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WS 2012/13

Probability Theory III - Homework 7

Due date: **Wednesday, December 5, 12:00 h**

Solutions to the assigned homework problems must be deposited in Daniel Altemeier's drop box 161 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Unless stated otherwise let $B = (B_u)_{u\geq 0}$ be a continuous standard Brownian Motion on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_u\}_{u\geq 0}, \mathbb{P}\big)$.

Exercise 7.I

Let $\big(B_t^x\big)_{t\geq 0}$ be a one-dimensional Brownian motion starting at $x > 0$. Consider the first hitting time of the origin:

$$\tau = \inf\big\{t > 0; B_t^x = 0\big\}.$$

a) Prove that $\tau < \infty$ almost surely for all $x > 0$.

b) Prove that $\mathbb{E}^x\big[\tau\big] = \infty$.

Exercise 7.II

Let $\big(X_t\big)_{t\geq 0}$ be an Itô diffusion in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a function such that

$$\mathbb{E}\left[\int_0^\infty \big|f(X_t)\big| dt\right] < \infty \qquad \text{for all } x \in \mathbb{R}^n.$$

Let τ be a stopping time with $\mathbb{P}[\tau < \infty] = 1$. Use the strong Markov property to prove that

$$\mathbb{E}^x\left[\int_\tau^\infty f(X_t) dt\right] = \mathbb{E}^x\big[g(X_\tau)\big],$$

where

$$g(y) = \mathbb{E}^y\left[\int_0^\infty f\big(X_t\big) dt\right].$$

Exercise 7.III

Let X_t be a geometric Brownian motion, i.e.

$$dX_t = rX_t dt + \alpha X_t dB_t, \quad X_0 = x > 0$$

where $r, \alpha \in \mathbb{R}$ are constants and $(B_t)_{t \geq 0}$ is a one-dimensional Brownian motion.

a) Find the generator A of X_t and compute $Af(x)$ for $f(x) = x^\gamma$, $x > 0, \gamma$ constant.

b) From the lecture we know that if $r < \alpha^2/2$ then $X_t \to 0$ as $t \to \infty$ a.s. But what is the probability p that X_t, when starting in $x < R$, ever reaches the value R? Use Dynkin's formula with a suitable modification of $f(x) = x^{\gamma_1}$, $\gamma_1 = 1 - \frac{2r}{\alpha^2}$, to prove that

$$p = \left(\frac{x}{R}\right)^{\gamma_1}.$$

c) If $r > \alpha^2/2$, then $X_t \to \infty$ as $t \to \infty$, a.s. Define

$$\tau = \inf\{t > 0 | X_t \geq R\}.$$

Use Dynkin's formula with a suitable modification of $f(x) = \ln(x), x > 0$, to prove that

$$\mathbb{E}^x[\tau] = \frac{\ln\left(\frac{R}{x}\right)}{r - \frac{1}{2}\alpha^2}.$$

Hints: First consider first exit times from (ρ, R), $\rho > 0$, and then let $\rho \to 0$. Derive estimates for

$$\Big(1 - \mathbb{P}^x\big[X_t \text{ reaches the value } R \text{ before } \rho\big]\Big) \ln(\rho).$$

from the computations in a) and b).

Exercise 7.IV (*Preparation for a mini-presentation on Tuesday, December 11*)

Prepare a short talk on the topic given below. You should present the material in your own words without relying on your notes. The presentation should be made using the blackboard, writing down keywords and the essential formulae.

Dynkin's formula and recurrence for a Brownian motion in $\mathbb{R}^d, d \leq 2$, respectively transience for a Brownian motion in $\mathbb{R}^d, d > 2$.