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WS 2012/13

Probability Theory III - Homework 2

Due date: **Wednesday, October 31, 12:00 h**

Solutions to the assigned homework problems must be deposited in Daniel Altemeier's drop box 161 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Let $B = (B_s)_{s\geq 0}$ be a continuous Brownian Motion on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\geq 0}, \mathbb{P}\big)$.

Exercise 2.I:

Consider $f \in \mathcal{V}$. We have shown that

$$M_t := \int_0^t f(s,\omega) dW(s), \quad t \geq 0$$

defines a continuous $\mathcal{L}_2$-martingale w.r.t. $\{\mathcal{F}_s\}_{s\geq 0}$. Prove that the process $\big(N_t\big)_{t\geq 0}$ defined by

$$N_t = M_t^2 - \int_0^t f^2(s,\omega) dW(s), \quad t \geq 0.$$

defines a continuous martingale w.r.t. $\{\mathcal{F}_s\}_{s\geq 0}$ with $N_0 = 0$.

Hints: First show that the Itô isometry is also valid for stochastic integrals with upper integration limit given by a bounded stopping time. Then apply a suitable criterion for a stochastic process to be a martingale.

Exercise 2.II

a) Suppose $f \in \mathcal{V}(0,T)$ and that $t \mapsto f(t,\omega)$ is continuous for a.a. ω. Compute the Stratonovich integral

$$\int_0^t B_u \circ dB_u \quad \text{for } t \in [0,T]$$

directly from the definition.
Hint: You might find example 1.9 helpful.

b) For $f \in \mathcal{V}(0,T)$ assume $K < \infty$ constant and $\varepsilon > 0$ such that

$$\mathbb{E}\Big[\big|f(s,\cdot) - f(t,\cdot)\big|^2\Big] \leq K|s-t|^{1+\varepsilon}; \quad \text{for } 0 \leq s,t \leq T.$$

Prove that

$$\int_0^T f(t,\omega)dB_t = \lim_{\Delta t_j \to 0} \sum_j f(t_j^\star,\omega)\Delta B_j \quad (\text{ limit in } \mathcal{L}^1(\mathbb{P}))$$

for any choice of $t_j^\star \in [t_j, t_{j+1}]$. In particular this implies that Itô integral and Stratonovich integral coincide in this case.
Hint: Consider $\mathbb{E}\big[\big|\sum_j f(t_j,\omega)\Delta B_j - \sum_j f(t_j^\star,\omega)\Delta B_j\big|\big]$.

Exercise 2.III

Denote by $\circ dB_t$ the Stratonovich differentials.

(i) Use equation (S$\leftrightarrow$I) from the lecture to transform the following Stratonovich differential equations into Itô differential equation:

a) $dX_t = \gamma X_t dt + \alpha X_t \circ dB_t$

b) $dX_t = \sin(X_t)\cos(X_t)dt + \big(t^2 + \cos(X_t)\big) \circ dB_t$

(ii) Transform the following Itô differential equations into Stratonovich differential equations:

a) $dX_t = \gamma X_t dt + \alpha X_t dB_t$

b) $dX_t = 2\exp\big(-X_t\big)dt + X_t^2 dB_t$

Exercise 2.IV (*Preparation for a mini-presentation on Tuesday, November 6*)

Prepare a short talk on the topic given below. You should present the material in your own words without relying on your notes. The presentation should be made using the blackboard, writing down keywords and the essential formulae.

Continuity and the martingale property for the Itô-Integral. Consider only integrands that fulfill (i),(ii),(iii).