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WS 2012/13

Probability Theory III - Homework 6

Due date: **Wednesday, November 28, 12:00 h**

Solutions to the assigned homework problems must be deposited in Daniel Altemeier's drop box 161 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Unless stated otherwise let $B = (B_u)_{u \geq 0}$ be a continuous standard Brownian Motion on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_u\}_{u \geq 0}, \mathbb{P}\big)$.

Exercise 6.I

Find the generator of the following Itô diffusions:

a) $dX_t = \mu X_t + \lambda dB_t$ (Geometric Brownian motion)

b) $dY_t = \begin{pmatrix} dt \\ dX_t \end{pmatrix}$ where $(X_t)_{t \geq 0}$ is the Ornstein-Uhlenbeck process given by $dX_t = \mu X_t dt + \sigma dB_t$, $\mu, \sigma \in \mathbb{R}$.

c) $Y = (Y_1, Y_2)$ where $\begin{cases} dY_1(t) = \exp\big(Y_1(t) - Y_2(t)\big)dt + dB_1(t) \\ dY_2(t) = dB_2(t) \end{cases}$,

where $\big(B_1(\cdot), B_2(\cdot)\big)$ is a 2-dimensional Brownian motion.

Exercise 6.II

Find an Itô diffusion (i.e. give a stochastic differential equation for it) whose generator is the following:

a) $Af(x) = \alpha f'(x) + \beta f''(x)$; $f \in C_0^2(\mathbb{R})$, $\alpha, \beta \in \mathbb{R}$,

b) $Af(t,x) = \dfrac{\partial f}{\partial t} + x\dfrac{\partial f}{\partial x} + \dfrac{1}{2}x^2\dfrac{\partial^2 f}{\partial x^2}$; $f \in C_0^2(\mathbb{R}^2)$,

c) $Af(x_1,x_2) = 2x_2\dfrac{\partial f}{\partial x_1} + \ln\big(1 + x_1^2 + x_2^2\big)\dfrac{\partial f}{\partial x_2} + \dfrac{1}{2}(1 + x_1^2)\dfrac{\partial^2 f}{\partial x_1^2} + x_1\dfrac{\partial^2 f}{\partial x_1 \partial x_2} + \dfrac{1}{2}\dfrac{\partial^2 f}{\partial x_2^2}$;
for all $f \in C_0^2\big(\mathbb{R}^2\big)$. Give at least two different choices for the pair of drift and diffusion coefficient if possible. How many different choices are there?

Exercise 6.III

For $x \in \mathbb{R}$, define

$$X_t = X_t^x = x \exp\left(ct + \alpha B_t\right), \quad t \geq 0,$$

where $c, \alpha \in \mathbb{R}$. Prove directly from the definition that X_t is a Markov process.

Exercise 6.IV (***Preparation for a mini-presentation on Tuesday, December 4***)

Prepare a short talk on the topic given below. You should present the material in your own words without relying on your notes. The presentation should be made using the blackboard, writing down keywords and the essential formulae.

Strong Markov property for diffusions.