The Nagumo Equation with Comsol Multiphysics

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1. Traveling Front in the Nagumo equation

Consider the **Nagumo equation**

$$u_t = u_{xx} + u(1-u)(u-b), \quad x \in \mathbb{R},\, t \geqslant 0$$

for $0 < b < 1$, where $u = u(x,t) \in \mathbb{R}$. We want to solve this equation numerically for a suitable initial value function u_0 with Comsol Multiphysics. Therefore, we have to restrict the equation on a sufficiently large bounded domain $\Omega \subset \mathbb{R}$ with homogeneous Neumann boundary conditions, i.e. we solve the initial boundary value problem

$$\begin{aligned} u_t &= u_{xx} + u(1-u)(u-b) && , x \in \Omega,\, t \in (0,T], \\ u(\cdot,0) &= u_0 && , x \in \bar{\Omega},\, t = 0, \\ u_x &= 0 && , x \in \partial\Omega,\, t \in [0,T], \end{aligned} \tag{1.1}$$

on the spatial domain $\Omega = (-50, 50)$ for end time $T = 100$, initial data

$$u_0(x) = \frac{1}{1 + \exp(-\frac{x}{\sqrt{2}})}, \quad x \in \mathbb{R}$$

and parameter $b = \frac{1}{4}$. For the space discretization we use linear Lagrange elements with maximal element size $\triangle x = 0.1$. For the time discretization we use the BDF method of maximum order 2 with intermediate time steps, time stepsize $\triangle t = 0.1$, relative tolerance rtol $= 10^{-3}$ and absolute tolerance atol $= 10^{-4}$ with global method set to be unscaled. The nonlinear equations should be solved by the Newton method. i.e. automatic (Newton).

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2. Model Wizard

Start Comsol Multiphysics.
To start Comsol Multiphysics 5.2 open the **Terminal** and enter

- **comsol -ckl**

Model Wizard.
Space dimension

- In the **New** window, click **Model Wizard**.
- In the **Model Wizard** window, click **1D** in the **Select Space Dimension** menu.

Equation

- In the **Select Physics** tree, select **Mathematics>PDE Interfaces>Coefficient Form PDE (c)**.
- Click **Add**.
- Next, locate the **Dependent Variables** section.
- In the **Field name** text field, type **u**.
- In the **Dependent variables** text field, type also **u**.

Study settings

- Click **Study** and choose **Preset Studies>Time Dependent**.
- Click **Done**.

Some Advanced Settings.
Hint: In the **Model Builder** window you should click on the **Show** icon and enable everything that is possible from the menu: **Equation Sections** (**Equation View**, **Override and Contribution**, **Discretization**, **Stabilization**, **Advanced Physics Options**, **Advanced Study Options** and **Advanced Results Options**). Done this, click **Expand All** icon.

3. Geometry

- In the **Model Builder** tree, expand the **Component 1 (comp1)** node, right-click **Geometry 1** and select **Interval**.
- In the **Settings** window for Interval, locate the **Interval** section.
- In the **Left endpoint** text field, type **-50**.
- In the **Right endpoint** text field, type **50**.
- In the **Model Builder** tree, right-click on the **Component 1 *(comp1)*→Geometry 1** node and select **Build all**. (Alternatively, press the short cut **F8**.)

4. Partial differential equation

General Settings.

- Click on **Component 1 *(comp1)*→Coefficient Form PDE *(c)***.
- Locate the **Settings** window for Coefficient Form PDE.
- In the **Label** text field, type **Nagumo Equation**.
- In the **Discretization** section choose
  - **Shape function type**: **Lagrange**,
  - **Element order**: **Linear**.

Partial differential equation. We define the PDE:

- Switch to **Component 1 *(comp1)*→Nagumo Equation *(c)*→Coefficient Form PDE 1**

$$e_a \frac{\partial^2 u}{\partial t^2} + d_a \frac{\partial u}{\partial t} + \nabla \cdot (-c\nabla u - \alpha u + \gamma) + \beta \cdot \nabla u + au = f$$

with $\nabla = \frac{\partial}{\partial x}$, and enter the following values
  - **Mass Coefficient** e_a: **0**,
  - **Damping or Mass Coefficient** d_a: **1**,
  - **Diffusion coefficient** c: **1**,
  - **Conservative Flux Convection Coefficient** α: **0**,
  - **Conservative Flux Source** γ: **0**,
  - **Convection Coefficient** β: **0**,
  - **Absorption Coefficient** a: **0**,
  - **Source Term** f: **fu**.

Boundary Conditions. Since the PDE requires homogeneous Neumann boundary conditions at both end points of the interval, we do not must change anything. Hint: By default, there is implemented a zero flux boundary condition on the whole boundary, that corresponds to a homogeneous Neumann boundary condition.

Initial Values. We define the initial value $u(\cdot, 0) = u_0$ for the partial differential equation:

- Click on **Component 1 *(comp1)*→Nagumo Equation *(c)*→Initial Values 1**.
- In the **Initial Values** section enter
  - **Initial value for u**: **u0**,
  - **Initial time derivative of u**: **0**.

The quantity **u0** will be defined below in Section 5. This completes the implementation of the initial boundary value problem.

5. Parameters and Variables

Parameters. We first define the parameters and constants arising in our model as 'global parameters':

- In the **Model Builder** tree, right-click on the **Global Definitions** node and select **Parameters**. (Alternatively: On the **Model** toolbar, click **Parameters**.)
- In the **Settings** window for Parameters, locate the **Parameters** section.
- In the table add the following entry:

Name	Expression	Value	Description
b	1/4	0.25	constant of Nagumo equation
T	100	100	end time

Variables 1. We now define all functions which appear in our model as 'local variables'.

- In the **Model Builder** tree, right-click on the **Component 1 *(comp1)*→Definitions** node and select **Variables**.
- In the **Settings** window for Variables, locate the **Variables** section.
- In the table add the following entries:

Name	Expression	Unit	Description
u0	1/(1+exp(-x/sqrt(2)))		initial value
fu	u*(1-u)*(u-b)		nonlinearity

6. Mesh

- In the **Model Builder** tree, click on **Component 1 *(comp1)*→Mesh 1**.
- In the **Settings** window for Mesh, locate the **Mesh Settings** section.
- Set the **Sequence type** on **User-controlled mesh**.
- In the **Model Builder** tree, switch to **Component 1 *(comp1)*→Mesh 1→Size**.
- In the **Settings** window for Size, locate the **Element Size Parameters** section.
- In the **Maximum element size** text field, type **0.1**.
- In the **Model Builder** tree, right-click on **Component 1 *(comp1)*→Mesh** and select **Build All**.

7. Studies and Computation

Study 1. Study 1

- Click on **Study 1**.
- Locate the **Settings** window for Study.
- In the **Label** text field, type **Study 1: Nagumo Equation**.

Step 1

- Click on **Study 1: Nagumo Equation→Step 1: Time Dependent**.
- Locate the **Settings** window for Time Dependent.
- In the **Study Settings** section enter
  - **Time unit**: **s**,
  - **Times**: **range(0,0.1,T)**,
  - **Relative tolerance**: **0.001**.

  The last input requires to enable the corresponding checkbox.

Solver Configurations

- Right-click on **Study 1: Nagumo Equation→Solver Configurations** and select **Show Default Solver**.
- Click on **Study 1: Nagumo Equation→Solver Configurations→Solution 1 *(sol1)*→Time-Dependent Solver 1**.
- Locate the **Settings** window for Time Dependent Solver.
- In the **Absolute Tolerance** section enter
  - **Global method**: **Unscaled**,
  - **Tolerance**: **0.0001**.
- In the **Time Stepping** section enter
  - **Method**: **BDF**,
  - **Steps taken by solver**: **intermediate**,
  - **Maximum BDF order**: **2**.
- Click on **Study 1: Nagumo Equation→Solver Configurations→Solution 1 *(sol1)*→Time-Dependent Solver 1→Fully Coupled 1**.
- Locate the **Settings** window for Fully Coupled.
- In the **Method and Termination** section, choose
  - **Nonlinear Method**: **Automatic (Newton)**,

Solution Store

- Right-click on **Study 1: Nagumo Equation**→**Solver Configurations**→**Solution 1** ***(sol1)*** and select **Other**>**Solution Store** from the list.
- Click on **Study 1: Nagumo Equation**→**Solver Configurations**→**Solution 1** ***(sol1)***→**Solution Store 1** ***(sol2)***.
- Locate the **Settings** window for Solution Store.
- In the **Label** text field, type **Nagumo Equation Solution**.

7.1. Computation.

- Right-click on **Study 1: Nagumo Equation** and select **Compute** from the list.

8. Postprocessing and graphical output

In this section we generate 2 Plot groups and a movie for visualizing our results.

8.1. Results for the Nagumo equation.

Plot Group 1: Traveling Front, View 1

- Click on **Results**→**1D Plot Group 1**. Hint: If **1D Plot Group 1** does not exists, right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Traveling Front, View 1**.
- In the **Data** section select **Data set** **Study 1: Nagumo Equation/Nagumo Equation Solution (sol2)**, **Time selection** **Interpolated** and **Times (s)** **0 20 40 60 80 100**.
- In the **Title** section select **Title type** **None**.
- In the **Plot Settings** section select **x-axis label** **x** and **y-axis label** **u(x,t)**.
- Click on **Results**→**Traveling Front, View 1**→**Line Graph 1**. Hint: If **Line Graph 1** does not exists, right-click on **Results**→**Traveling Front, View 1** and select **Line Graph** from the list.
- Locate the **Settings** window for Line Graph 1.
- In the **Data** section select **Data set** **From parent**.
- In the **Selection** section select **Selection** **All domains**.
- In the **y-Axis Data** section select **Expression** **u**.
- In the **x-Axis Data** section select **Parameters** **Expression** and **Expression** **x**.
- In the **Coloring and Style** section select **Line** **Solid**, **Color** **Cycle** and **Width** **2** in the **Line style** subsection.
- In the **Legends** section enable the **Show legends** checkbox, select **Legends** **Manual** and enter the legends **t=0**, **t=20**, **t=40**, **t=60**, **t=80** and **t=100**.

Plot Group 2: Traveling Front, View 2

- Click on **Results**→**1D Plot Group 2**. Hint: If **1D Plot Group 2** does not exists, right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group.
- In the **Label** text field, type **Traveling Front, View 2**.
- In the **Data** section select **Data set** **Study 1: Nagumo Equation/Nagumo Equation Solution (sol2)** and **Time selection** **All**.

- In the **Title** section select **Title type** **None**.
- In the **Plot Settings** section select **x-axis label** **x** and **y-axis label** **t**.
- Click on **Results**→**Traveling Front, View 2**→**Line Graph 1**. Hint: If **Line Graph 1** does not exists, right-click on **Results**→**Traveling Front, View 2** and select **Line Graph** from the list.
- Locate the **Settings** window for Line Graph 1.
- In the **Data** section select **Data set** **From parent**.
- In the **Selection** section select **Selection** **All domains**.
- In the **y-Axis Data** section select **Expression** **t**.
- In the **x-Axis Data** section select **Parameter** **Expression** and **Expression** **x**.
- Right-click on **Results**→**Traveling Front, View 2**→**Line Graph 1** and select **Color Expression**.
- Click on **Results**→**Traveling Front, View 2**→**Line Graph 1**→**Color Expression 1**.
- Locate the **Settings** window for Color Expression.
- In the **Expression** section select **Expression** **u**.

Plot Group 3: Plot for Animation

- Right-click on **Results** and select **1D Plot Group** from the list.
- Locate the **Settings** window for 1D Plot Group 3.
- In the **Label** text field, type **Plot Group for Animation**.
- In the **Data** section select **Data set** **Study 1: Nagumo Equation/Nagumo Equation Solution (sol2)**, **Time selection** **First**.
- In the **Title** section select **Title type** **None**.
- In the **Plot Settings** section select **x-axis label** **x** and **y-axis label** **u(x,t)**.
- Click on **Results**→**Plot Group for Animation**→**Line Graph 1**. Hint: If **Line Graph 1** does not exists, right-click on **Results**→**Plot Group for Animation** and select **Line Graph** from the list.
- Locate the **Settings** window for Line Graph 1.
- In the **Data** section select **Data set** **From parent**.
- In the **Selection** section select **Selection** **All domains**.
- In the **y-Axis Data** section select **Expression** **u**.
- In the **x-Axis Data** section select **Parameters** **Expression** and **Expression** **x**.
- In the **Coloring and Style** section select **Line** **Solid**, **Color** **Cycle** and **Width** **2** in the **Line style** subsection.
- In the **Legends** section enable the **Show legends** checkbox, select **Legends** **Automatic**.

Animation 1: Traveling Front, Animation

- Click on the **Animation** icon on top of the **Settings** window and select **Player**.
- Locate the **Setting** window for Animation 1.
- In the **Label** text field, type **Traveling Front, Animation**.
- In the **Target** section select **File**.
- In the **Output** section select **Format** **GIF** and in the **Filename** text field type **NagumoEquation.gif**.

- In the **Frames** section enter **Number of Frames** **100**.
- In the **Layout** section enable the checkbox for **Include** and then the checkboxes for **Title, Legend** and **Axes** and enter **Font size** **10**.
- In the **Advanced** section disable the checkbox **Synchronize scales between frames**.
- Right-click **Results**→**Export**→**Traveling Front, Animation** and select **Export**.