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WS 2012/13

Probability Theory III - Homework 11

Due date: **Wednesday, January 16, 12:00 h**

Solutions to the assigned homework problems must be deposited in Daniel Altemeier's drop box 161 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

- Unless stated otherwise let $B = (B_u)_{u\geq 0}$ be a continuous standard Brownian Motion on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_u\}_{u\geq 0}, \mathbb{P}\big)$.
- Denote the equivalence of two probability measures $\mathbb{P}$ and Q on $(\Omega, \mathcal{F})$ by $\mathbb{P} \sim Q$, which means $\mathbb{P} \ll Q$ and $Q \ll \mathbb{P}$ where $Q \ll \mathbb{P}$ means *absolute continuity of Q w.r.t. $\mathbb{P}$ (i.e. for all $A \in \mathcal{F} : \mathbb{P}[A] = 0 \Rightarrow Q[A] = 0$).*

Exercise 11.I

Let B be 1-dimensional.

a) Let $Y_t = t + B_t, t \geq 0$. For each $T > 0$ find a probability measure Q_T on $\mathcal{F}_T$ such that $Q_T \sim \mathbb{P}$ and $(Y_t)_{t\in[0,T]}$ is a Brownian motion w.r.t. Q_T. Prove that there is a probability measure Q on $\mathcal{F}_\infty$ such that

$$Q\big|_{\mathcal{F}_T} = Q_T \text{ for all } T > 0.$$

Hint: You might find the remarks on the Girsanov theorem from the lecture useful.

b) Show that

$$\mathbb{P}\left[\lim_{t\to\infty} Y_t = \infty\right] = 1$$

while

$$Q\left[\lim_{t\to\infty} Y_t = \infty\right] = 0.$$

Why does this not contradict the Girsanov theorem?

Exercise 11.II

Let $B_t = \big(B_t^{(1)}, B_t^{(2)}\big), t \geq 0$, and $T > 0$ fixed. Consider $Y = \big(Y_t\big)_{t\geq 0}$, defined by

$$dY_t = \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} dB_t^{(1)} \\ dB_t^{(2)} \end{pmatrix} \quad \text{for } t \in [0,T] \quad \text{and} \quad Y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Find a probability measure Q on $\mathcal{F}_T$ such that $Q \sim \mathbb{P}$ and such that for $t \geq 0$:

$$dY_t = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} d\tilde{B}_t^{(1)} \\ d\tilde{B}_t^{(2)} \end{pmatrix} \quad \text{where} \quad \tilde{B}_t := \begin{pmatrix} -3t \\ t \end{pmatrix} + \begin{pmatrix} B_t^{(1)} \\ B_t^{(2)} \end{pmatrix},$$

and $\big(\tilde{B}_t\big)_{t\in[0,T]}$ is a Brownian motion w.r.t. Q.

Exercise 11.III

Let $b : \mathbb{R} \to \mathbb{R}$ be a Lipschitz-continuous function and define $X_t = X_t^x \in \mathbb{R}$ for $t \geq 0$ by

$$\begin{cases} dX_t = b(X_t)dt + dB_t & \text{for } t > 0, \\ X_0 = x. \end{cases}$$

a) Use the Girsanov theorem to prove that for all $M < \infty$, $x \in \mathbb{R}$ and $t > 0$ we have

$$\mathbb{P}\big[X_t^x \geq M\big] > 0.$$

b) Choose $b(x) = -r$ for $r > 0$ constant. Prove that for all $x \in \mathbb{R}$,

$$X_t^x \xrightarrow[\mathbb{P}\text{-a.s.}]{t\to\infty} -\infty.$$

Exercise 11.IV (*Preparation for a mini-presentation on Tuesday, January 22*)

Prepare a short talk on the topic given below. You should present the material in your own words without relying on your notes. The presentation should be made using the blackboard, writing down keywords and the essential formulae.

The Girsanov theorem.