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WS 2012/13

Probability Theory III - Homework 9

Due date: **Wednesday, December 19, 12:00 h**

Solutions to the assigned homework problems must be deposited in Daniel Altemeier's drop box 161 located in V3-128 no later than 12:00 h on the due date. Homework solutions must be completely legible, on A4 paper, in the correct order and stapled, with your name neatly written on the first page.

Unless stated otherwise let $B = (B_u)_{u \geq 0}$ be a continuous standard Brownian Motion on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_u\}_{u \geq 0}, \mathbb{P}\big)$.

Exercise 9.I

Let $\big(B_t\big)_{t \geq 0}$ be an n-dimensional Brownian motion. Consider $F \in \mathcal{B}\big(\mathbb{R}^n\big)$ a Borel set in $\mathbb{R}^n$. Prove that the expected total length of time that B_t stays in F is zero if and only if the Lebesgue-measure of F is zero.

Hint: Consider the resolvent R_α for $\alpha > 0$ and let $\alpha \to 0$.

Exercise 9.II

Let $M = (M_t)_{t \geq 0}$ be a continuous L^2-martingale on a filtered probability space $\big(\Omega, \mathcal{F}, \big(\mathcal{F}_t\big)_{t \geq 0}, \mathbb{P}\big)$ and let almost surely $M_0 = 0$. Consider $f \in C^1$. Show that for all $t \geq 0$,

$$\left\langle \int_0^\cdot f(M_s) dM_s \right\rangle_t = \int_0^t f^2(M_s) d\langle M \rangle_s.$$

Hint: Prepare Excercise 9.IV first. You can use the following replacement of the Itô isometry:

$$\mathbb{E}\left[\left(\int_0^t f(s, \cdot) dM_s\right)^2\right] = \mathbb{E}\left[\int_0^t f^2(s, \cdot) d\langle M \rangle_s\right].$$

Exercise 9.III

Let $\big(X_t\big)_{t\geq 0}$ be an Itô diffusion in $\mathbb{R}^n$ with generator

$$Af(y) = \sum_{i,j} a_{ij}(y)\frac{\partial^2 f}{\partial y_i \partial y_j} + \sum_i b_i(y)\frac{\partial f}{\partial y_i}, \qquad \text{for } f \in C_0^2$$

and assume that the transition measure of $\big(X_t\big)_{t\geq 0}$ has a *density* $p_t(x,y)$, i.e. that

$$\mathbb{E}^x\big[f(X_t)\big] = \int_{\mathbb{R}^n} f(y)p_t(x,y)dy; \quad f \in C_0^2.$$

Assume that $y \mapsto p_t(x,y)$ is smooth for each t, x. Prove that $p_t(x,y)$ satisfies the Kolmogorov forward equation

$$\frac{d}{dt}p_t(x,y) = A_y^\star p_t(x,y) \qquad \text{for all } x, y,$$

where $A_y^\star$ operates on the variable y and is given by

$$A_y^\star \varphi(y) = \sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j}(a_{ij}\varphi) - \sum_i \frac{\partial}{\partial y_i}(b_i\varphi); \quad \varphi \in C^2,$$

i.e. $A_y^\star$ is the *adjoint* of A_y.

Hint: Apply Dynkin's formula to derive

$$\int_{\mathbb{R}^n} f(y)p_t(x,y)dy = f(x) + \int_0^t \int_{\mathbb{R}^n} A_y f(y) p_s(x,y)dyds; \quad f \in C_0^2.$$

Now differentiate w.r.t. t and use that

$$\big\langle A\varphi, \psi\big\rangle = \big\langle \varphi, A_y^\star \psi\big\rangle \qquad \text{for } \varphi \in C_0^2, \psi \in C^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(dy)$.

Exercise 9.IV (*Preparation for a mini-presentation on Tuesday, January 8*)

Prepare a short talk on the topic given below. You should present the material in your own words without relying on your notes. The presentation should be made using the blackboard, writing down keywords and the essential formulae.

Construction of the stochastic integral w.r.t to a continuous L^2-martingale.

You can base your presentation on Section 5.4 (Stochastic integrals over square integrable martingales) in "Statistics of Random Processes I, General Theory" by Liptser and Shiryayev.