilde{B}_t,$$

where $(\tilde{B}_t)_{t\geq 0}$ is a Brownian motion as well.

Exercise 10.IV (***Preparation for a mini-presentation on Tuesday, January 15***)

Prepare a short talk on the topic given below. You should present the material in your own words without relying on your notes. The presentation should be made using the blackboard, writing down keywords and the essential formulae.

Explain the martingale problem and deduce the relationship to weak solutions of SDEs.