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WS 2012/13

# Probability Theory III - Homework 4

Due date: **Wednesday, November 14, 12:00 h**

**Solutions to the assigned homework problems must be deposited in Daniel Altemeier's drop box 161 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.**

Unless stated otherwise let $B = (B_u)_{u \geq 0}$ be a standard Brownian Motion on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_u\}_{u \geq 0}, \mathbb{P}\big)$.

### Exercise 4.I

a) Let $\mathbb{B} = \big(\mathbb{B}_t\big)_{t \geq 0}$ be a $d$-dimensional Brownian motion and let $f : \mathbb{R}^d \to \mathbb{R}$ be twice continuously differentiable ($f \in C^2$). Use Itô's formula to prove that

$$f(\mathbb{B}_t) = f(\mathbb{B}_0) + \int_0^t \nabla f(\mathbb{B}_s) d\mathbb{B}_s + \frac{1}{2} \int_0^t \Delta f(\mathbb{B}_s) ds,$$

where $\Delta = \sum_{i=1}^d \frac{\partial^2 g^2}{\partial x_i^2}$ is the Laplace operator.

b) Assume that $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable everywhere ($g \in C^1$) and even twice continuously differentiable outside finitely many points $z_1, \dots, z_n$. Assume that $|g''(x)| \leq M$ for some finite $M \geq 0$ and for all $x \notin \{z_1, \dots, z_n\}$. For $d = 1$ show that the result of part a) is still valid.

*Hints:* Choose a sequence $\big(g_k\big)_{k \in \mathbb{N}}$ of functions such that $g_k \in C^2$ $\forall\, k$, $g_k \to g$ uniformly, $g_k' \to g'$ uniformly, $|g_k''| \leq M$ and $g_k'' \to g''$ outside $\{z_1, \dots, z_n\}$. Apply a) to $g_k$ and study the limit as $n \to \infty$.

**Exercise 4.II** (Tanaka's formula and local time)

We would like to apply Itô's formula to $g(B_t) = \big|B_t\big|$ where $B = \big(B_t\big)_{t\geq 0}$ is a 1-dimensional Brownian motion. As $g \notin C^2$ (of course, in $x = 0$, $g$ is not differentiable), we approximate $g$ by functions $g_\varepsilon \in C^2$, given by

$$g_\varepsilon(x) = \begin{cases} |x| & \text{if } |x| \geq \varepsilon \\ \\ \frac{1}{2}\left(\varepsilon + \frac{x^2}{\varepsilon}\right) & \text{if } |x| < \varepsilon \end{cases}$$

a) Apply Exercise 4.I b) to show that

$$g_\varepsilon\big(B_t\big) = g_\varepsilon\big(0\big) + \int_0^t g_\varepsilon'\big(B_s\big)dB_s + \frac{1}{2\varepsilon}\lambda\Big(\Big\{s \in [0,t] : B_s \in (-\varepsilon, \varepsilon)\Big\}\Big),$$

where $\lambda\{A\}$ denotes the Lebesgue measure of set $A$.

b) Prove that

$$\int_0^t g_\varepsilon'\big(B_s\big)\mathbb{1}_{\{B_s\in(-\varepsilon,\varepsilon)\}}dB_s = \int_0^t \frac{B_s}{\varepsilon}\mathbb{1}_{\{B_s\in(-\varepsilon,\varepsilon)\}}dB_s \xrightarrow[\varepsilon\to 0]{\text{in } \mathcal{L}^2} 0.$$

*Hint:* Apply the Itô isometry to $\mathbb{E}\big[\big(\int_0^t B_s/\varepsilon\mathbb{1}_{\{\ldots\}}dB_s\big)^2\big]$.

c) By letting $\varepsilon \to 0$ prove that

$$|B_t| = |B_0| + \int_0^t \text{sign}\big(B_s\big)dB_s + L_t, \qquad (\star)$$

where $L_t : \Omega \to \mathbb{R}$, defined by

$$L_t = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\lambda\big\{s \in [0,t]; B_s \in (-\varepsilon,\varepsilon)\big\} \quad (\text{limit in } \mathcal{L}^2(\mathbb{P})).$$

and

$$\text{sign}(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ \\ 1 & \text{if } x < 0 \end{cases}.$$

$L_t$ is called the *local time* for Brownian motion at 0 and $(\star)$ is called the *Tanaka formula* (for Brownian motion).

**Exercise 4.III** (Application of Itô's and Martingale representation theorem)

a) In each of the cases below find the admissible stochastic process $\big(f(s,\cdot)\big)_{s\geq 0}$, such that

$$X = \mathbb{E}[X] + \int_0^t f(s,\cdot)dB_s.$$

(i) $X = \int_0^t B_s ds$

(ii) $X = \exp\big(B_t\big)$

b) Let $T > 0$. We know that for every $\mathcal{F}_T$-measurable RV $Y$ with $\mathbb{E}[|Y|^2] < \infty$, we can define a martingale $(M_t)_{t\geq 0}$ by

$$M_t := \mathbb{E}[Y|\mathcal{F}_t] \quad \text{for } 0 \leq t \leq T.$$

b.1) Show that $\mathbb{E}[M_t^2] < \infty$ for all $t \in [0,T]$.

b.2) According to the martingale representation theorem there is a unique admissible process $f = \big(f(s,\cdot)\big)_{s\in[0,T]}$ such that

$$M_t = \mathbb{E}[M_0] + \int_0^t f(s,\cdot)dB_s \quad \text{for } t \in [0,T].$$

Find that $f$ in the following cases:

(i) $Y = B_T^2$

(ii) $Y = \exp\big(\lambda B_T\big)$,

where $\lambda \in \mathbb{R}$ is a constant.

**Exercise 4.IV** (***Preparation for a mini-presentation on Tuesday, November 20***)

**Prepare a short talk on the topic given below. You should present the material in your own words without relying on your notes. The presentation should be made using the blackboard, writing down keywords and the essential formulae.**

The Martingale representation theorem.