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WS 2012/13

Probability Theory III - Homework 8

Due date: **Wednesday, December 12, 12:00 h**

Solutions to the assigned homework problems must be deposited in Daniel Altemeier's drop box 161 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Unless stated otherwise let $B = (B_u)_{u\geq 0}$ be a continuous standard Brownian Motion on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_u\}_{u\geq 0}, \mathbb{P}\big)$.

Exercise 8.I

Let the functions b, σ satisfy the *linear growth condition*

$$\big|b(t,x)\big| + \big|\sigma(t,x)\big| \leq C\big(1+|x|\big) \quad \text{for } x \in \mathbb{R}^n,\ t \geq 0,$$

where the constant $C \in \mathbb{R}$ is independent of t. Let $(X_t)_{t\geq 0}$ be a solution of

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 \in \mathcal{L}^2.$$

Show that

$$\mathbb{E}\big[|X_t^2|\big] \leq \Big(1 + \mathbb{E}\big[|X_0|^2\big]\Big) \exp\big(Kt\big) - 1$$

for some constant K independent of t.

Hint: Dynkin's formula.

Exercise 8.II

Let Δ denote the Laplace operator on $\mathbb{R}^n$.

a) Write down (in term of a Brownian motion) a bounded solution g of the Cauchy problem

$$\begin{cases} \frac{\partial g(t,x)}{\partial t} - \frac{1}{2}\Delta_x g(t,x) = 0 & \text{for } t > 0, x \in \mathbb{R}^n \\ g(0,x) = \varphi(x). \end{cases}$$

where $\varphi \in C_0^2$ is given. (From general theory it is known that the solution is unique.)

b) Let $\psi \in C_b\big(\mathbb{R}^n\big)$ and $\alpha > 0$. Find a bounded solution u of the equation

$$\Big(\alpha - \frac{1}{2}\Delta\Big) u = \psi \quad \text{in } \mathbb{R}^n.$$

Prove that the solution is unique.

Exercise 8.II

Let $\mathbb{X} = \big(X_t\big)_{t\geq 0}$ be a one-dimensional Itô diffusion with characteristic operator $\mathcal{A}$. Assume that $\mathbb{X}$ solves the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x.$$

Let $f \in C^2(\mathbb{R})$ be a solution to the ODE

$$\mathcal{A}f(x) = b(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x) = 0, \quad x \in \mathbb{R}.$$

Let $(a,b) \subset \mathbb{R}$ be an open interval such that $x \in (a,b)$ and set

$$\tau = \inf\{t > 0 : X_t \notin (a,b)\}.$$

Assume that $\tau < \infty$ holds $\mathbb{P}^x$-almost surely and define

$$p = \mathbb{P}^x\big[X_\tau = b\big].$$

a) Use Dynkin's formula to prove that

$$p = \frac{f(x) - f(a)}{f(b) - f(a)}.$$

In other words, the *harmonic measure* $\mu^x_{(a,b)}$ of X on $\partial(a,b) = \{a,b\}$ is given by

$$\mu^x_{(a,b)}(b) = \frac{f(x) - f(a)}{f(b) - f(a)}, \quad \mu^x_{(a,b)}(a) = \frac{f(b) - f(x)}{f(b) - f(a)}.$$

b) Now specialize to $b(\cdot) = 0$ and $\sigma(\cdot) = 1$, hence, $X_t = x + B_t, t \geq 0$. Prove that

$$p = \frac{x - a}{b - a}.$$

c) Find p if $X_t = x + \mu t + \sigma B_t, t \geq 0$, where μ and σ are nonzero constants

Exercise 8.IV (*Preparation for a mini-presentation on Tuesday, December 18*)

Prepare a short talk on the topic given below. You should present the material in your own words without relying on your notes. The presentation should be made using the blackboard, writing down keywords and the essential formulae.

Derive the Feynman–Kac formula and discuss killing of a diffusion.