Prof. Dr. Barbara Gentz
Daniel Altemeier
Faculty of Mathematics
University of Bielefeld

WS 2012/13

Probability Theory III - Homework 5

Due date: **Wednesday, November 21, 12:00 h**

Solutions to the assigned homework problems must be deposited in Daniel Altemeier's drop box 161 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Unless stated otherwise let $B = (B_u)_{u\geq 0}$ be a standard Brownian Motion on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_u\}_{u\geq 0}, \mathbb{P}\big)$.

Exercise 5.I

a) Verify that the given processes solve the given corresponding stochastic differential equations:

(i) $\big(X_1(t), X_2(t)\big) = \big(t, \exp(t)B_t\big)$ solves

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} 1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} 0 \\ \exp(X_1) \end{pmatrix} dB_t.$$

(ii) $\big(X_1(t), X_2(t)\big) = \big(\cosh(B_t), \sinh(B_t)\big)$ solves

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} dt + \begin{pmatrix} X_2 \\ X_1 \end{pmatrix} dB_t.$$

b) Solve the following stochastic differential equations

(i) $$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & X_1 \end{pmatrix}\begin{pmatrix} dB_1 \\ dB_2 \end{pmatrix}$$

(ii) $dX_t = X_t dt + \exp(-t) dB_t$.

(iii) $dY_t = r dt + \alpha Y_t dB_t$,

where r, α are real constants, $B_t \in \mathbb{R}$.

Hint: Multiply the equation by the 'integrating factor'

$$F_t = \exp\Big(- \alpha B_t + \frac{1}{2}\alpha^2 t\Big).$$

Exercise 5.II (The Brownian bridge)

For fixed $a, b \in \mathbb{R}$ consider following 1-dimensional equation

$$dY_t = \frac{b - Y_t}{1-t} dt + dB_t, \quad 0 \le t < 1, \quad Y_0 = a,$$

Verify that

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s} dB_s, \quad 0 \le t < 1$$

solves the equation and prove that $\lim_{t\to 1} Y_t = b$ a.s. The process Y_t is called the *Brownian bridge* (from a to b).

Hints: Use the Borel–Cantelli Lemma for the events

$$A_n = \left\{ \omega \in \Omega : \sup_{t \in \left[1-2^{-n}, 1-2^{-n-1}\right]} (1-t) \left| \int_0^t \frac{1}{1-s} dB_s \right| > 2^{-\frac{n}{4}} \right\}, \quad n \in \mathbb{N}.$$

to show that for a.a. $\omega \in \Omega$ there is $N(\omega) < \infty$ such that

$$n > N(\omega) \Rightarrow w \notin A_n.$$

Exercise 5.III

The technique used in Exercise 5.I b) (iii) can be applied to more general nonlinear stochastic differential equations of the form

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t, \qquad X_0 = x \tag{$\star$}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R} \to \mathbb{R}$ are given continuous, deterministic functions.

a) Define the 'integrating factor'

$$F_t = \exp\left(-\int_0^t c(s) dB_s + \frac{1}{2}\int_0^t c^2(s) ds \right).$$

Show that ($\star$) can be written

$$d\big(F_t X_t\big) = F_t f(t, X_t) dt. \tag{$\star\star$}$$

b) Now define $Y_t = F_t X_t$ so that $X_t = F_t^{-1} Y_t$. Deduce that equation ($\star\star$) is transformed into

$$\frac{dY_t}{dt} = F_t f\big(t, F_t^{-1} Y_t\big); \quad Y_0 = x. \tag{$\star\star\star$}$$

Note that this is just a deterministic differential equation in the function $t \mapsto Y_t$ for each $\omega \in \Omega$. We can therefore solve ($\star\star\star$) for given ω and then obtain X_t.

c) Apply this method to solve the stochastic differential equation

$$dX_t = \frac{1}{X_t}dt + \alpha X_t dB_t; \quad X_0 = x > 0$$

where α is constant.

d) Apply the method to study the solutions of the stochastic differential equation

$$dX_t = X_t^\gamma dt + \alpha X_t dB_t; \quad X_0 = x > 0$$

where α and γ are constants. For what values of γ do we get explosion?

Exercise 5.IV (***Preparation for a mini-presentation on Tuesday, November 27***)

Prepare a short talk on the topic given below. You should present the material in your own words without relying on your notes. The presentation should be made using the blackboard, writing down keywords and the essential formulae.

For stochastic differential equations we know the concept of *strong* and *weak* solutions. Explain these two concepts. Give one example with a strong solution and one example with a weak solution that is not a strong solution.