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WS 2012/13

Probability Theory III - Homework 3

Due date: **Wednesday, November 7, 12:00 h**

Solutions to the assigned homework problems must be deposited in Daniel Altemeier's drop box 161 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Unless stated otherwise let $B = (B_u)_{u\geq 0}$ be a continuous Brownian Motion on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_u\}_{u\geq 0}, \mathbb{P}\big)$.

Exercise 3.I (Application of Itô's formula)

(i) Use Itô's formula to write the following stochastic processes $\big(Y_u\big)_{u\geq 0}$ in the *standard form* $dY_u(\omega) = \mu(u,\omega)du + \sigma(u,\omega)dB_u(\omega)$ for suitable choices of $\mu : [0,\infty)\times\Omega \to \mathbb{R}^n$, $\sigma : [0,\infty)\times\Omega \to \mathbb{R}^{m\times n}$ and dimensions m and n. For $B = (B_u)_{u\geq 0}$, a 1-dimensional Brownian Motion, consider for $u \geq 0$

(a) $Y_u = B_u^2$

(b) $Y_u = 2 + u + \exp\big(B_u\big)$

for $B = \big(B_u^{(1)}, B_u^{(2)}\big)_{u\geq 0}$, 2-dimensional Brownian Motion, consider for $u \geq 0$

(c) $Y_u = \big(B_u^{(1)}\big)^2 + \big(B_u^{(2)}\big)^2$

For $B = \big(B_u^{(1)}, B_u^{(2)}, B_u^{(3)}\big)_{u\geq 0}$, a 3-dimensional Brownian Motion, consider for $u \geq 0$

(d) $Y_u = \Big(B_u^{(1)} + B_u^{(2)} + B_u^{(3)}, \big(B_u^{(2)}\big)^2 - B_u^{(1)}B_u^{(3)}\Big)$

(ii) Use Itô's formula to prove that the following stochastic processes are $\big\{\mathcal{F}_u\big\}_{u\in[0,\infty)}$-martingales:

(a) $X_u = \exp\left(\frac{1}{2}u\right)\cos\big(B_u\big)$

(b) $X_u = \big(B_u + u\big)\exp\left(-B_u - \frac{1}{2}u\right)$.

(iii) In exercise 1.I.(ii) we have seen that for $u \geq 0$,

$$\int_0^t B_u^2 dB_u = \frac{1}{3}B_t^3 - \int_0^t B_u du,$$

Give an alternative proof of this result by using Itô's formula.

Exercise 3.II (Exponential martingales)

For given $n \in \mathbb{N}$, suppose $\sigma(t,\omega) = \big(\sigma_1(t,\omega), \dots, \sigma_n(t,\omega)\big) : [0,\infty) \times \Omega \to \mathbb{R}^n$ with $\sigma_k(\cdot,\cdot) \in \mathcal{V} = \mathcal{V}(0,\infty)$ for $k = 1, \dots n$. Define the process $Z = \big(Z_t\big)_{t \geq 0}$ by

$$Z_t = \exp\left(\int_0^t \sigma(u,\omega) d\mathbb{B}_u - \frac{1}{2}\int_0^t \sigma^2(u,\omega)du\right) \quad \text{for } u \geq 0,$$

where $\mathbb{B} = \big(\mathbb{B}_u\big)_{u \geq 0}$ is an n-dimensional Brownian Motion.

(a) Use Itô's formula to prove that $dZ_t = Z_t \sigma(t,\omega) dB_t$.

(b) Deduce that Z_t is a martingale for all $t \geq 0$, provided that

$$\big(Z_t \sigma(t,\omega)\big)_{t \geq 0, \omega \in \Omega} \in \mathcal{V} \quad \text{for } k = 1, \dots, n.$$

Remark: A sufficient condition that $\big(Z_t\big)_{t \geq 0}$ is a martingale is the *Novikov* condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^\infty \sigma^2(u,\omega)du\right)\right] < \infty.$$

Exercise 3.III (First approach to *Girsanov's theorem*)

Let $dX_t = \mu(t,\omega) + dB_t$, where we assume $u \in \mathcal{V}$ to be bounded. We know that $X = \big(X_t\big)_{t \geq 0}$ is in general not a $\big\{\mathcal{F}_t\big\}_{t \geq 0}$-martingale, unless $\mu = 0$. However, we can construct an $\big\{\mathcal{F}_t\big\}_{t \geq 0}$-martingale from X by multiplying by a suitable exponential martingale. Therefore define

$$Y_t = X_t M_t \quad \text{for } t \geq 0,$$

where

$$M_t = \exp\left(-\int_0^t \mu(u,\omega) dB_u - \frac{1}{2}\int_0^t \mu^2(u,\omega)du\right).$$

Use Itô's formula to prove that Y_t is an $\mathcal{F}_t$-martingale.

Remark: The result of Exercise 3.III can be interpreted as a special case of the important *Girsanov theorem* if one understands $\big(X_t\big)_{t \geq 0}$ as a martingale w.r.t. the probability measure $\mathbb{Q}$, defined by

$$d\mathbb{Q}\big|_{\mathcal{F}_t} = M_t d\mathbb{P}\big|_{\mathcal{F}_t} \qquad \text{for } t \geq 0,$$

which implicitly means that the exponential martingale $(M_t)_{t \geq 0}$ is the process of *Radon–Nikodym* derivatives of $\mathbb{Q}\big|_{\mathcal{F}_t}$ w.r.t. $\mathbb{P}\big|_{\mathcal{F}_t}$ for all $t \geq 0$.

Exercise **3.IV** (*Preparation for a mini-presentation on Tuesday, November 13*)

Prepare a short talk on the topic given below. You should present the material in your own words without relying on your notes. The presentation should be made using the blackboard, writing down keywords and the essential formulae.

The Itô formula.