Analytical and numerical aspects of nonlinear evolution equations with application to incompressible fluid flow

DIPLOMARBEIT

zur Erlangung des akademischen Grades

DIPL.-MATH.

an der

Fakultät II – Mathematik und Naturwissenschaften
der Technischen Universität Berlin

– Institut für Mathematik –

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Berlin, 14. Juli 2009

Zusammenfassung

In dieser Arbeit untersuchen wir die schwache Lösbarkeit eines Systems partieller Differentialgleichungen, das das Verhalten verallgemeinerter Newtonscher Fluide beschreibt. Die interne Spannung in dem Fluid wird durch einen nichtlinearen, elliptischen Operator beschrieben, für den wir p-Koerzitivität, strikte Monotonie und eine $(p-1)$-Wachstumsbedingung voraussetzen.

Wir präsentieren zunächst einen Lösungsansatz von Wolf [67], der auf einer Approximation des Konvektionsterms basiert. Das so erhaltene System partieller Differentialgleichungen kann lokal in der Zeit mit Methoden der Theorie monotoner Operatoren, Kompaktheitsargumenten und einem Fixpunktargument für alle $p > 2d/(d+2)$ gelöst werden, wobei d die Raumdimension ist. Diese zunächst zeitlich lokale Lösung kann dann mit Hilfe einer Energiegleichung und A-priori-Abschätzungen zu einer globalen Lösung des approximativen Systems fortgesetzt werden. Wir diskutieren diesen Lösungsansatz und skizzieren den Beweis der Konvergenz gegen eine Lösung des ursprünglichen Systems.

Der Hauptteil dieser Arbeit befasst sich mit der Analyse eines Verfahrens zur numerischen Approximation. Mit Hilfe einer impliziten Semidiskretisierung in der Zeit konstruieren wir aus exakten Lösungen des stationären Problems stückweise polynomielle Prolongationen $u_{\Delta t}$ und $v_{\Delta t}$. Geeignete A-priori-Abschätzungen sichern dann die Konvergenz dieser Prolongationen gegen eine Grenzfunktion u. Um zu zeigen, dass diese Grenzfunktion tatsächlich eine Lösung des Problems ist, ist es erforderlich, hinreichend reguläre Testfunktionen zu konstruieren. In Analogie zum Vorgehen von Diening, Růžička und Wolf [22], leiten wir eine Lipschitz-Abschneide-Technik her und führen eine geeignete Zerlegung des Drucks für das System ein. Dies soll es uns ermöglichen, die Konvergenz der Gradienten von $u_{\Delta t}$ fast überall zu zeigen um so sicherzustellen, dass u in der Tat eine Lösung unseres Problems ist.

Die größte Schwierigkeit, die sich dabei ergibt, ist es, eine geeignete Variante der Lipschitz-Abschneide-Technik zu finden, die auf stückweise konstante Funktionen angewendet werden kann. Dafür betrachten wir fraktionale Sobolev-Räume und leiten Ungleichungen vom Poincaré-Typ für diese her. Die Konvergenz der Lipschitz-Abschneidungen wird dann durch die Beschränktheit der Folge der Prolongationen im gebrochenen Sobolew-Raum gesichert. Wie sich herausstellt, können wir eine solche Schranke unter der etwas stärkeren Bedingung $p > \max((d + \sqrt{d^2 + 2d(d+2)})/(d+2), 3d/(d+2))$ finden. Diese Bedingung beinhaltet den interessanten Fall $p < 2$ in Raumdimensionen $d = 2$ und $d = 3$.

Des weiteren präsentieren wir eine angepasste Version des Beweises von Di-

ening, Malék und Steinhauer [19] zur schwachen Lösbarkeit des entsprechenden stationären Problems. Hier wird ein regularisierender Term zur Gleichung addiert, der sicherstellt, dass wir mit der Lösung selbst testen dürfen. Dies ermöglicht die Herleitung von A-priori-Abschätzungen, mit denen die Konvergenz der Approximationen gezeigt werden kann. Eine stationäre Lipschitz-Abschneide-Technik wird daraufhin benutzt, um die Konvergenz der Gradienten fast überall zu zeigen. Damit beweisen wir, dass der Grenzwert der Approximationen eine Lösung des stationären Problems ist.

Offen bleibt jedoch der Beweis einer Abschätzung für die Zeitlableitung in instationären Fall, die benötigt wird um die Konvergenz der Gradienten der Approximationen zu zeigen. Wir diskutieren kurz die Ursachen für die Schwierigkeiten bei dieser Abschätzung.

Abstract

In this work, we study a system of partial differential equations describing the motion of a generalized Newtonian fluid. The nonlinear elliptic operator related to the stress is assumed to be p-coercive, strictly monotone and to fulfil a $(p-1)$-growth condition.

We present the ideas of Wolf [67] to approximate the convection term and discuss the weak solvability of the approximate system for $p > 2d/(d+2)$, where d is the space dimension. Employing methods from the theory of monotone operators, compactness arguments and a fixed point theorem, we show the local-in-time existence of a solution. By means of an energy equality and a priori estimates, we are able to extend this to a global approximate solution. We then sketch the proof of convergence towards a solution to the original problem.

The main part of this work consists of the introduction of a numerical approach to prove the existence of weak solutions involving an implicit semi-discretization in time. We construct piecewise polynomial prolongations $u_{\Delta t}$ and $v_{\Delta t}$ from exact solutions of the corresponding stationary problem and show weak convergence towards a limit function u. For the identification of the limits with the terms in the differential equation, it is necessary to construct sufficiently regular test functions. Following the ideas of Diening, Růžička and Wolf [22], we establish a Lipschitz truncation technique and recover pressure functions for the system. This should help us showing the almost everywhere convergence of the gradients of $u_{\Delta t}$, which implies that u is indeed a solution to our problem.

The obstacle here lies in a suitable adaptation of the Lipschitz truncation method for piecewise constant functions. We overcome this by considering fractional order Sobolev spaces and deriving suitable Poincaré-type inequalities for them. The boundedness of the sequence of prolongations in such Sobolev-Slobodeckii spaces then yields the convergence of the truncations. The range of admissible p's for this turns out to be $p > \max((d + \sqrt{d^2 + 2d(d+2)})/(d+2), 3d/(d+2))$. This covers the interesting case $p < 2$ in space dimensions $d = 2$ and $d = 3$.

Additionally, we present an adaptation of a proof by Diening, Málek and Steinhauer [19] for the existence of weak solutions to the corresponding stationary problem. In this proof, a regularizing term is added to the equation to permit testing the equation with the solution itself. This is important for the derivation of a priori estimates, with which one can then show convergence of the approximate solutions. A stationary Lipschitz truncation finally yields the almost everywhere convergence of the gradients and hence the existence of a weak solution.

However, there remains an open problem of estimating a term involving the time-derivatives of the prolongations in the non-stationary case. We shortly discuss the diffculties in handling this term.

Acknowledgements

First and foremost, I would like to express my deep gratitude to my supervisor, Priv.-Doz. Dr. Etienne Emmrich, who introduced me to the field of partial differential equations. With his constant support over the past years and especially while writing this thesis I was able to learn an incredible amount. Without his encouragement and clever ideas when I was stuck this thesis would have been in a much poorer state. I would like to thank him for his great confidence in me and the numerous possibilities he provided for me to obtain invaluable experiences in international research during conferences, in schools and at mathematical talks.

I am also very grateful to Christian Kreusler whose door was always open for scientific and non-scientific discussions over a cup of freshly made espresso. His extensive knowledge in functional analysis and his great personal library helped me out in a lot of situations.

Finally, many thanks go to my fellow student Max Klimm for the great time I had working next to him and his support with everything that occurs when writing a diploma thesis, from his daily supply of sweets to his help with technical issues in Latex.

Contents

1

Introduction

Many phenomena and processes in natural sciences can be described by means of partial differential equations. The questions arising for these equations are ones of solvability, uniqueness of solutions, stability, regularity and approximation of solutions. Although many complex dynamical systems can be represented by a small number of short equations, in general it is not at all simple to find solutions to them. In fluid mechanics, the Navier-Stokes equations, describing the motion of a Newtonian fluid, provides a prominent example for this: Having been introduced by Navier in 1827, the Navier-Stokes system has been subject of mathematical research until today. Up to now, the question of existence and smoothness of solutions in three spatial dimensions has not been solved satisfactorily.[1]

In the field of fluid mechanics, one is interested in the description of fluids depending on factors like e.g. external forces, temperature, pressure or properties of the fluid itself. When external factors are fixed, the nature of a fluid and with it the corresponding model is determined by the behavior of its internal stress. Assuming a linear relation between the stress and strain, i.e. assuming a constant viscosity, results in the Navier-Stokes equations and the description of so called Newtonian fluids like water for example. However, there are many examples of fluids that exhibit "unusual" behavior and cannot be modeled with these equations. Those fluids are called non-Newtonian. They appear in nature, e.g. blood, lava, snow or mud slurries, in our everyday life, e.g. toothpaste, tomato ketchup or paints, as well as in many industrial environments, e.g. melts or polymeric liquids. In order to model non-Newtonian fluids, one has to derive suitable constitutive laws for the internal stress which means relations between the internal stress, strain, external forces, etc.

Non-Newtonian fluid mechanics is a vast subject, that has gained much attention in mathematics, chemical engineering, biology and even geophysics (see [8]). Over the time, many different models incorporating the several phenomena exhibited by non-Newtonian fluids have been introduced and investigated. Flows with shear-rate dependent viscosity can be described by the models of generalized Newtonian fluids such as Power-Law fluids and fluids of the Carreau-Yasuda-type. Viscoelastic properties are captured by Maxwell's and Oldroyd's models and can describe elastic responses of polymeric fluids. In contrast to these macroscopic laws, one can also investigate the microscopic behavior of the

[1] In fact, the Clay Mathematical Institute counts the (incompressible) Navier-Stokes problem to one of the so called Millennium Prize Problems.

fluid to derive suitable representations of the internal stress. This results for example in the Navier-Stokes-Fokker-Planck equations. A general discussion of non-Newtonian fluids and their mechanics can be found in the monographs [13], [6] or [10].

1.1 Physical background

Let us state the governing equations that describe the flow of fluids. Since many properties of non-Newtonian fluids arise from microstructures, we assume that the length scale of the flow field is greater than that of its microstructures. Given this continuum hypothesis, one can derive the equations representing the conservation of mass, momentum and energy. Since we will only consider isothermal fluid flow, the balance of energy can be omitted. The other two equations are given in terms of Eulerian-coordinates by

$$\text{Mass:} \qquad \frac{\partial}{\partial t}\varrho + \operatorname{div}(\varrho u) = 0, \tag{1.1}$$

$$\text{Momentum:} \qquad \frac{\partial}{\partial t}(\varrho u) - \operatorname{div}\sigma + \operatorname{div}(\varrho u \otimes u) = \varrho f, \tag{1.2}$$

where ϱ is the density, u the velocity field and f represents external forces. The "total stress tensor" σ describes the change of momentum by virtue of molecular motions and interactions within the fluid and thus will be important for the description of the nature of the fluid. The dyadic product is defined by $(u \otimes u)_{i,j} = u_i u_j$. The equations are to be understood row-wise.

So far, the system is valid for any isothermal fluid flow. We will now restrict ourselves to incompressible flows, meaning that the density is constant in time and space. Thus, equation (1.1) is equivalent to

$$\operatorname{div} u = 0.$$

In this case, the total stress tensor σ splits into the isotropic part πI with the thermodynamic pressure π and the deviatoric stress tensor τ:

$$\sigma = \pi I + \tau.$$

With this, (1.2) can be written as

$$\varrho\frac{\partial}{\partial t}u - \operatorname{div}\tau + \varrho\operatorname{div}(u \otimes u) + \nabla\pi = \varrho f. \tag{1.3}$$

Except for the incompressibility, we have not yet stated any properties of the fluid itself. These can now be expressed by so-called "constitutive equations" or "constitutive laws" for the stress tensor τ. Throughout this work, we assume, that the stress tensor is symmetric.

In the case of Newtonian fluids, Stokes' law

$$\tau = -\mu\, Du + \left(\frac{2}{3}\mu - \kappa\right)(\operatorname{div} u)I$$

has been established experimentally for many "ordinary fluids". Here, $Du = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$ denotes the symmetric part of the velocity gradient which is also called rate-of-strain tensor in this context. The constant μ is the dynamic viscosity and κ is the dilatational viscosity. Considering incompressible fluids, this gives the well known diffusion term of the incompressible Navier-Stokes equation

$$-\operatorname{div}\tau = -\mu\Delta u.$$

In the last years, the development and analysis of coupled microscopic-macroscopic models have been of special interest. In those models, the stress tensor does not underly a simple macroscopic law, but is coupled to a microscopic equation, which arises from the kinetic (often stochastic) theory for the polymer molecules. One example is the Navier-Stokes-Fokker-Planck model for dilute polymers (see [9]).

In industrial chemical engineering, a very important characteristic of non-Newtonian fluids is the fact that they have a "shear-rate dependent" viscosity. This means, their viscosity increases or decreases noticeably as the shear rate increases. We then call the fluid shear-thickening or shear-thinning, respectively.

So called generalized Newtonian constitutive equations incorporate these effects in a simple way. They are derived as small modifications to the Newtonian constitutive equations. However, they are not able to describe other features of non-Newtonian fluids as, for instance, normal stress effects or any time-dependent elastic effects.

The idea is to regard the viscosity as a function of the temperature, particle concentration and the scalar invariants of the rate-of-strain tensor. In our case, we omit the dependence on temperature and let the particulate concentration be a material constant. If the fluid is incompressible, the viscosity depends only on the second invariant of the rate-of-strain tensor, which is the "shear-rate" $|Du|$. The constitutive law for generalized Newtonian fluids then reads

$$\tau = \mu(|Du|)Du.$$

The popular power-law model is included in this type of constitutive laws. This is a two-parameter model, where the viscosity is of the form

$$\mu(|Du|) = \mu_0|Du|^{p-2}.$$

This expression can reliably describe the viscosity of many non-Newtonian fluids in the cases interesting for industrial chemical engineers (see [13, Chapter4.b]). The parameter p is crucial for the characteristic behaviour of the fluid. If $p < 2$, the fluid appears shear-thinning, for $p > 2$, it is shear-thickening. The case $p = 2$ results in the linear relation of a Newtonian fluid.

Another example of a generalized Newtonian fluid is the slightly more complex Carreau-Yasuda model. This model can describe the transition between the zero-strain-rate region and the power-law region of a fluid (see [13, Chapter4.a]). The constitutive relation of this model is given by

$$\mu(|Du|) = \mu_0(1 + |Du|^2)^{(p-2)/2}.$$

1.2 Mathematical formulation

To develop the system of equations that we will consider in this work, we divide (1.3) by ϱ and relabel τ/ϱ by S and π/ϱ again by π. We describe the motion of an isothermal, incompressible generalized Newtonian fluid in a domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, and a time interval $[0, T]$ by the initial-boundary value problem

$$\partial_t u - \operatorname{div} S(Du) + \operatorname{div}(u \otimes u) + \nabla\pi = f \quad \text{in } \Omega \times (0,T), \tag{1.4}$$
$$\operatorname{div} u = 0, \tag{1.5}$$
$$u = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.6}$$
$$u(\cdot, 0) = u_0 \quad \text{in } \Omega, \tag{1.7}$$

where $u = (u_1(x,t), \dots, u_d(x,t))$ is the velocity field with the prescribed initial velocity $u_0 = u_0(x)$ with $\operatorname{div} u_0 = 0$. We denote by $\pi = \pi(x,t)$ the pressure of the system, by $f = (f_1(x,t), \dots, f_d(x,t))$ the external force per unit mass and by $S = (S_{ij}(x, Du))_{i,j=1}^d$ the symmetric deviatoric stress tensor.

We assume that $S : \overline{\Omega} \times \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}^{d\times d}_{\text{sym}}$, $(x, z) \mapsto S(x, z)$ is measurable in x, continuous in z and fulfils the following growth condition:

There exists a constant $c > 0$ such that for any $z \in \mathbb{R}^{d\times d}_{\text{sym}}$ holds

$$|S(z)| \le c(1 + |z|)^{p-1}. \tag{1.8}$$

Furthermore, the stress should be strictly monotone, i.e. for every $y, z \in \mathbb{R}^{d\times d}_{\text{sym}}$, $y \ne z$, holds

$$(S(z) - S(y)) : (z - y) > 0, \tag{1.9}$$

and p-coercive, which means that there is a constant $\tilde{c_0} > 0$ such that for any $z \in \mathbb{R}^{d\times d}_{\text{sym}}$ we have

$$S(z) : z \ge \tilde{c_0}|z|^p. \tag{1.10}$$

Here, $y : z = \sum_{i,j=1}^d y_{ij} z_{ij}$ denotes the inner product in $\mathbb{R}^{d\times d}$.

A function S of this kind incorporates all power-law models and models of the Carreau-Yasuda-type described above.

Remark 1.1. Note, that we do not incorporate time-dependence in the viscosity term S. In this way, we avoid some technical difficulties and unnecessary complexity. We are confident that the results in this work hold also true in the time-dependent case.

1.3 State of the art

A comprehensive view on the mathematical theory for the Navier-Stokes system, which can be seen as the special case $p = 2$, can be found in Ladyzhenskaya [38], Lions [41], Temam [62], Sohr [60] and Galdi [33]. Concerning time-discretizations for the Navier-Stokes system, we refer to Marion/Temam [46], Girault/Raviart [35] and again Temam [62] for the implicit Euler-scheme and to Emmrich [24] for the BDF(2) method (two step backward difference formula). An overview over new achievements in the theory of the Navier-Stokes equations can be found in Temam [63].

The Oldroyd model for viscoelastic fluids (see Oldroyd [48]) has been studied in e.g. Fernández-Cara/Guillén/Ortega [30], Renardy [52] and Lions/Masmoudi [43]. Some recent work has been done on coupled microscopic-macroscopic multiscale models for dilute polymeric fluids, see Barret/Knezevic/Süli [9], Keunings [37] or Renardy [51] and the references cited therein.

Analysis of compressible models and incompressible limits are studied e.g. in Lions/Masmoudi [42], Feireisl [28] and Feireisl/Novotný [29].

The mathematical analysis of the model (1.4)-(1.7) with the conditions (1.8), (1.9) and (1.10) was initiated by Ladyzhenskaya [38, 39] and Lions [41]. They proved the existence of weak solutions for the space periodic and Dirichlet-boundary problem combining the theory of monotone operators and compactness arguments for

$$p \geq 1 + \frac{2d}{d+2}, \tag{1.11}$$

where $d \in \{2, 3\}$ is the spatial dimension. In the case $d = 3$, this means $p > \frac{11}{5}$ and especially the Newtonian case $p = 2$ is not covered. The case $p < 2$ has many applications (see [44, Examples 1.78, 1.80, 1.83]) and is not covered even in two space dimensions.

Results on measure-valued solutions in the case

$$p > \frac{2d}{d+2} \tag{1.12}$$

for the space periodic as well as Dirichlet-boundary problems have been presented in Málek/Nečas/Rokyta/Růžička [44].

Just recently, the existence of weak solutions for $p > 2\frac{d+1}{d+2}$ for the Dirichlet-boundary problem has been proven in Wolf [67] using an approximation of the convection term and a L^∞-truncation technique. This result could be generalized to the case (1.12) by Diening/Růžička/Wolf in [22] employing the same approximation as in [67] and then using a parabolic Lipschitz truncation technique. We will discuss this approach in Chapter 4.

The Lipschitz truncation technique for Sobolev-functions have already been successfully employed before in different contexts, see e.g. Acerbi/Fusco [1], Landes [40], and in the context of generalized Newtonian fluids for the stationary case in Frehse/Málek/Steinhauer [31] and Diening/Málek/Steinhauer [19]. We will give a short presentation of the application of the stationary Lipschitz truncation to our problem in Chapter 7.

Another approximation approach is introduced by Zhikov [69]. In this article, the approximation is done in the diffusion term rather than in the convection term. Introducing the approximation on the diffusion term A_ε and the corresponding approximate solution u_ε, the weak solvability of (1.4)-(1.7) is shown by means of a convergence result for the measure $A_\varepsilon \cdot \nabla u_\varepsilon \, \mathrm{d}x \, \mathrm{d}t$. The limit of this measure is splitted into an absolute continuous part with some density function a and a singular part. Studying the density a finally yields the existence of a weak solution for the same range of parameters

$$p > \max\left(\frac{d + \sqrt{d^2 + 2d(d+2)}}{d+2}, \frac{3d}{d+2}\right),$$

that we will encounter in our proof.

A temporal semi-discretization with a fully- as well as semi-implicit BDF(2) method has been studied in Emmrich [26]. There, the convergence of a piecewise polynomial prolongation of the discrete solution towards an exact solution is shown for the case (1.11). This is, in fact, the method of proof, we will be considering in this work.

Based on the approximation in [67], a full discretization of (1.4)-(1.7) has recently been analysed in Carelli/Haehnle/Prohl [18] employing an implicit Euler scheme in time coupled with an implicit finite element discretization in space. For solving the discrete problem, a fixed point scheme is proposed.

In the context of strong solutions, error estimates for fully- as well as semi-implicit Euler schemes coupled with a finite element approximation have been investigated in Prohl/Růžička [50], Diening/Prohl/Růžička [20] and more recently in Berselli/Diening/Růžička [12]. The existence of strong solutions for arbitrary data in the case (1.11) is assured by the results in Málek/Nečas/Růžička [45] and in Bellout/Bloom/Nečas [11].

1.4 Outline and aim of this work

This work is organized as follows. In Chapter 2 we introduce the basic notation and function spaces as well as some basic results such as the Lions-Aubin compactness lemma and Poincaré-type inequalities. The following chapter is dedicated to the definition and analysis of the relevant operators that we will encounter. Furthermore, the weak formulation of the system (1.4)–(1.7) is stated and an equivalent operator differential equation is derived. In Chapter 4, we present the ideas of Wolf [67] and introduce an approximation of the convection term. We prove the solvability of the approximate system and give an outline of the proof of convergence towards a solution of the original system. Following the steps in Diening/Růžička/Wolf [22], we reconstruct pressure functions which vanished in the weak formulation of (1.4)-(1.7) in Chapter 5. This provides a representation of our problem in which we can test with any (sufficiently smooth) not necessarily solenoidal function. In Chapter 6, we establish a Lipschitz truncation method for functions of fractional order in time, i.e. belonging to some Sobolev-Slobodeckii space, similar to the one in Diening/Růžička/Wolf [22]. We then investigate the corresponding stationary problem and present an adaptation of the proof of existence by Diening/Málek/Steinhauer [19]. With this result, we are able to introduce a well-defined temporal semi-discretization applying an implicit Euler-scheme in Chapter 8. After deriving suitable a priori estimates, we show the convergence of polynomial prolongations of the discrete solutions towards a weak solution of (1.4)-(1.7). In the course of this, we apply the results on the Lipschitz truncation and the pressure representation in order to obtain almost everywhere convergence of the gradients, which replaces the use of Minty's monotonicity trick as in [26].

The aim of this work is twofold. On the one hand, we provide an alternative prove of existence of weak solutions to the system (1.4)-(1.7). The condition on the parameter p required for this proof is

$$p > \max\left(\frac{d + \sqrt{d^2 + 2d(d+2)}}{d+2}, \frac{3d}{d+2}\right),$$

which covers the interesting case $p < 2$ in dimension $d \in \{2, 3\}$.

On the other hand, we show the convergence of a numerical approximation in time. For a full discretization, we need to combine this implicit Euler-scheme with an appropriate discretization in space. One can expect, that the methods applied here will carry over to suitable full discretizations.

Our method of proof is sometimes called Rothe method (see e.g. [53, Section 8.2]). We construct piecewise constant and piecewise linear functions from the discrete (stationary) solutions approximating an exact solution. With the help of a priori estimates and compactness arguments, the convergence of each term in the weak formulation can be shown.

New in this context is the adaptation of the Lipschitz truncation method introduced in Diening/Růžička/Wolf [22] to functions with fractional order regularity in time and the application of this technique to the Rothe method.

2

Function spaces and preliminaries

Before we start looking at the equation itself, let us introduce some of the relevant function spaces. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and let $(0, T)$ be the considered time interval. As usual, we denote by $C_0^\infty(\Omega)$ the space of real-valued functions that have derivatives of all orders and compact support in Ω. Here and in what follows, c denotes a generic positive constant that can change from line to line.

2.1 Lebesgue and Sobolev spaces

Real-valued functions We denote by $L^s(\Omega)$, $1 \leq s < \infty$, the Lebesgue space of (equivalence classes of) real-valued functions with s-th power absolutely integrable over Ω. This is a Banach space with the norm

$$\|u\|_{L^s(\Omega)} = \left(\int_\Omega |u(x)|^s \,\mathrm{d}x \right)^{1/s}.$$

In the case $s = \infty$, we consider the space of essentially bounded functions and the norm

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|.$$

In the special case $s = 2$, $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = \int_\Omega u(x) v(x) \,\mathrm{d}x\,.$$

The d-dimensional Lebesgue-measure will be denoted by μ_d. For a Lebesgue-measurable set A and a function $g \in L^1_{\mathrm{loc}}(\Omega)$ (absolutely integrable on any compact subset of Ω) we write

$$g_A = \fint_A g(x) \,\mathrm{d}x = \frac{1}{\mu_d(A)} \int_A g(x) \,\mathrm{d}x$$

for the integral mean of g on A.

The Sobolev spaces $W^{k,s}(\Omega)$, $k \in \mathbb{N}$, $s \in [1, \infty)$, contain real-valued functions for which the weak derivatives of order up to k exist and belong to $L^s(\Omega)$. With the usual notation for multiindices $\alpha \in \mathbb{N}^d$ we consider the norm

$$\|u\|_{W^{k,s}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^s_{L^s(\Omega)} \right)^{1/s}.$$

Equipped with this norm, $W^{k,s}(\Omega)$ is a Banach space. In the case $s = \infty$, we consider the norm

$$\|u\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha|\leq k} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

As before, $s = 2$ gives a Hilbert space equipped with the inner product

$$(\!(u, v)\!)_k = \sum_{|\alpha|\leq k} (D^\alpha u, D^\alpha v).$$

We will consider Dirichlet-boundary conditions and thus deal with functions vanishing on the boundary. We define the space $W_0^{1,s}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,s}(\Omega)$. If Ω is bounded, one can show that

$$\|u\|_{W_0^{1,s}(\Omega)} = \left(\sum_{|\alpha|=1} \|D^\alpha u\|^s_{L^s(\Omega)} \right)^{1/s}$$

defines an equivalent norm to $\|\cdot\|_{W_0^{1,s}(\Omega)}$. Introducing the trace operator $\gamma_0 : W^{k,s}(\Omega)^d \to L^s(\partial\Omega)$ giving the restriction of a function on the boundary $\partial\Omega$, one finds the convenient representation

$$W_0^{1,s}(\Omega) = \left\{ v \in W^{1,s}(\Omega) : \gamma_0 v = 0 \right\},$$

see [3, Theorem 7.55] or [47, Chapter 2, Theorem 4.10].

$\mathbb{R}^d$- and $\mathbb{R}^{d\times d}$-valued functions In the setting of fluid flows, we shall often be concerned with $\mathbb{R}^d$-valued functions.

We denote by $a \cdot b = \sum_{i=1}^d a_i b_i$ the usual inner product in $\mathbb{R}^d$ and with $A : B = \sum_{i,j=1}^d A_{ij} B_{ij}$ the one in $\mathbb{R}^{d\times d}$.

Elements of the spaces $L^s(\Omega)^d$ and $W^{k,s}(\Omega)^d$ are d-dimensional vector-valued functions with components in the respective Sobolev or Lebesgue spaces defined above. We impose a usual product norm on the product space. Hence, we receive, for instance, the norm

$$\|u\|_{L^s(\Omega)^d} = \left(\sum_{i=1}^d \|u_i\|^s_{L^s(\Omega)} \right)^{1/s},$$

which can also be written as

$$\|u\|_{L^s(\Omega)^d} = \left(\int_\Omega |u(x)|^s \,\mathrm{d}s \right)^{1/s}$$

with the norm $|u(x)| = \left(\sum_{i=1}^d |u_i(x)|^s \right)^{1/s}$ on $\mathbb{R}^d$.

Analogously one defines the Lebesgue and Sobolev spaces for $\mathbb{R}^{d\times d}$-valued functions.

Considering $k = 1$, we have especially

$$\|u\|_{W_0^{k,s}(\Omega)} = \|\nabla u\|_{L^s(\Omega)^d}$$

and

$$\|u\|_{W_0^{1,s}(\Omega)^d} = \left(\sum_{i=1}^{d} \|\nabla u_i\|_{L^s(\Omega)}^s \right)^{1/s} = \|\nabla u\|_{L^s(\Omega)^{d\times d}}$$

where ∇u is defined row-wise. The reader is referred to [3] for a more detailed view on the theory of Sobolev spaces.

2.2 Solenoidal function spaces

Since we will mostly deal with solenoidal (i.e. divergence-free) functions, we define the space of smooth solenoidal functions with compact support in Ω by

$$\mathcal{V} = \{v \in C_0^\infty(\Omega)^d : \operatorname{div} v = 0\}.$$

We denote by $H_s(\Omega)$ and $V_s(\Omega)$ the closure of $\mathcal{V}$ with respect to the $L^s(\Omega)^d$-norm and the $W_0^{1,s}(\Omega)^d$-norm, respectively. In view of readability, we may omit the declaration of the domain, if the context is clear. The norms of H_s and V_s shall be denoted with $\|\cdot\|_{V_s} = \|\cdot\|_{W_0^{1,s}(\Omega)^d}$ and $\|\cdot\|_{H_s} = \|\cdot\|_{L^s(\Omega)^d}$, respectively. Since the Hilbert space H_2 plays a special role, we may write $H = H_2$.

One can show, that the spaces H_s and V_s can be characterized without using the closure of $\mathcal{V}$. For Lipschitz-domains Ω it is

$$\begin{aligned} V_s &= \left\{ v \in W_0^{1,s}(\Omega)^d : \operatorname{div} v = 0 \right\}, \\ H_s &= \left\{ u \in L^s(\Omega)^d : \operatorname{div} v = 0, \gamma_\eta v = 0 \right\}. \end{aligned} \tag{2.1}$$

In the case of H_s, the divergence is to be understood in the weak sense. The operator γ_η is the trace operator for L^s-functions on Ω giving the trace of v in the normal direction. For more details, see [62, Theorem 1.4, Theorem 1.6] or [33, Paragraph III.4.1, p. 143, Paragraph III.2, pp. 113ff.].

By virtue of the Sobolev embedding theorems (c.f. [3, Theorem 5.4] or [27, Theorem 5.6.3]), V_p, H and the dual space V_p' form a Gelfand triple with compact embeddings, if we assume $p > \frac{2d}{d+2}$. We denote the conjugated exponent by $p' = \frac{p}{p-1}$.

2.3 Abstract function spaces for time-dependent functions

Let $(X, \|\cdot\|_X)$ be a Banach space, $T \in (0, \infty)$. By $L^s(0,T;X)$, $1 \le s \le \infty$ we denote the space of Bochner-measurable functions $u : (0,T) \to X$ such that

$$\|u\|_{L^s(0,T;X)} = \left(\int_0^T \|u(t)\|_X^s \,\mathrm{d}t \right)^{1/s} < \infty$$

and respectively

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|u(t)\|_X < \infty.$$

For $1 < s < \infty$, we will usually identify the dual spaces $L^{s'}(0,T;X') \cong (L^s(0,T;X))'$.

Many problems become rather easy, if the time derivative of a function lies in the dual space. As the natural space for solutions to evolution problems in the variational setting, one often considers the space

$$W^s(0,T) = \{u \in L^s(0,T;V_s) : u' \in L^{s'}(0,T;V_s')\}$$

with the norm

$$\|u\|_{W^s(0,T)} := \|u\|_{L^s(0,T;V_s)} + \|u'\|_{L^{s'}(0,T;V_s')}.$$

However, we will mainly deal with weaker assumptions on our equation that admit functions with time derivatives in larger spaces.

Lemma 2.1. *Let* $s > \frac{2d}{d+2}$. *Then* $V_s \overset{c}{\hookrightarrow} H \hookrightarrow V_s'$ *is a Gelfand triple and* $W^s(0,T)$ *is a Banach space. Here, the symbol* $\overset{c}{\hookrightarrow}$ *means "compactly embedded in". Furthermore, every* $u \in W^s(0,T)$ *is almost everywhere in* $(0,T)$ *equal to a continuous function on* $[0,T]$ *with values in* H *and*

$$W^s(0,T) \hookrightarrow C([0,T];H).$$

The space $C^\infty([0,T];\mathcal{V})$ *lies dense in* $W^s(0,T)$ *and the rule of integration by parts*

$$\int_s^t (\langle v'(\tau), w(\tau)\rangle + \langle v(\tau), w'(\tau)\rangle)\, d\tau = (v(t), w(t)) - (v(s), w(s))$$

holds for all $0 \le s \le t \le T$ *and* $v, w \in W^s(0,T)$.

Proof. See [25, Theorem 8.4.1], [32, Chapter IV, Paragraph 1, Theorem 1.17] or [53, Lemma 7.3]. □

This rule of integration by parts is a crucial part in the proof of convergence for the approximate solutions in Chapter 4. In particular, we make use of the following corollary.

Corollary 2.2. *For any* $u \in W^p(0,T)$ *there holds*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 = \langle u'(t), u(t)\rangle \tag{2.2}$$

in the weak sense and almost everywhere in $(0,T)$.

Proof. See [53, Remark 7.5]. □

2.4 Nikolskii and Sobolev-Slobodeckii fractional spaces

We will work with piecewise constant functions taking values in Banach spaces to approximate the solution to our problem. For the Lipschitz truncation theorem (see Chapter 6) and also for some Poincaré inequalities (see Lemma 2.10), it will be important that these functions are somehow regular in time.

One way to measure this regularity lies in the so called Nikolskii spaces. It can be shown that piecewise constant functions on an equidistant time-grid belong to these spaces of order $1/q$ with L^q-regularity. For $0 < \sigma < 1$, $1 \leq q < \infty$, we define the space

$$N^{\sigma,q}(0,T;X) = \big\{ v \in L^q(0,T;X) : |v|_{N^{\sigma,q}(0,T;X)} < \infty \big\}$$

with

$$|v|_{N^{\sigma,q}(0,T;X)} = \operatorname*{ess\,sup}_{0<h<T} \left(\int_0^{T-h} \frac{\|v(t+h) - v(t)\|_X^q}{h^{\sigma q}} \,\mathrm{d}t \right)^{1/q}$$

and for $q = \infty$

$$|v|_{N^{\sigma,\infty}(0,T;X)} = \operatorname*{ess\,sup}_{0<h<T} \frac{\operatorname{ess\,sup}_{t\in(0,T-h)} \|v(t+h) - v(t)\|_X}{h^{\sigma}}.$$

Equipped with the norm

$$\|v\|_{N^{\sigma,q}(0,T;X)} = \|v\|_{L^q(0,T;X)} + |v|_{N^{\sigma,q}(0,T;X)}$$

this is a Banach space.

Since these spaces are not very easy to handle in our applications, we introduce another class of fractional order spaces. These are called fractional Sobolev or Sobolev-Slobodeckii spaces. For $0 < \sigma < 1$, $1 \leq q < \infty$ we define

$$W^{\sigma,q}(0,T;X) = \big\{ v \in L^q(0,T;X) : |v|_{W^{\sigma,q}(0,T;X)} < \infty \big\}$$

with

$$|v|_{W^{\sigma,q}(0,T;X)} = \left(\int_0^T \int_0^T \frac{\|v(t) - v(s)\|_X^q}{|t-s|^{1+\sigma q}} \,\mathrm{d}s \,\mathrm{d}t \right)^{1/q}$$

and for $q = \infty$

$$|v|_{W^{\sigma,\infty}(0,T;X)} = \operatorname*{ess\,sup}_{\substack{t,s\in(0,T)\\ t\neq s}} \frac{\|v(t) - v(s)\|_X}{|t-s|^{\sigma}}.$$

This is a Banach space if equipped with the norm

$$\|v\|_{W^{\sigma,q}(0,T;X)} = \|v\|_{L^q(0,T;X)} + |v|_{W^{\sigma,q}(0,T;X)}.$$

If X is a Hilbert space and $q = 2$, then $W^{\sigma,2}(0,T;X)$ is a Hilbert space with the inner product

$$(v,w)_{\sigma,2,X} = (v,w)_{L^2(0,T;X)} + (v,w)_{\sigma,2,X}$$

for

$$(v,w)_{\sigma,2,X} = \int_0^T \int_0^T \frac{(v(t) - v(s), w(t) - w(s))_X}{|t-s|^{1+2\sigma}} \,\mathrm{d}t \,\mathrm{d}s\,.$$

More about Sobolev-Slobodeckii spaces can be found in [3], [55], [59] and [23].

A connection between Nikolskii and Sobolev-Slobodeckii spaces has been studied in [59, Corollary 24]. There, it has been shown that the Nikolskii spaces are continuously embedded in the Sobolev-Slobodeckii spaces, if the order is slightly decreased.

Lemma 2.3. *Suppose $\bar{\sigma} > \sigma$ and $1 \le q \le \infty$. Then*

$$W^{\bar{\sigma},q}(0,T;X) \hookrightarrow N^{\bar{\sigma},q}(0,T;X) \hookrightarrow W^{\sigma,q}(0,T;X).$$

It turns out, that a fractional derivative in the above sense gives a sufficient amount of information to obtain compactness results in L^p-spaces. Indeed, there is a result from Simon [58] which is a generalization of the famous Lions-Aubin lemma (cf. Lemma 2.9) in such fractional spaces. Let us consider the following special case:

Lemma 2.4. *For $0 < \sigma < 1$, $1 \le q \le \infty$ and $p > \frac{2d}{d+2}$, the space*

$$L^p(0,T;W^{1,p}(\Omega)^d) \cap W^{\sigma,q}(0,T;L^2(\Omega)^d)$$

equipped with the norm

$$\|\cdot\|_{L^p(0,T;W^{1,p}(\Omega)^d)} + \|\cdot\|_{W^{\sigma,q}(0,T;L^2(\Omega)^d)}$$

is compactly embedded into $L^s(0,T;L^2(\Omega)^d)$ for all $s < q/(1-\sigma q)$. In particular, we can always take $s = q$.

Proof. The proof is an application of [58, Corollary 2] with $s_0 = 0$, $r_0 = p$, $s = \sigma$ and $r = q$. ☐

In fact, one can even achieve a compact embedding into fractional Sobolev spaces instead of Lebesgue spaces. Again, we will only consider a special case which will be of interest in this work.

Lemma 2.5. *For $1 \le q \le \infty$ and $p > \frac{2d}{d+2}$, the space*

$$L^p(0,T;W^{1,p}(\Omega)^d) \cap W^{\sigma,q}(0,T;L^2(\Omega)^d)$$

equipped with the norm

$$\|\cdot\|_{L^p(0,T;W^{1,p}(\Omega)^d)} + \|\cdot\|_{W^{\sigma,q}(0,T;L^2(\Omega)^d)}$$

is compactly embedded into $W^{\bar{\sigma},q}(0,T;L^2(G)^d)$ for all $0 \le \bar{\sigma} < \sigma$.

Proof. The proof is a consequence of an embedding result of Amann [5, Theorem 5.2]. There, we choose $E_1 = W^{1,p}(\Omega)^d$, $E_0 = L^2(\Omega)^d$ and E some interpolation space with θ sufficiently close to zero. Then we take $n = 1$, $s_0 = \sigma$, $p_0 = q$, $s_1 = 0$, $p_1 = p$, $s = \bar{\sigma}$ and $p = q$ to show the compact embedding into $W^{\bar{\sigma},q}(0,T;E)$. This works if

$$\bar{\sigma} < (1-\theta)\sigma, \qquad \bar{\sigma} < (1-\theta)\sigma + \theta\left(\frac{1}{q} - \frac{1}{p}\right)$$

which is true for θ close enough to zero. The embedding $E \hookrightarrow L^2(\Omega)^d$ then completes the proof. ☐

Remark 2.6. Note that the Sobolev-Slobodeckii and Nikolskii spaces (and even Lebesgue and Hölder spaces) may be captured in a larger scale of function spaces called Besov spaces $B^s_{p,q}$ and Triebel-Lizorkin spaces $F^s_{p,q}$. Interesting for us are the identities

$$\begin{aligned} W^{m,p} &= F^m_{p,2} && \text{for } 1 < p < \infty, \quad m = 1, 2, \dots, \\ W^{s,p} &= F^s_{p,p} = B^s_{p,p} && \text{for } 1 \le p \le \infty, \quad 0 < s \ne \text{integer}, \\ N^{s,p} &= B^s_{p,\infty}. \end{aligned}$$

For a closer look on Besov, Triebel-Lizorkin spaces and other interpolation spaces, we refer to [64] and [55].

2.5 Basic inequalities and functional analytic tools

Due to the discrepancy between the derivatives in the convection term and in the diffusion term, one can work with two different norms of $W^{1,p}_0(\Omega)^d$. In [22], the authors use the norm

$$\|Dv\|_{L^p(\Omega)^{d\times d}}$$

in contrast to the usual norm

$$\|\nabla v\|_{L^p(\Omega)^{d\times d}},$$

that we will utilize. By Korn's well-known first inequality (see [44, Theorem 1.10, p. 196]) these two norms are equivalent on $W^{1,p}_0(\Omega)^d$:

Lemma 2.7. *There is a constant $c_1 > 0$ such that*

$$\|\nabla v\|_{L^p(\Omega)^{d\times d}} \le c_1 \|Dv\|_{L^p(\Omega)^{d\times d}}$$

for all $v \in W^{1,p}_0(\Omega)^d$.

It is often useful to interpolate between Lebesgue spaces. Especially when working with the convection term, we will employ the following interpolatory inequality.

Lemma 2.8. *Let $1 \le q \le p \le \infty$ and $v \in L^p(\Omega)^d$. Then for $\vartheta \in [0,1]$*

$$\|v\|_{L^s(\Omega)^d} \le \|v\|^{\vartheta}_{L^p(\Omega)^d} \|v\|^{1-\vartheta}_{L^q(\Omega)^d} \tag{2.3}$$

holds if $\frac{1}{s} = \frac{\vartheta}{p} + \frac{1-\vartheta}{q}$.

Proof. This inequality directly follows from the Hölder's inequality with $|v|^s = |v|^{\vartheta s}|v|^{s(1-\vartheta)}$. □

We will conclude this section with the well-known Lions-Aubin compactness lemma (see for example [41, Chapter 1, Theorem 5.1, p. 58], [25, Theorem 8.1.12] [56, Proposition III.2.1.2], [54, Lemma 3.74] or for generalized versions [53, Section 7.3]).

Lemma 2.9. *Let X_1, X_0 and X_{-1} be Banach spaces and let X_1, X_{-1} be reflexive. If $X_1 \overset{c}{\hookrightarrow} X_0 \hookrightarrow X_{-1}$, then for $1 < r, s < \infty$, the Banach space*

$$\{v \in L^r(0,T;X_1) : v' \in L^s(0,T;X_{-1})\}$$

imposed with the norm

$$\|v\|_{L^r(0,T;X_1)} + \|v'\|_{L^s(0,T;X_{-1})}$$

is compactly embedded into $L^r(0,T;X_0)$.

2.6 Critical exponents

At this point, we should make some remarks on possible values of the parameter p. As mentioned above, the assumption

$$p > \frac{2d}{d+2} \tag{2.4}$$

is needed for the embedding $V_p \overset{c}{\hookrightarrow} H$. For smaller p, the variational approach does not make sense and particularly the convection term $u \otimes u$ is not integrable, see Example 3.7.

In order to derive suitable a priori estimates for a steady state approximate solution, one usually tests the weak formulation of the problem with the approximate solution itself. For us in particular, this means to evaluate the term

$$\int_\Omega u \otimes u : \nabla u \, \mathrm{d}x$$

for $u \in V_p$. Since $\nabla u \in L^p(\Omega)^{d\times d}$, we have to ensure that $u \otimes u$ belongs to $L^{p'}(\Omega)^{d\times d}$. This is true for $u \in L^{2p'}(\Omega)^d$. The required exponent for this is

$$p \geq \frac{3d}{d+2}.$$

In the time-dependent case, one has to consider a suitable representation of the convection term, i.e.

$$\int_0^T \int_\Omega u \otimes u : \nabla \varphi \, \mathrm{d}x \; \mathrm{d}t \, .$$

One of the important techniques to show existence of solutions is the "decisive monotonicity trick" or "Minty's trick". In the original version of this method, one has to ensure that the convection term with $\varphi = u$ is well-defined. As we will only be able to show $u \in L^\infty(0,T;H) \cap L^p(0,T;V_p)$, we need additional assumptions on p to employ this technique. By parabolic interpolation (see Lemma 3.8) one can show $u \in L^{2p'}(0,T;L^{2p'}(\Omega)^d)$ for

$$p \geq 1 + \frac{2d}{d+2}.$$

The Lipschitz truncation method, that will be employed in this work, delivers an alternative to the usual decisive monotonicity trick without using this assumption.

2.7 Poincaré inequalities

One advantage of the Sobolev-Slobodeckii space is, that one can rather easily derive Poincaré-type inequalities. Note here, that the exponent of the constant in front of the right-hand side equals the order of the space.

Lemma 2.10. *Let $(X, \|\cdot\|_X)$ be a Banach space and let $u \in W^{\sigma,1}(a,b;X)$. Then*

$$\int_a^b \|u(t) - u_{[a,b]}\|_X \, \mathrm{d}t \leq (b-a)^\sigma |u|_{W^{\sigma,1}(a,b;X)}.$$

Proof. There holds

$$\begin{aligned}\int_a^b \|u(t) - u_{[a,b]}\|_X \,\mathrm{d}t &= \int_a^b \left\| \frac{1}{|b-a|} \int_a^b u(t) - u(s)\,\mathrm{d}s \right\|_X \mathrm{d}t \\ &\le \frac{1}{|b-a|} \int_a^b \int_a^b \|u(t) - u(s)\|_X \,\mathrm{d}s\ \mathrm{d}t \\ &\le \frac{1}{|b-a|} \int_a^b \int_a^b \frac{\|u(t) - u(s)\|_X}{|t-s|^{1+\sigma}} |t-s|^{1+\sigma} \,\mathrm{d}s\ \mathrm{d}t \\ &\le \frac{1}{|b-a|} (b-a)^{1+\sigma} \int_a^b \int_a^b \frac{\|u(t) - u(s)\|_X}{|t-s|^{1+\sigma}} \,\mathrm{d}s\ \mathrm{d}t \\ &\le (b-a)^\sigma |u|_{W^{\sigma,1}(a,b;X)}.\end{aligned}$$

□

This result can be extended to a Poincaré-type inequality in space and time.

Lemma 2.11. *For $a < b \in \mathbb{R}$ and $B_r \subset \mathbb{R}^d$ a Euklidean ball with radius r, let $u \in L^1(a,b;W^{1,1}(B_r)^d) \cap W^{\sigma,1}(a,b;L^1(B_r)^d)$. Then for $Q = B_r \times (a,b)$ holds*

$$\int_Q |u(x,t) - u_Q| \,\mathrm{d}(x,t) \le c\,r\,\|\nabla u\|_{L^1(a,b;L^1(B_r)^{d\times d})} + (b-a)^\sigma |u|_{W^{\sigma,1}(a,b;L^1(B_r)^d)},$$

where c only depends on the dimension d.

Proof. Let us start by inserting $u_{B_r}(t) = ⨍_{B_r} u(y,t)\,\mathrm{d}y$. This gives

$$\int_Q |u(x,t) - u_Q| \,\mathrm{d}(x,t) \le \int_Q |u(x,t) - u_{B_r}(t)| \,\mathrm{d}(x,t) + \int_Q |u_{B_r}(t) - u_Q| \,\mathrm{d}(x,t).$$

Then we write for the first integral

$$\int_Q |u(x,t) - u_{B_r}(t)| \,\mathrm{d}(x,t) = \int_a^b \int_{B_r} |u(x,t) - u_{B_r}(t)| \,\mathrm{d}x\ \mathrm{d}t$$

and use the usual Poincaré inequality (see for example [27, Section 5.8.1, Theorem 2]) to bound this term by

$$c\,r \int_a^b \int_{B_r} |\nabla u(x,t)| \,\mathrm{d}x\ \mathrm{d}t = c\,r\,\|\nabla u\|_{L^1(a,b;L^1(B_r)^{d\times d})}.$$

We note, that the integrand of the second integral does not depend on x. Thus the spatial integral gives a factor $\mu_d(B_r)$. We then find

$$\begin{aligned}\int_Q |u_{B_r}(t) - u_Q| \,\mathrm{d}(x,t) &= \mu_d(B_r) \int_a^b \left| ⨍_{B_r} u(y,t)\,\mathrm{d}y - ⨍_a^b ⨍_{B_r} u(y,s)\,\mathrm{d}y\ \mathrm{d}s \right| \mathrm{d}t \\ &\le \int_a^b \int_{B_r} |u(y,t) - u_{[a,b]}(y)| \,\mathrm{d}y\ \mathrm{d}t \\ &\le \int_a^b \|u(t) - u_{[a,b]}\|_{L^1(B_r)^d} \,\mathrm{d}t.\end{aligned}$$

The Poincaré inequality in time (see Lemma 2.10) then yields

$$\int_Q |u_{B_r}(t) - u_Q| \,\mathrm{d}(x,t) \;\le (b-a)^\sigma |u|_{W^{\sigma,1}(a,b;L^1(B_r)^d)}.$$

This finishes the proof. □

3

Weak formulation and operator differential equation

Let $p > \frac{2d}{d+2}$ and $T \in (0, \infty)$ throughout this chapter.

Before we state the weak formulation of (1.4)-(1.7), let us make a remark about the space $C_w([0,T];X)$ of demicontinuous abstract functions taking values in a Banach space X (see for example [53, Definition 2.3] for a definition of demicontinuity).

Lemma 3.1. *Let X, Y be Banach spaces with $X \hookrightarrow Y$ and let X be reflexive. Then from*

$$v \in L^\infty(0,T;X) \cap C_w([0,T];Y)$$

follows

$$v \in C_w([0,T];X).$$

In particular, if a function $v \in L^\infty(0,T;H)$ possesses a weak derivative $v' \in L^{r'}(0,T;V_r')$ for some $\frac{2d}{d+2} \le r < \infty$, then $v \in C_w([0,T];H)$.

Proof. A proof of the first part can be found in [24, Lemma A 2.1].

For the second part, we consider

$$\begin{aligned} v &\in L^\infty(0,T;H) \hookrightarrow L^{r'}(0,T;V_r'), \\ v' &\in L^{r'}(0,T;V_r') \end{aligned}$$

and by [24, Remark A 1.5] we then have

$$v \in C([0,T];V_r') \subset C_w([0,T];V_r').$$

Application of the first part gives

$$v \in C_w([0,T];H).$$

□

To derive the weak formulation of the system (1.4)-(1.7), one multiplies (1.4) by a function $\varphi \in \mathcal{V}$ and integrates over the domain Ω. With application of the rule of integration by parts, the pressure term $\nabla\pi$ vanishes because φ is solenoidal (see [62, Paragraph 1, Section 1.4]). Equations (1.5) and (1.6) are contained in the formulation of the spaces V_p and H.

Looking at the diffusion term, we note that for an arbitrary $\mathbb{R}^{d\times d}$-valued function $A = A(x)$ and smooth φ holds

$$-\int_\Omega (\operatorname{div} A)\cdot\varphi \,\mathrm{d}x = \int_\Omega A^T : \nabla\varphi \,\mathrm{d}x\,.$$

Thus the symmetry of the stress tensor implies

$$-\int_\Omega (\operatorname{div} S(Du))\cdot\varphi \,\mathrm{d}x = \int_\Omega S(Du) : \nabla\varphi \,\mathrm{d}x = \int_\Omega S(Du) : D\varphi \,\mathrm{d}x\,.$$

The weak formulation of (1.4)-(1.7) then is as follows.

Definition 3.2. *Let $Q = \Omega\times(0,T)$, $f \in L^{p'}(0,T;(W_0^{1,p}(G)^d)')$ and $u_0 \in H$. Suppose, that $S = S(x,z)$ is a Carathéodory-function satisfying the growth condition (1.8). A $\mathbb{R}^d$-valued function $u \in C_w([0,T];H)\cap L^p(0,T;V_p)$ is called a weak solution to (1.4)-(1.7), iff $u(0) = u_0$ in H and*

$$-\int_Q u\cdot\varphi' \,\mathrm{d}(x,t) + \int_Q S(Du) : D\varphi \,\mathrm{d}(x,t) + \int_Q u\otimes u : \nabla\varphi \,\mathrm{d}(x,t) = \langle f,\varphi\rangle \quad (3.1)$$

holds for every $\varphi \in C_0^\infty(0,T;\mathcal{V})$.

3.1 Diffusion term

With the diffusion term appearing in equation (3.1), we associate the nonlinear operator

$$A : V_p \to V_p' \quad \text{with} \quad \langle Av, w\rangle = \int_\Omega S(Dv) : Dw \,\mathrm{d}x\,.$$

Lemma 3.3. *Let $S = S(x,z)$ be a Carathéodory function satisfying assumptions (1.8), (1.9) and (1.10). Then the nonlinear operator $A : V_p \to V_p'$ associated with it is strictly monotone, hemicontinuous[1], p-coercive and fulfils a growth condition, such that there are constants $c, c_0 > 0$ with*

$$\langle Av, v\rangle \geq c_0\|v\|_{V_p}^p, \quad \|Av\|_{V_p'} \leq c(1+\|v\|_{V_p})^{p-1} \quad (3.2)$$

holds for all $v \in V_p$.

Via $(Au)(t) = A(u(t))$ this operator $A : V_p \to V_p'$ extends to a strictly monotone, hemicontinuous, coercive operator

$$A : L^p(0,T;V_p) \to L^{p'}(0,T;V_p'),$$

that fulfils a growth condition, such that for any $v \in L^p(0,T;V_p)$ there are constants $c, c_0 > 0$ with

$$\langle Av, v\rangle \geq c_0\|v\|^p_{L^p(0,T;V_p)}, \quad \|Av\|_{L^{p'}(0,T;V_p')} \leq c(1+\|v\|_{L^p(0,T;V_p)})^{p-1}.$$

[1]See [53, Definition 2.3] for a definition of hemicontinuity.

Proof. The operator $A : V_p \to V_p'$ is well-defined, since with Hölder's inequality, Korn's inequality and the growth condition (1.8) we have for any $v, w \in V_p$

$$\begin{aligned}
|\langle Av, w\rangle| &\le \int_\Omega |S(Dv) : Dw|\,\mathrm{d}x \\
&\le \left(\int_\Omega |S(Dv)|^{p'}\,\mathrm{d}x\right)^{1/p'} \|Dw\|_{L^p(\Omega)^{d\times d}} \\
&\le \left(\int_\Omega c^{p'}(1+|Dv|))^p\,\mathrm{d}x\right)^{1/p'} c_1\|w\|_{V_p} \\
&\le c\,\|1+|Dv|\|^{p-1}_{L^p(\Omega)^{d\times d}}\|w\|_{V_p} \\
&\le c\,(1+\|v\|_{V_p})^{p-1}\|w\|_{V_p}.
\end{aligned}$$

This also shows the growth condition for A.

The monotonicity of A directly follows from the monotonicity of S (see (1.9)). For arbitrary $v, w \in V_p$, $v \neq w$, we calculate

$$\langle Av - Aw, v - w\rangle = \int_\Omega (S(Dv) - S(Dw)) : (Dv - Dw)\,\mathrm{d}x\; > 0.$$

From (1.10) directly follows the coercivity of A. Let $v \in V_p$ be arbitrary, then

$$\langle Av, v\rangle = \int_\Omega S(Dv) : Dv\,\mathrm{d}x\; \ge \int_\Omega \tilde{c_0}|Dv|^p\,\mathrm{d}x\; \le \tilde{c_0}c_1\|v\|^p_{V_p}$$

holds and we write $c_0 = \tilde{c_0}c_1$.

It remains to show the hemicontinuity of A. For this purpose, we consider $u, v, w \in V_p$ and $\tau \in [0,1]$ and employ Hölder's inequality to calculate

$$\begin{aligned}
&|\langle A(u+\tau v), w\rangle - \langle Au, w\rangle| \\
&\quad\le \int_\Omega |S(D(u+\tau v)) : Dw - S(Du) : Dw|\,\mathrm{d}x \\
&\quad\le \int_\Omega (S(D(u+\tau v) - S(Du)) : Dw|\,\mathrm{d}x \\
&\quad\le \left(\int_\Omega |S(D(u+\tau v)) - S(Du)|^{p'}\,\mathrm{d}x\right)^{1/p'} \|Dw\|_{L^p(\Omega)^{d\times d}}.
\end{aligned}$$

Now (1.8) provides an integrable majorant function, which together with the continuity of S makes it possible to apply Lebesgue's theorem of dominated convergence and show the convergence of

$$\int_\Omega |S(D(u+\tau v)) - S(Du)|^{p'}\,\mathrm{d}x\,.$$

This implies, that A maps Bochner-measurable functions $u : (0,T) \to V_p$ into Bochner-measurable functions $Au : (0,T) \to V_p'$.

The second part of the proof is standard (see for exmample [68, Theorem 30.A]). ☐

For simplicity, we will always write $Au(t)$ instead of $(Au)(t)$.

3.2 Convection term

3.2.1 Definition of the trilinear form b

Let us consider for any smooth functions u, v, w the trilinear form

$$b(u, v, w) = -\int_\Omega u \otimes v : \nabla w \, \mathrm{d}x \,.$$

Lemma 3.4. *Let $u, w \in C_0^\infty(\Omega)^d$ and $v \in \mathcal{V}$. Then*

$$b(u, v, w) = -b(w, v, u)$$

and in particular $b(v, v, v) = 0$.

Proof. We start by employing the product rule to show

$$\begin{aligned} b(u, v, w) &= -\int_\Omega \sum_{i,j=1}^{d} u_i v_j \partial_j w_i \, \mathrm{d}x \\ &= -\int_\Omega \sum_{i,j=1}^{d} \partial_j (u_i v_j w_i) + (\partial_j u_i) v_j w_i + u_i (\partial_j v_j) w_i \, \mathrm{d}x \\ &= -\int_\Omega \sum_{i=1}^{d} \operatorname{div} (v \, u_i w_i) + \sum_{i,j=1}^{d} w_i v_j \partial_j u_i + \sum_{i=1}^{d} u_i \operatorname{div} v \, w_i \, \mathrm{d}x \end{aligned}$$

Employing the Gauß-Green divergence theorem and $v \in \mathcal{V}$ yields

$$b(u, v, w) = -\int_{\partial\Omega} v \cdot \eta \, u \cdot w \, \mathrm{d}x \, - b(w, v, u) = -b(w, v, u),$$

where η is the outward normal on the boundary of Ω. □

Lemma 3.5. *Let $1 \le \alpha, \beta, r \le \infty$. For smooth u, v, w, the trilinear form b satisfies the estimates*

$$|b(u, v, w)| \le c \, \|u\|_{H_\alpha} \|v\|_{H_\beta} \|w\|_{V_r}$$

for $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{r'}$ and

$$|b(v, v, w)| \le c \, \|v\|_{H_{2r'}} \|w\|_{V_r}.$$

Proof. The inequality follows by application of Hölder's inequality. □

Remark 3.6. Lemma 3.5 implies, that the trilinear form b can be uniquely extended to a bounded trilinear form

$$b : H_\alpha \times H_\beta \times V_r \to \mathbb{R}.$$

Then the results of Lemma 3.4 hold also true for $u \in H_\alpha$, $v \in H_\beta$ and $w \in V_r$. In particular for $u \in V_r \cap H_{2r'}$, we have

$$b(u, u, u) = 0.$$

For $u \in V_p$, this is certainly true, if $p \ge 3d/(d + 2)$ as then $V_p \hookrightarrow H_{2p'}$.

3.2.2 Counter example

A simple example can show that in general, the expression $b(u, u, u)$ is not well-defined for $p < 3d/(d + 2)$.

Example 3.7. Let us consider the domain $\Omega = \{(x, y)^T \in \mathbb{R}^2 : \sqrt{2x^2 + y^2} < 1\}$ and the function $u : \Omega \to \mathbb{R}^2$ defined through

$$u\begin{pmatrix} x \\ y \end{pmatrix} = \left(1 - \left|\begin{pmatrix} \sqrt{2}x \\ y \end{pmatrix}\right|\right) \left|\begin{pmatrix} \sqrt{2}x \\ y \end{pmatrix}\right|^{\alpha-1} \begin{pmatrix} y \\ -2x \end{pmatrix}.$$

This function vanishes on the boundary of Ω and has a singularity of order $\alpha < 0$ in the origin. Without the factor 2, the example would be trivial, since then the convective term would vanish pointwise. For better readability, let us write $R = \sqrt{2x^2 + y^2}$. One easily calculates the partial derivatives

$$\begin{aligned}
\partial_x u_1 \begin{pmatrix} x \\ y \end{pmatrix} &= -2xy\, R^{\alpha-2} + 2xy\,(\alpha - 1)(1 - R)R^{\alpha-3} \\
\partial_y u_1 \begin{pmatrix} x \\ y \end{pmatrix} &= -y^2\, R^{\alpha-2} + y^2\,(\alpha - 1)(1 - R)R^{\alpha-3} + (1 - R)R^{\alpha-1} \\
\partial_x u_2 \begin{pmatrix} x \\ y \end{pmatrix} &= 4x^2\, R^{\alpha-2} - 4x^2\,(\alpha - 1)(1 - R)R^{\alpha-3} - 2(1 - R)R^{\alpha-1} \\
\partial_y u_2 \begin{pmatrix} x \\ y \end{pmatrix} &= 2xy\, R^{\alpha-2} - 2xy\,(\alpha - 1)(1 - R)R^{\alpha-3}.
\end{aligned}$$

We see, that $\operatorname{div} u = 0$ and $u \in W_0^{1,p}(\Omega)^d$ if the gradient belongs to $L^p(\Omega)^{d\times d}$, which is true, basically, if $R^{\alpha-1} \in L^p(\Omega)$. Since $\Omega \subset \mathbb{R}^2$, we have to assume

$$p(\alpha - 1) > -2 \Leftrightarrow \alpha > -\frac{2}{p} + 1$$

for this to hold.

The convection term for this function reads

$$\int_\Omega u \otimes u : \nabla u \, \mathrm{d}(x, y) \; = \int_\Omega -2xy(1 - R)^3 R^{3\alpha-3} \, \mathrm{d}(x, y)\,.$$

We substitute $\tilde{x} = \sqrt{2}x$ and receive, again writing x instead of $\tilde{x}$,

$$\int_{B_1(0)} -xy(1 - \sqrt{x^2 + y^2})^3 (\sqrt{x^2 + y^2})^{3\alpha-3} \, \mathrm{d}(x, y)\,.$$

Splitting this integral into positive and negative parts I_+, I_- yields after changing to polar coordinates and applying Fubini's theorem

$$\begin{aligned}
I_+ &= \int_{\frac{\pi}{2}}^{\pi} \int_0^1 -rr\cos(\varphi)\, r\sin(\varphi)(1 - r)^3 r^{3\alpha-3} \, \mathrm{d}r \; \mathrm{d}\varphi \\
&\quad + \int_{\frac{3\pi}{2}}^{2\pi} \int_0^1 -rr\cos(\varphi)\, r\sin(\varphi)(1 - r)^3 r^{3\alpha-3} \, \mathrm{d}r \; \mathrm{d}\varphi \\
&= -\int_{\frac{\pi}{2}}^{\pi} \cos(\varphi)\sin(\varphi) \int_0^1 (1 - r)^3 r^{3\alpha} \, \mathrm{d}r \; \mathrm{d}\varphi \\
&\quad - \int_{\frac{3\pi}{2}}^{2\pi} \cos(\varphi)\sin(\varphi) \int_0^1 (1 - r)^3 r^{3\alpha} \, \mathrm{d}r \; \mathrm{d}\varphi
\end{aligned}$$

and analogously

$$
\begin{aligned}
I_- = &-\int_0^{\frac{\pi}{2}} \cos(\varphi)\sin(\varphi)\int_0^1 (1-r)^3 r^{3\alpha}\, \mathrm{d}r\ \mathrm{d}\varphi \\
&-\int_\pi^{\frac{3\pi}{2}} \cos(\varphi)\sin(\varphi)\int_0^1 (1-r)^3 r^{3\alpha}\, \mathrm{d}r\ \mathrm{d}\varphi\,.
\end{aligned}
$$

By the definition of the Lebesgue-integral, this is only well-defined, if both parts I_- and I_+ are finite. Thus, there will be problems for

$$3\alpha \le -1 \Leftrightarrow \alpha \le -\frac{1}{3}.$$

This means, that we can find a counter example, if there is an α such that

$$-\frac{2}{p} + 1 < \alpha \le -\frac{1}{3}$$

which is possible for $p < 3/2 = 3d/(d+2)$.

3.2.3 Parabolic interpolation

Before we define the operator corresponding to the convection term in (3.1), we remark, that the Rellich-Kondrachov theorem (c.f. [3, Theorem 6.2] or [27, Section 5.7, Theorem 1]) states

$$W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega),\ \text{for } 1 \le s \le \frac{pd}{d-p}$$

for $p < d$. In particular, the embedding is compact if $s < \frac{pd}{d-p}$. Our assumption $p > \frac{2d}{d+2}$ immediately implies $2 < \frac{pd}{d-p}$. Thus, for any number $2 < 2r' < \frac{pd}{d-p}$, we know

$$W_0^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^{2r'}(\Omega). \tag{3.3}$$

For $p \ge d$ the space $W_0^{1,p}(\Omega)$ is compactly embedded into the space of continuous functions and thus also into every $L^s(\Omega)$ for $1 \le s \le \infty$.

Let us now determine a suitable number $1 < r' < \frac{pd}{2(d-p)}$. For this, we consider the following parabolic interpolation result:

Lemma 3.8. *Let $\{u_\nu\}_{\nu\in\mathbb{N}} \subset L^\infty(0,T;H) \cap L^p(0,T;V_p)$ be a bounded sequence.*

The embedding $L^\infty(0,T;H) \cap L^p(0,T;V_p) \hookrightarrow L^{p\frac{d+2}{d}}(0,T;H_{p\frac{d+2}{d}})$ holds with parabolic interpolation and we have

$$\|v\|_{L^{p\frac{d+2}{d}}(0,T;H_{p\frac{d+2}{d}})} \le c\,\|v\|_{L^p(0,T;V_p)}^{\frac{d}{d+2}} \|v\|_{L^\infty(0,T;H)}^{\frac{2}{d+2}} \tag{3.4}$$

for every $v \in L^\infty(0,T;H) \cap L^p(0,T;V_p)$. Thus, the sequence $\{u_\nu\}_{\nu\in\mathbb{N}}$ is bounded in $L^{p\frac{d+2}{d}}(0,T;H_{p\frac{d+2}{d}})$.

Further, if there is a function $u \in L^\infty(0,T;H) \cap L^p(0,T;V_p)$ such that $u_\nu \to u$ in $L^2(0,T;H)$ as ν tends to infinity, then

$$u_\nu \to u \quad \text{in } L^{2r'}(0,T;H_{2r'}) \quad \text{as} \quad \nu \to \infty,$$

for $1 < r' < p\frac{d+2}{2d}$.

Proof. Let $\{u_\nu\}_{\nu\in\mathbb{N}}$ and u be as in the assumptions of the lemma. With (3.3) and $p > \frac{2d}{d+2}$, we can use the interpolatory inequality (2.3) for almost every $t \in (0,T)$:

$$\begin{aligned}\|u_\nu(t)-u(t)\|_{H_{2r'}} &\leq \|u_\nu(t)-u(t)\|^{\vartheta}_{H_{\frac{pd}{d-p}}} \|u_\nu(t)-u(t)\|^{1-\vartheta}_{H} \\ &\leq c\,\|u_\nu(t)-u(t)\|^{\vartheta}_{V_p} \|u_\nu(t)-u(t)\|^{1-\vartheta}_{H}.\end{aligned}$$

for $\vartheta \in [0,1]$ and

$$\frac{1}{2r'} = \frac{\vartheta(d-p)}{pd} + \frac{1-\vartheta}{2}.$$

Choosing

$$\vartheta = \frac{d}{d+2}$$

implies $p = 2r'\vartheta$ and $2r' = p\frac{d+2}{d}$. In this case, we have

$$\begin{aligned}\|u_\nu - u\|^{2r'}_{L^{2r'}(0,T;H_{2r'})} &= \int_0^T \|u_\nu(t)-u(t)\|^{2r'}_{H_{2r'}}\,\mathrm{d}t \\ &\leq \int_0^T \|u_\nu(t)-u(t)\|^{2r'\vartheta}_{V_p} \|u_\nu(t)-u(t)\|^{2r'(1-\vartheta)}_{H}\,\mathrm{d}t \\ &\leq \|u_\nu - u\|^{p}_{L^p(0,T;V_p)} \|u_\nu - u\|^{2r'(1-\vartheta)}_{L^\infty(0,T;H)}\end{aligned}$$

and hence boundedness in $L^{p\frac{d+2}{d}}(0,T;H_{p\frac{d+2}{d}})$. The interpolation inequality is analogous.

We further want to show convergence and this means choosing

$$\vartheta < \frac{d}{d+2},$$

which results in

$$p > 2r'\vartheta \quad \text{and} \quad 2r' < p\frac{d+2}{d}.$$

Then we apply Hölder's inequality with $q = p/(2r'\vartheta)$ to estimate

$$\begin{aligned}\|u_\nu - u\|^{2r'}_{L^{2r'}(0,T;H_{2r'})} &= \int_0^T \|u_\nu(t)-u(t)\|^{2r'}_{H_{2r'}}\,\mathrm{d}t \\ &\leq \int_0^T \|u_\nu(t)-u(t)\|^{2r'\vartheta}_{V_p} \|u_\nu(t)-u(t)\|^{2r'(1-\vartheta)}_{H}\,\mathrm{d}t \\ &\leq \|u_\nu - u\|^{2r'\vartheta}_{L^p(0,T;V_p)} \|u_\nu - u\|^{2r'(1-\vartheta)q'}_{L^{2r'(1-\vartheta)q'}(0,T;H)}.\end{aligned}$$

Since $u_\nu \to u$ in $L^2(0,T;H)$ and $\{u_\nu\}_{\nu\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H)$, we know

$$u_\nu \to u \quad \text{in } L^s(0,T;H)$$

for any $s \in [1,\infty)$, especially for $s = 2r'(1-\vartheta)q'$. Thus, the boundedness of $\{u_\nu\}_{\nu\in\mathbb{N}}$ in $L^p(0,T;V_p)$ implies that

$$\|u_\nu - u\|^{2r'}_{L^{2r'}(0,T;H_{2r'})} \leq \|u_\nu - u\|^{2r'\vartheta}_{L^p(0,T;V_p)} \|u_\nu - u\|^{2r'(1-\vartheta)}_{L^{2(1-\vartheta)r'}(0,T;H)},$$

converges to zero as ν tends to infinity. □

3.2.4 Definition of the operator B

Let us now fix such a number $1 < r' < p\frac{d+2}{2d}$. This number will stay fixed throughout this article. We define the operator

$$B : H_{2r'} \to V_r', \quad \langle Bv, w\rangle = b(v,v,w) = \int_\Omega v \otimes v : \nabla w \,\mathrm{d}x \tag{3.5}$$

representing the convection term in the weak formulation (3.1). If

$$p > 1 + \frac{2d}{d+2},$$

we will always choose $r = p$. In any other case, it is $r > p$.

Lemma 3.9. *Assuming* (2.4)*, the operator B as a mapping of $H_{2r'}$ into V_r' is well-defined, bounded and continuous. There is a constant $c > 0$ such that*

$$\|Bv\|_{V_r'} \leq c\,\|v\|_{H_{2r'}}^2. \tag{3.6}$$

Via $(Bv)(t) = B(v(t))$ this operator extends to a bounded continuous operator

$$B : L^{2r'}(0,T;H_{2r'}) \to L^{r'}(0,T;V_r')$$

fulfilling the growth estimate

$$\|Bv\|_{L^{r'}(0,T;V_r')} \leq c\,\|v\|_{L^{2r'}(0,T;H_{2r'})}^2.$$

Proof. Using Hölder's inequality, we find for arbitrary $v \in H_{2r'}$ and $w \in V_p$

$$\begin{aligned}
|\langle Bv, w\rangle| &\leq \int_\Omega |v \otimes v : \nabla w| \,\mathrm{d}x \\
&\leq \left(\int_\Omega |v \otimes v|^{r'} \,\mathrm{d}x\right)^{1/r'} \left(\int_\Omega |\nabla w|^r \,\mathrm{d}x\right)^{1/r} \\
&\leq \left(\int_\Omega |v|^{2r'} \,\mathrm{d}x\right)^{1/r'} \|w\|_{V_r} \\
&\leq \|v\|_{H_{2r'}}^2 \|w\|_{V_r}.
\end{aligned}$$

Hence, B is well-defined, bounded and fulfils the growth condition (3.6).

Let $\{v_\nu\}_{\nu\in\mathbb{N}} \subset H_{2r'}$, $v \in H_{2r'}$ such that $\|v_\nu - v\|_{H_{2r'}} \to 0$ for $\nu \to \infty$. Then with Hölder's inequality, it is easy to show

$$\|Bv_\nu - Bv\|_{V_r'} \leq \|v_\nu \otimes v_\nu - v \otimes v\|_{L^{r'}(\Omega)^{d\times d}}.$$

With the definition of the dyadic product we estimate pointwise

$$\begin{aligned}
|v_\nu \otimes v_\nu - v \otimes v|^{r'} &= \sum_{i,j=1}^d |(v_\nu)_i (v_\nu)_j - v_i v_j|^{r'} \\
&\leq c \sum_{i,j=1}^d |(v_\nu)_i|^{r'} |(v_\nu)_j - v_j|^{r'} + |v_j|^{r'} |(v_\nu)_i - v_i|^{r'} \\
&= c \sum_{i=1}^d |(v_\nu)_i|^{r'} \sum_{j=1}^d |(v_\nu)_j - v_j|^{r'} + \sum_{j=1}^d |v_j|^{r'} \sum_{i=1}^d |(v_\nu)_i - v_i|^{r'} \\
&= c\left(|v_\nu|^{r'} |v_\nu - v|^{r'} + |v|^{r'} |v_\nu - v|^{r'}\right),
\end{aligned}$$

where $(v_\nu)_i$ denotes the i-th component of the vector $v_\nu(x,t) \in \mathbb{R}^d$. The declaration of the point (x,t) has been omitted in favour of better readability.

Thus, employing Hölder's inequality yields

$$\begin{aligned}
&\|v_\nu \otimes v_\nu - v \otimes v\|_{L^{r'}(\Omega)^{d\times d}} \\
&\leq c \left(\int_\Omega |v_\nu|^{r'} |v_\nu - v|^{r'} + |v|^{r'} |v_\nu - v|^{r'} \,\mathrm{d}x \right)^{1/r'} \\
&\leq c \left(\left(\int_\Omega |v_\nu|^{2r'} \,\mathrm{d}x \right)^{1/2} \left(\int_\Omega |v_\nu - v|^{2r'} \,\mathrm{d}x \right)^{1/2} \right)^{1/r'} \\
&\quad + c \left(\left(\int_\Omega |v|^{2r'} \,\mathrm{d}x \right)^{1/2} \left(\int_\Omega |v_\nu - v|^{2r'} \,\mathrm{d}x \right)^{1/2} \right)^{1/r'} \\
&= c \left(\|v_\nu\|_{H_{2r'}} \|v_\nu - v\|_{H_{2r'}} + \|v\|_{H_{2r'}} \|v_\nu - v\|_{H_{2r'}} \right).
\end{aligned}$$

This implies the continuity of $B : H_{2r'} \to V_r'$.

Let $v = v(t)$, $v \in L^{2r'}(0,T;H_{2r'})$, then $Bv : (0,T) \to V_r'$ is Bochner-measurable by virtue of the continuity of the operator $B : H_{2r'} \to V_r'$. The boundedness and the growth estimate follow easily from the results above. The continuity can be shown analogously to above. □

For simplicity, we will always write $Bu(t)$ instead of $(Bu)(t)$.

Remark 3.10. With the compact embedding $V_p \overset{c}{\hookrightarrow} H_{2r'}$ for $p > \frac{2d}{d+2}$, we can regard B also as a strongly continuous operator mapping V_p into V_r'.

Remark 3.11. The condition $r' < p\frac{d+2}{d}$ is only necessary for the time-dependent case. If one is only interested in the stationary case, the suitable condition is $r' < \frac{pd}{2(d-p)}$ as then (3.3) holds. The results for the stationary case in Lemma 3.9 then still hold true.

3.3 Operator differential equation

With the above definitions, we can bring (3.1) in a convenient operator-formulation. However, we first rewrite Definition 3.1 in terms of the newly defined operators. Equation (3.1) is equivalent to

$$-\langle u, \varphi' \rangle + \langle Au, \varphi \rangle + \langle Bu, \varphi \rangle = \langle f, \varphi \rangle, \quad \varphi \in C_0^\infty(0,T;\mathcal{V}). \tag{3.7}$$

The dense embedding $\mathcal{V} \overset{d}{\hookrightarrow} V_r$ implies, that the distributional time derivative of u belongs to $L^{r'}(0,T;V_r')$ and that the equation (3.7) is equivalent to the operator differential equation

$$u' + Au + Bu = f \quad \text{in } L^{r'}(0,T;V_r').$$

This gives rise to the following definition of a weak solution to (1.4)-(1.7):

Definition 3.12. *Let $f \in L^{p'}(0,T;(W_0^{1,p}(G)^d)')$, $u_0 \in H$ and the operators A and B as above. The function $u \in C_w([0,T];H) \cap L^p(0,T;V_p)$ is said to be a weak solution to* (1.4)-(1.7)*, iff $u(0) = u_0$ in H and*

$$u' + Au + Bu = f \quad \text{in } L^{r'}(0,T;V_r'). \tag{3.8}$$

4

Approximation of the convection term

For our first approach to find a solution to (3.8), we follow the ideas of Wolf [67, Section 3] and approximate the convection term B. This will allow us to solve the approximated equation with standard methods of the theory of monotone operators paired with a fixed point argument. The approximate problem will be solvable for all $p > \frac{2d}{d+2}$.

4.1 Approximation

The idea behind the approximation is cutting off the integrand of the convection term to gain enough regularity to be able to employ a fixed point argument in H. In particular, we would like to construct an operator $B_\varepsilon : H \to V_p'$ approximating $B : H_{2r'} \to V_r'$ for small $\varepsilon > 0$.

For this, let $\Phi \in C^\infty([0,\infty))$ be a non-decreasing function with $0 \le \Phi \le 1$, $\Phi = 1$ in $[0,1]$, $\Phi = 0$ in $[2,\infty)$ and $0 \le -\Phi' \le 2$. Further, we define for $\varepsilon > 0$

$$\Phi_\varepsilon(t) := \Phi(\varepsilon t), \quad t \in [0,\infty).$$

Now let us consider the approximate system

$$\partial_t u_\varepsilon - \operatorname{div} S(Du_\varepsilon) + \operatorname{div}(\Phi_\varepsilon(|u_\varepsilon|^2) u_\varepsilon \otimes u_\varepsilon) + \nabla p = f \quad \text{in } \Omega \times (0,T), \tag{4.1}$$

$$\operatorname{div} u_\varepsilon = 0 \quad \text{in } \Omega \times (0,T), \tag{4.2}$$

$$u_\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{4.3}$$

$$u_\varepsilon(0) = u_0 \quad \text{in } \Omega. \tag{4.4}$$

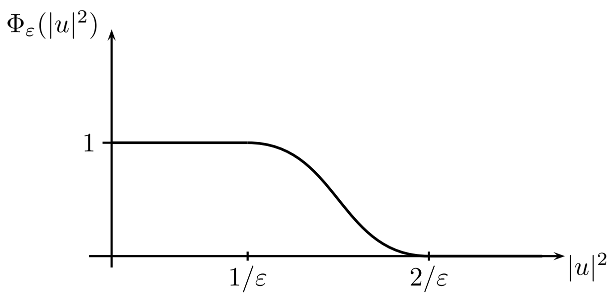

Figure 4.1: Cutoff function

In the weak formulation of this problem, the convection term is represented by the form

$$b_\varepsilon(u,v,w) = -\int_\Omega \Phi_\varepsilon(|u|^2) u \otimes v : \nabla w \, \mathrm{d}x\,.$$

Note, that in contrast to [67], we follow the approach of [18] and insert $|u_\varepsilon|^2$ into the cut-off function instead of $|u_\varepsilon|$.

The following lemma shows, that the convection term vanishes, when we insert only one divergence-free function into b_ε. With this property we will be able to derive suitable a priori estimates for the weak solution to (4.1)–(4.4).

Lemma 4.1. *For $v \in \mathcal{V}$ it is*

$$b_\varepsilon(v,v,v) = 0.$$

Proof. Let $v \in \mathcal{V}$. It easy to see, that the identity

$$\mathrm{div}\,(\Phi_\varepsilon(|w|^2) w \otimes w) = \mathrm{div}\,(\Phi_\varepsilon(|w|^2) w) w + \Phi_\varepsilon(|w|^2)(w\cdot\nabla)w$$

holds for $w \in C^1(\Omega)^d$. Thus, we have with integration by parts

$$\begin{aligned}
b_\varepsilon(v,v,v) &= \int_\Omega \Phi_\varepsilon(|v|^2) v\otimes v : \nabla v \, \mathrm{d}x \\
&= -\int_\Omega \mathrm{div}\,(\Phi_\varepsilon(|v|^2) v \otimes v)\cdot v \, \mathrm{d}x \\
&= -\int_\Omega \mathrm{div}\,(\Phi_\varepsilon(|v|^2) v) v \cdot v \, \mathrm{d}x - \int_\Omega \Phi_\varepsilon(|v|^2)(v\cdot\nabla)v\cdot v\, \mathrm{d}x \\
&= \int_\Omega \Phi_\varepsilon(|v|^2) v \cdot \nabla(|v|^2)\, \mathrm{d}x - \int_\Omega \Phi_\varepsilon(|v|^2) v\otimes v : \nabla v\, \mathrm{d}x \\
&= \int_\Omega \Phi_\varepsilon(|v|^2) v \cdot \nabla(|v|^2)\, \mathrm{d}x - b_\varepsilon(v,v,v).
\end{aligned}$$

Introducing the antiderivative $\tilde{\Phi}_\varepsilon$ of Φ_ε, we obtain with integration by parts

$$\begin{aligned}
b_\varepsilon(v,v,v) &= \frac{1}{2}\int_\Omega \Phi_\varepsilon(|v|^2) v\cdot\nabla(|v|^2)\, \mathrm{d}x \\
&= \frac{1}{2}\int_\Omega \nabla\big(\tilde{\Phi}_\varepsilon(|v|^2)\big)\cdot v\, \mathrm{d}x \\
&= -\frac{1}{2}\int_\Omega \tilde{\Phi}_\varepsilon(|v|^2)\, \mathrm{div}\, v\, \mathrm{d}x \\
&= 0,
\end{aligned}$$

since $\mathrm{div}\, v = 0$. □

Lemma 4.2. *For every $\varepsilon > 0$, the operator*

$$B_\varepsilon : H \to V_p', \quad \langle B_\varepsilon v, w\rangle = b_\varepsilon(v,v,w)$$

is well-defined, bounded and continuous. For $v \in V_p$ it is

$$\langle B_\varepsilon v, v\rangle = 0. \tag{4.5}$$

There exists a constant $c_\varepsilon > 0$ depending on ε, such that

$$|\langle B_\varepsilon v, w\rangle| = |b_\varepsilon(v,v,w)| \le c_\varepsilon \|v\|_H^{2/p'} \|w\|_{V_p}. \tag{4.6}$$

Via $(B_\varepsilon v)(t) \coloneqq B_\varepsilon(v(t))$ we extend $B_\varepsilon : H \to V_p$ to a bounded, continuous operator

$$B_\varepsilon : L^2(0,T;H) \to L^{p'}(0,T;V_p')$$

with

$$\|B_\varepsilon v\|_{L^{p'}(0,T;V_p')} \le c_\varepsilon \|v\|_{L^2(0,T;H)}^{2/p'},$$

where $c_\varepsilon = c\left(\frac{2}{\varepsilon}\right)^{1/(p-1)}$ for some $c > 0$. For a function $v \in L^p(0,T;V_p)$ it is

$$\langle B_\varepsilon v, v\rangle = 0.$$

Remark 4.3. In fact, we can show the following growth condition for any $w \in W_0^{1,s}(\Omega)^d$ and $v \in L^1(\Omega)^d$, $1 \le s \le \infty$:

$$|b_\varepsilon(v,v,w)| \le c\,\|\Phi_\varepsilon(|v|^2)^{1/2} v\|_{L^{2s'}(\Omega)^d}^2 \|w\|_{W_0^{1,s}(\Omega)^d} \le c\left(\frac{2}{\varepsilon}\right)\|w\|_{W_0^{1,s}(\Omega)^d},$$

where the constant c is independent of ε.

Proof. Let $v \in H$, $w \in V_p$. Eventually gaining some positive constants, we may estimate equivalent norms on $\mathbb{R}^d$ without explicitly writing them down. Hölder's inequality yields

$$\begin{aligned} |b_\varepsilon(v,v,w)| &\le \int_\Omega |\Phi_\varepsilon(|v|^2) v \otimes v : \nabla w| \,\mathrm{d}x \\ &\le \left(\int_\Omega |\Phi_\varepsilon(|v|^2) v\otimes v|^{p'} \,\mathrm{d}x\right)^{1/p'} \left(\int_\Omega |\nabla w|^p \,\mathrm{d}x\right)^{1/p}. \end{aligned} \tag{4.7}$$

For the first term we can estimate pointwise (omitting the argument x)

$$\begin{aligned} |\Phi_\varepsilon(|v|^2) v\otimes v|^{p'} &= \sum_{i,j=1}^d |\Phi_\varepsilon(|v|^2) v_i v_j|^{p'} \\ &= \Phi_\varepsilon(|v|^2)^{p'} \sum_{i=1}^d |v_i|^{p'} \sum_{j=1}^d |v_j|^{p'} \\ &= \Phi_\varepsilon(|v|^2)^{p'} |v|^{2p'}. \end{aligned}$$

To prove the remark, we simply follow the above argument with arbitrary $1 \le s \le \infty$ instead of p and consider

$$\Phi_\varepsilon(|v|^2)^{s'} |v|^{2s'} = |\Phi_\varepsilon(|v|^2)^{1/2} v|^{2s'} = (\Phi_\varepsilon(|v|^2)|v|^2)^{s'} \le \left(\frac{2}{\varepsilon}\right)^{s'}.$$

We now split the norm in one part resulting in the H-norm and another part being bounded:

$$\Phi_\varepsilon(|v|^2)^{p'} |v|^{2p'} = \Phi_\varepsilon(|v|^2)^{p'} |v|^2 |v|^{2p'-2} = |v|^2 \left(\Phi_\varepsilon(|v|^2)^p |v|^2\right)^{1/(p-1)}.$$

The properties of Φ_ε imply, that $\Phi_\varepsilon(|v|^2)^p|v|^2$ is bounded by $2/\varepsilon$ almost everywhere and thus we receive by estimating equivalent norms on $\mathbb{R}^d$

$$|\Phi_\varepsilon(|v|^2)v \otimes v|^{p'} \leq c \left(\frac{2}{\varepsilon}\right)^{1/(p-1)} |v|^2 = c_\varepsilon |v|^2.$$

Inserting this result into (4.7) yields the desired growth estimate

$$|b_\varepsilon(v,v,w)| \leq \left(\int_\Omega c_\varepsilon |v|^2 \,\mathrm{d}x\right)^{1/p'} \left(\int_\Omega |\nabla w|^p \,\mathrm{d}x\right)^{1/p} = c_\varepsilon \, \|v\|_H^{2/p'} \|w\|_{V_p}.$$

Let now $\{v_\nu\}_{\nu\in\mathbb{N}} \subset H$, $v \in H$ with $\|v_\nu - v\|_H \to 0$ as ν tends to infinity. To prove the continuity of $B : H \to V_p$ it is sufficient to show the convergence

$$\Phi_\varepsilon(|v_\nu|^2)v_\nu \otimes v_\nu \to \Phi_\varepsilon(|v|^2)v \otimes v \quad \text{in } L^{p'}(\Omega)^{d\times d}.$$

We will accomplish this by employing Lebesgue's theorem on dominated convergence.

We know, that $\{v_\nu\}_{\nu\in\mathbb{N}}$ contains a subsequence converging towards v almost everywhere in Ω. For simplicity, we will denote this subsequence again by $\{v_\nu\}_{\nu\in\mathbb{N}}$. It is easy to show, that the function $z \mapsto \Phi_\varepsilon(|z|^2)z \otimes z$ is continuous and hence,

$$\Phi_\varepsilon(|v_\nu|^2)v_\nu \otimes v_\nu \to \Phi_\varepsilon(|v|^2)v \otimes v \quad \text{a.e. in } \Omega.$$

In order to find an integrable majorant function for $|\Phi_\varepsilon(|v_\nu|^2)v_\nu \otimes v_\nu - \Phi_\varepsilon(|v|^2)v \otimes v|^{p'}$, we estimate pointwise (again omitting the argument x)

$$\begin{aligned}
&|\Phi_\varepsilon(|v_\nu|^2)v_\nu \otimes v_\nu - \Phi_\varepsilon(|v|^2)v \otimes v|^{p'} \\
&= \sum_{i,j=1}^d |\Phi_\varepsilon(|v_\nu|^2)(v_\nu)_i(v_\nu)_j - \Phi_\varepsilon(|v|^2)v_iv_j|^{p'} \\
&\leq \sum_{i,j=1}^d c \left(|\Phi_\varepsilon(|v_\nu|^2)(v_\nu)_i(v_\nu)_j|^{p'} + |\Phi_\varepsilon(|v|^2)v_iv_j|^{p'}\right) \\
&\leq c \left((\Phi_\varepsilon(|v_\nu|^2)|v_\nu|^2)^{p'} + (\Phi_\varepsilon(|v|^2)|v|^2)^{p'}\right) \\
&\leq c \left(\frac{2}{\varepsilon}\right)^{p'}.
\end{aligned}$$

In the last step, we again used $\Phi_\varepsilon(|v_\nu|^2) = 0$ for $|v_\nu|^2 \geq 2/\varepsilon$. Whence, by the dominated convergence theorem, we conclude

$$\|B_\varepsilon v_\nu - B_\varepsilon v\|_{V_p'} \leq \|\Phi_\varepsilon(|v_\nu|^2)v_\nu \otimes v_\nu - \Phi_\varepsilon(|v|^2)v \otimes v\|_{L^{p'}(\Omega)^{d\times d}} \to 0.$$

Since the limit is unique, the usual argumentation with contradiction shows, that this convergence holds true for the whole sequence. This implies the continuity of $B : H \to V_p$.

The identity

$$\langle B_\varepsilon v, v\rangle = 0$$

for $v \in V_p$ follows by density from Lemma 4.1.

Due to the continuity of $B : H \to V_p'$, a Bochner-measurable function $u : (0,T) \to H$ is mapped to a Bochner-measurable function $Bu : (0,T) \to V_p'$. Moreover, the mapping $t \mapsto \|(Bu)(t)\|_{V_p'}^{p'}$ is integrable due to the growth condition (4.6). Indeed, we have

$$\begin{aligned}\|B_\varepsilon v\|_{L^{p'}(0,T;V_p')}^{p'} &= \int_0^T \|B_\varepsilon v(t)\|_{V_p'}^{p'} \,\mathrm{d}t \\ &\le c_\varepsilon \int_0^T (\|v(t)\|_H^{2/p'})^{p'} \,\mathrm{d}t \\ &= c_\varepsilon \|v\|_{L^2(0,T;H)}^2.\end{aligned}$$

□

From now on, we will write $B_\varepsilon u(t)$ instead of $(B_\varepsilon u)(t)$.

4.2 Existence of solutions to the approximate system

With the above definition for the convection term, we are able to formulate the evolution problem representing (4.1)-(4.4) for any $f \in L^{p'}(0,T;V_p')$ and $u_0 \in H$:

$$u_\varepsilon' + Au_\varepsilon + B_\varepsilon u_\varepsilon = f \text{ in } L^{p'}(0,T;V_p'), \quad u_\varepsilon(0) = u_0 \text{ in } H. \tag{4.8}$$

The aim of this section is to prove the existence of solutions to this problem for any $\varepsilon > 0$. We follow the steps of Wolf in [67, Theorem 3.1]. The general idea behind it is using the standard theory of monotone operators to show the existence of a unique solution to the problem $u' + Au = f - B_\varepsilon w$ for every suitable w. Since we can show boundedness of solutions in $L^\infty(0,T;H)$ it will be possible to employ a fixed point argument locally in time in $L^2(0,T;H)$ for the operator mapping w to the solution u. After this, global estimates allow us to extend this fixed point to a solution of (4.8) on $(0,T)$.

Let $T_* \in (0,T]$ be arbitrary for the time being.

Lemma 4.4. *For any $g \in L^{p'}(0,T_*;V_p')$ and $u_0 \in H$ there exists a unique solution $u \in W^p(0,T_*)$ to the evolution problem*

$$u' + Au = g \text{ in } L^{p'}(0,T_*;V_p'), \quad u(0) = u_0 \text{ in } H.$$

Proof. See e.g. [41, Chapter 2, Section 1.4, Theorem 1.2]. □

We define the set

$$M_{T_*} := \left\{ w \in L^2(0,T_*;H) : \|w\|_{L^2(0,T_*;H)} \le 1 \right\}$$

on which we will be employing Schauder's fixed point theorem.

Obviously, the restriction of any $f \in L^{p'}(0,T;V_p')$ to the interval $(0,T_*)$ belongs to $L^{p'}(0,T_*;V_p')$. For simplicity, we will not introduce a new notation for this restriction. For a fixed $w \in L^2(0,T_*;H)$, Lemma 4.2 ensures that $f - B_\varepsilon w$ is a suitable right-hand side to the problem in Lemma 4.4. Let us denote by $K(w) \in W^p(0,T_*) \hookrightarrow L^2(0,T_*;H)$ the unique solution to

$$u' + Au = f - B_\varepsilon w \quad \text{in } L^{p'}(0,T_*;V_p'), \quad u(0) = u_0 \text{ in } H. \tag{4.9}$$

With this we can define the mapping

$$K : M_{T_*} \to L^2(0, T_*; H), \quad K(w) = u.$$

Since $K(w)$ is a solution to (4.9), the function $K(w)$ actually belongs to the space $W^p(0, T_*)$.

Lemma 4.5. *There exists a constant $c = c(\varepsilon, f, u_0) > 0$ such that*

$$\|K(w)\|^2_{L^\infty(0,T_*;H)} + c_0\|K(w)\|^p_{L^p(0,T_*;V_p)} \le c(\varepsilon, f, u_0)$$

for every $w \in M_{T_*}$. Precisely, we have

$$c(\varepsilon, f, u_0) = \tilde{c}_\varepsilon + \delta'\|f\|^{p'}_{L^{p'}(0,T_*;V_p')} + \|u_0\|^2_H$$

where δ' only depends on c_0 and p and $\tilde{c}_\varepsilon$ depends on the constant from Lemma 4.2, c_0 and p.

Proof. Let $u = K(w)$. For $t \in (0, T)$, we test (4.9) with the function $\chi_{(0,t)}u$. This gives

$$\int_0^t \langle u'(\tau), u(\tau)\rangle + \langle Au(\tau), u(\tau)\rangle \,\mathrm{d}\tau = -\int_0^t \langle B_\varepsilon w(\tau), u(\tau)\rangle \,\mathrm{d}\tau + \int_0^t \langle f(\tau), u(\tau)\rangle \,\mathrm{d}\tau\,. \quad (4.10)$$

We exploit the rule of integration by parts (2.2) and the coercivity of A to show that the left-hand side of this equation is greater or equal to

$$\frac{1}{2}\left(\|u(t)\|^2_H - \|u(0)\|^2_H\right) + c_0\|u\|^p_{L^p(0,t;V_p)}.$$

On the right-hand side of (4.10) we need the growth estimate (4.6) for B and Young's inequality to estimate

$$\begin{aligned} -\int_0^t \langle B_\varepsilon w(\tau), u(\tau)\rangle \,\mathrm{d}\tau &\le \int_0^t c_\varepsilon \|w(\tau)\|^{2/p'}_H \|u(\tau)\|_V \,\mathrm{d}\tau \\ &\le \int_0^{T_*} c_\varepsilon^{p'}\delta'\|w(\tau)\|^2_H + \delta\|u(\tau)\|^p_V \,\mathrm{d}\tau \\ &\le c_\varepsilon^{p'}\delta'\|w\|^2_{L^2(0,T_*;H)} + \delta\|u\|^p_{L^p(0,T_*;V_p)}. \end{aligned}$$

Here δ can be chosen freely and $\delta' = \frac{(\delta p)^{-p'/p}}{p'}$. We choose $\delta = c_0/4$ and for simplicity $c_\varepsilon^{p'}\delta'$ will be called $\tilde{c}_\varepsilon$.

For the term involving f we proceed analogously to estimate

$$\begin{aligned} \int_0^t \langle f(\tau), u(\tau)\rangle \,\mathrm{d}\tau &\le \int_0^{T_*} \|f(\tau)\|_{V_p'}\|u(\tau)\|_{V_p} \,\mathrm{d}\tau \\ &\le \int_0^{T_*} \delta'\|f(\tau)\|^{p'}_{V_p'} + \delta\|u(\tau)\|^p_{V_p} \,\mathrm{d}\tau \\ &\le \delta'\|f\|^{p'}_{L^{p'}(0,T_*;V_p')} + \delta\|u\|^p_{L^p(0,T_*;V_p)}. \end{aligned}$$

Once again, we choose $\delta = c_0/4$.

Putting these estimates together results in the inequality

$$\begin{aligned}\frac{1}{2}\|u(t)\|_H^2 + \frac{c_0}{2}\|u\|_{L^p(0,t;V_p)}^p &\leq \tilde{c}_\varepsilon \|w\|_{L^2(0,T_*;H)}^2 + \delta' \|f\|_{L^{p'}(0,T_*;V_p')}^{p'} + \frac{1}{2}\|u_0\|_H^2 \\ &\leq \tilde{c}_\varepsilon + \delta' \|f\|_{L^{p'}(0,T_*;V_p')}^{p'} + \frac{1}{2}\|u_0\|_H^2\end{aligned}$$

with $\|w\|_{L^2(0,T_*;H)}^2 \leq 1$, since $w \in M_{T_*}$. Now we take $t = T_*$ to gain an estimate on $\|u\|_{L^p(0,T_*;V_p)}^p$ and then we take the supremum over all $t \in (0, T_*)$ for an estimate on $\|u\|_{L^\infty(0,T_*;H)}^2$. Eventually receiving a factor 2 on the right-hand side, the sum of those estimates and $u = K(w)$ gives the desired inequality. □

The application of Schauder's fixed point theorem requires that K maps the set M_{T_*} into itself. This is an easy implication of the foregoing lemma.

Corollary 4.6. *Choosing* $T_* < \min(c(\varepsilon, f, u_0)^{-1}, T)$ *results in*

$$\|K(w)\|_{L^2(0,T_*;H)}^2 \leq 1 \tag{4.11}$$

for every $w \in M_{T_*}$ *and hence* K *maps* M_{T^*} *into itself. Additionally, we have the estimate*

$$\|K(w)\|_{L^p(0,T_*;V_p)}^p \leq 2c_0^{-1} c(\varepsilon, f, u_0). \tag{4.12}$$

Proof. From Lemma 4.5 we know

$$\begin{aligned}\|K(w)\|_{L^2(0,T_*;H)}^2 &= \int_0^{T_*} \|K(w)(t)\|_H^2 \, dt \\ &\leq \|K(w)\|_{L^\infty(0,T_*;H)}^2 T_* \\ &\leq c(\varepsilon, f, u_0) T_*.\end{aligned}$$

Taking

$$T_* < \min(c(\varepsilon, f, u_0)^{-1}, T)$$

yields

$$\|K(w)\|_{L^2(0,T_*;H)}^2 \leq 1$$

and hence, K maps M_{T_*} into itself.

The other inequality follows from Lemma 4.5 with the special T_*. □

Lemma 4.7. *Let* $T_* \in (0, T)$ *be as in Corollary 4.6. Then the mapping* $K : M_{T_*} \to M_{T_*}$ *is compact.*

Proof. See also [67, Theorem 3.1]. The first part of the proof is based on the Lions-Aubin lemma (see Lemma 2.9). We will show the boundedness of the set $K(M_{T_*})$ in $W^p(0, T_*)$. The estimate (4.12) implies the boundedness in $L^p(0, T_*; V_p)$. To handle the time derivative, we use equation (4.9) and afterwards the growth estimates for A and B. For arbitrary $w \in M_{T_*}$ we write $u = K(w)$ and calculate

$$\begin{aligned}\|u'\|_{L^{p'}(0,T_*;V_p')} &= \|f - Au - B_\varepsilon w\|_{L^{p'}(0,T_*;V_p')} \\ &\leq \|f\|_{L^{p'}(0,T_*;V_p')} + \|Au\|_{L^{p'}(0,T_*;V_p')} + \|B_\varepsilon w\|_{L^{p'}(0,T_*;V_p')} \\ &\leq \|f\|_{L^{p'}(0,T_*;V_p')} + c(1 + \|u\|_{L^p(0,T_*;V_p)})^{p-1} + \|w\|_{L^2(0,T_*;H)}^{2/p'}.\end{aligned}$$

Whence, $K(M_{T_*})$ is bounded in $W^p(0,T_*)$ and by the Lions-Aubin compactness lemma relatively compact in $L^p(0,T_*;H)$. This is sufficient if $p \geq 2$ as then $L^p(0,T_*;H) \hookrightarrow L^2(0,T_*;H)$.

Otherwise, consider a sequence $\{w_\nu\}_{\nu\in\mathbb{N}} \subset M_{T_*}$ and with it $\{K(w_\nu)\}_{\nu\in\mathbb{N}}$. Let us again denote $u_\nu = K(w_\nu)$ for every $\nu \in \mathbb{N}$. From the boundedness of $\{u_\nu\}_{\nu\in\mathbb{N}}$ in $W^p(0,T_*)$ then follows the existence of a function $u \in L^p(0,T_*;H)$ and a convergent subsequence (again denoted by $\{u_\nu\}_{\nu\in\mathbb{N}}$) with $u_\nu \to u$ in $L^p(0,T;H)$. Due to the embedding

$$W^p(0,T_*) \hookrightarrow C([0,T_*];H),$$

which we stated in Lemma 2.1, $\{u_\nu\}_{\nu\in\mathbb{N}}$ is bounded in $C([0,T_*];H)$. For arbitrary exponents $s \in (p,\infty)$, particularly $s = 2$, thus follows

$$\begin{aligned}
\|u_\nu - u\|_{L^s(0,T_*;H)}^s &= \int_0^{T_*} \|u_\nu(t) - u(t)\|_H^s \,\mathrm{d}t \\
&\leq \operatorname*{ess\,sup}_{t\in[0,T_*]} \|u_\nu(t) - u(t)\|_H^{s-p} \int_0^{T_*} \|u_\nu(t) - u(t)\|_H^p \,\mathrm{d}t \\
&\leq \|u_\nu - u\|_{L^\infty(0,T_*;H)}^{s-p} \|u_\nu - u\|_{L^p(0,T_*;H)}^p \\
&\leq (\|u_\nu\|_{L^\infty(0,T_*;H)} + \|u\|_{L^\infty(0,T_*;H)})^{s-p} \|u_\nu - u\|_{L^p(0,T_*;H)}^p \\
&\to 0.
\end{aligned}$$

This means the relative compactness of $K(M_{T_*})$.

Let us now turn to the continuity of K. We consider a sequence $\{w_\nu\}_{\nu\in\mathbb{N}} \subset M_{T_*}$ converging to w in $L^2(0,T_*;H)$ as ν goes to infinity. As we have proven above, $\{K(w_\nu)\}_{\nu\in\mathbb{N}}$ is relatively compact in $L^2(0,T_*;H)$. Thus, there exists a limit $u \in L^2(0,T_*;H)$ of a (not relabeled) subsequence of $\{u_\nu\}_{\nu\in\mathbb{N}}$, i.e.

$$u_\nu \to u \quad \text{in} \quad L^2(0,T_*;H).$$

Our goal is to show $u = K(w)$. We achieve this by demonstrating, that u is indeed the solution of (4.9) pertaining to w.

Since $\{u_\nu\}_{\nu\in\mathbb{N}}$ is bounded in $W^p(0,T_*)$ we can, by means of reflexivity of $L^p(0,T_*;V_p)$ and $L^{p'}(0,T_*;V_p')$, extract subsequences still denoted by $\{u_\nu\}_{\nu\in\mathbb{N}}$ such that

$$\begin{aligned}
u_\nu &\rightharpoonup v \quad \text{in} \quad L^p(0,T_*;V_p), \\
u_\nu' &\rightharpoonup \chi \quad \text{in} \quad L^{p'}(0,T_*;V_p'),
\end{aligned} \tag{4.13}$$

see e.g. [66, Theorem III.3.7, p. 107] or [17, Theorem III.27, p. 50]. By density and uniqueness of weak limits follows $v = u$. Further, χ equals the time derivative of u since for all $\varphi \in C_0^\infty(0,T_*)$ and $v \in V_p$ we have

$$\begin{aligned}
&\int_0^{T_*} \langle \chi(t), v\rangle \varphi(t) \,\mathrm{d}t \\
&= \int_0^{T_*} \langle \chi(t) - u_\nu'(t), v\rangle \varphi(t) \,\mathrm{d}t + \int_0^{T_*} \langle u_\nu'(t), v\rangle \varphi(t) \,\mathrm{d}t \\
&= \int_0^{T_*} \langle \chi(t) - u_\nu'(t), v\varphi(t)\rangle \,\mathrm{d}t - \int_0^{T_*} \langle u_\nu(t) - u(t), v\varphi'(t)\rangle \,\mathrm{d}t \\
&\quad - \int_0^{T_*} \langle u(t), v\rangle \varphi'(t) \,\mathrm{d}t\,.
\end{aligned}$$

With $u_\nu \rightharpoonup u$ and $u'_\nu \rightharpoonup \chi$, we find

$$u'_\nu \rightharpoonup u' \quad \text{in} \quad L^{p'}(0, T_*; V'_p). \tag{4.14}$$

We have already demonstrated in Lemma 4.2 that B is continuous, which gives

$$B_\varepsilon w_\nu \to B_\varepsilon w \text{ in } L^{p'}(0, T_*; V'_p). \tag{4.15}$$

The growth condition (3.2) yields the boundedness of the sequence $\{Au_\nu\}_{\nu \in \mathbb{N}}$ in $L^{p'}(0, T_*; V'_p)$ and by reflexivity, we find a weakly convergent subsequence such that

$$Au_\nu \rightharpoonup a. \tag{4.16}$$

for some $a \in L^{p'}(0, T_*; V'_p)$.

Altogether, (4.9), (4.13), (4.14), (4.15) and (4.16) imply

$$\langle u', v\rangle + \langle a, v\rangle + \langle B_\epsilon w, v\rangle = \langle f, v\rangle \tag{4.17}$$

for all $v \in V_p$.

It remains to show $Au = a$. For this, we employ Minty's decisive monotonicity trick. We test (4.9) with u_ν for every $\nu \in \mathbb{N}$ and receive

$$\langle u'_\nu, u_\nu\rangle + \langle Au_\nu, u_\nu\rangle + \langle B_\varepsilon w_\nu, u_\nu\rangle = \langle f, u_\nu\rangle.$$

We now use the monotonicity of A and add $\langle Au_\nu - A\zeta, u_\nu - \zeta\rangle \geq 0$ for an arbitrary $\zeta \in L^p(0, T_*; V_p)$ to the left-hand side. Some easy calculations yield

$$\langle u'_\nu, u_\nu\rangle + \langle A\zeta, u_\nu - \zeta\rangle + \langle Au_\nu, \zeta\rangle + \langle B_\varepsilon w_\nu, u_\nu\rangle \leq \langle f, u_\nu\rangle. \tag{4.18}$$

With (4.13), (4.15) and (4.16) follows

$$\begin{aligned}
\langle A\zeta, u_\nu - \zeta\rangle &\to \langle A\zeta, u - \zeta\rangle, \\
\langle Au_\nu, \zeta\rangle &\to \langle a, \zeta\rangle, \\
\langle f, u_\nu\rangle &\to \langle f, u\rangle, \\
\langle B_\varepsilon w_\nu, u_\nu\rangle &\to \langle B_\varepsilon w, u\rangle,
\end{aligned}$$

as $\nu \to \infty$. This handles every term except for the time derivative. We will now show $\langle u', u\rangle \leq \liminf \langle u'_\nu, u_\nu\rangle$ using integration by parts.

The boundedness of $\{u_\nu\}_{\nu \in \mathbb{N}}$ in $W^p(0, T_*) \hookrightarrow C([0, T_*]; H)$ indicates that the sequence $\{u_\nu(T_*)\}_{\nu \in \mathbb{N}}$ is bounded in H. This means there exists a $\theta \in H$ such that

$$u_\nu(T_*) \rightharpoonup \theta \text{ in } H$$

for a suitable subsequence. With the rule of integration by parts in Lemma 2.1 we can show $\theta = u(T_*)$. Indeed, for any $v \in V_p$ and $\phi \in C^1([0, T_*])$ with $\phi(0) = 0$ and $\phi(T_*) = 1$ we can calculate

$$\begin{aligned}
&(u(T_*), v)\phi(T_*) - (u(0), v)\phi(0) \\
= \quad & \langle u', v\phi\rangle + \langle u, \phi'\rangle \\
= \quad & \langle f - a - B_\varepsilon(w), v\phi\rangle + \langle u, v\phi'\rangle \\
= \quad & \langle Au_\nu - a + B_\varepsilon w_\nu - B_\varepsilon w + u'_\nu, v\phi\rangle + \langle u, v\phi'\rangle \\
= \quad & \langle Au_\nu - a + B_\varepsilon w_\nu - B_\varepsilon w, v\phi\rangle + \langle u'_\nu, v\phi\rangle + \langle u, v\phi'\rangle \\
= \quad & \langle Au_\nu - a + B_\varepsilon w_\nu - B_\varepsilon w, v\phi\rangle + \langle u - u_\nu, v\phi'\rangle \\
& + (u_\nu(T_*), v)\phi(T_*) - (u_\nu(0), v)\phi(0).
\end{aligned}$$

We now let ν tend to infinity and use $u_\nu(0) = u_0 = u(0)$ to show $u(T_*) = \theta$. In particular, the weak convergence implies

$$\|u(T_*)\|_H \le \liminf_{\nu\to\infty} \|u_\nu(T_*)\|_H.$$

With this, (2.2) gives

$$\begin{aligned}\langle u', u\rangle &= \frac{1}{2}\left(\|u(T_*)\|_H^2 - \|u(0)\|_H^2\right)\\ &\le \liminf_{\nu\to\infty} \frac{1}{2}\left(\|u_\nu(T_*)\|_H^2 - \|u(0)\|_H^2\right)\\ &= \liminf_{\nu\to\infty}\langle u_\nu', u_\nu\rangle.\end{aligned}$$

We can now take the limes inferior in (4.18) to show

$$\langle u', u\rangle + \langle A\zeta, u-\zeta\rangle + \langle a, \zeta\rangle + \langle B_\varepsilon w, u\rangle \le \langle f, u\rangle.$$

Together with (4.17) follows

$$\langle A\zeta, u-\zeta\rangle \le \langle a, u-\zeta\rangle.$$

We now choose $\zeta = u \mp \tau v$ for an arbitrary $v \in L^p(0,T_*;V_p)$ and $\tau \in [0,1]$. Then

$$\mp\langle A(u\mp\tau v), v\rangle \le \mp\langle a, v\rangle$$

and the hemicontinuity of A yields

$$\mp\langle Au, v\rangle \le \mp\langle a, v\rangle.$$

for $\tau \to 0^+$. Since $v \in L^p(0,T_*;V_p)$ was arbitrary, this implies

$$Au = a \quad \text{in } L^{p'}(0,T_*;V_p')$$

and then (4.17) indicates that u is the unique solution to (4.9) corresponding to w and hence

$$u = K(w).$$

In particular, noticing that this limit is unique, the usual argumentation by contradiction shows the convergence of the whole sequence

$$K(w_\nu) \to K(w) \quad \text{in } L^2(0,T_*;H),$$

which concludes the proof. □

We will now come to the main result in this section.

Theorem 4.8. *Let $u_0 \in H$ and $f \in L^{p'}(0,T;V_p')$. Then for every $\varepsilon > 0$ there exists a solution $u_\varepsilon \in W^p(0,T)(\cap C([0,T];H))$ of the problem (4.8).*

The solution u_ε satisfies the energy equality

$$\frac{1}{2}\|u_\varepsilon(t)\|_H^2 + \int_0^t \langle Au_\varepsilon(\tau), u_\varepsilon(\tau)\rangle\,\mathrm{d}\tau = \frac{1}{2}\|u_0\|_H^2 + \int_0^t \langle f(\tau), u_\varepsilon(\tau)\rangle\,\mathrm{d}\tau \qquad (4.19)$$

for every $t \in [0,T]$.

Proof. A shorter argumentation can be found in [67, Theorem 3.1, Step 2°]. Let $T_* \in (0,T]$ be as in Corollary 4.6. Let K be the solution operator defined as above, mapping $w \in L^2(0,T_*;H)$ to the solution of (4.9). Then Lemma 4.7 enables us to employ Schauder's fixed point theorem which gives a fixed point $u_{T_*} \in L^2(0,T_*;H)(\cap W^p(0,T_*))$ that solves (4.8) on $(0,T_*) \times \Omega$.

We still need to show that this local solution can be extended to a solution on the whole time interval. We derive an energy estimate by testing (4.8) with the solution u_{T_*}, using integration by parts (2.2) and taking into account the coercivity of A. By Lemma 4.2, it is $\langle B_\varepsilon u_{T_*}, u_{T_*}\rangle = 0$ and hence

$$\frac{1}{2}\|u_{T_*}(T_*)\|_H^2 + c_0\|u_{T_*}\|^p_{L^p(0,T_*;V)} \le \langle f, u_{T_*}\rangle + \frac{1}{2}\|u_0\|_H^2.$$

By means of Young's inequality with $\delta \le c_0/4$, we have for the right-hand side f

$$\int_0^{T_*} \langle f(t), u_{T_*}(t)\rangle \,\mathrm{d}t \le \delta' \|f\|^{p'}_{L^{p'}(0,T_*;V_p')} + \delta\|u_{T_*}\|^p_{L^p(0,T_*;V_p)} < \infty.$$

Together, this gives a global energy estimate for the solution u_{T_*}

$$\frac{1}{2}\|u_{T_*}(T_*)\|_H^2 + \frac{c_0}{2}\|u_{T_*}\|^p_{L^p(0,T_*;V)} \le \delta' \|f\|^{p'}_{L^{p'}(0,T_*;V_p')} + \frac{1}{2}\|u_0\|_H^2.$$

Thus, the solution cannot blow up and by the usual arguments, we can extend the solution u_{T_*} to a global solution of (4.8).

□

4.3 Outline of the end of the proof

We only sketch the remaining parts of the proof here, since we will present another approach to obtain approximate solutions of (3.8) via temporal time semi-discretization (see Chapter 8), where we will then encounter the same or similar arguments. The details can be found in [22].

For $\nu \in \mathbb{N}$, Theorem 4.8 ensures the existence of solutions u_ν to (4.8) for every $\varepsilon = \frac{1}{\nu}$. Let us denote the operator B_ε for $\varepsilon = \frac{1}{\nu}$ by B_ν. With the energy equality (4.19) we obtain the following a priori estimates on the sequence $\{u_\nu\}_{\nu\in\mathbb{N}}$ of approximate solutions by using the coercivity (3.2) of the operator A:

$$\|u_\nu\|_{L^\infty(0,T;H)} + c_0\|u_\nu\|_{L^p(0,T;V_p)} \le c(f,u_0),$$

where the constant is independent of ν.

This implies the boundedness of the sequence $\{u_\nu\}_{\nu\in\mathbb{N}}$ in $L^\infty(0,T;H) \cap L^p(0,T;V_p)$ and Lemma 3.8 then shows boundedness in $L^{2r'}(0,T;H_{2r'})$. By reflexivity and separability, there exists a function $u \in L^\infty(0,T;H) \cap L^p(0,T;V_p)$ and a (for simplicity not relabeled) subsequence, such that

$$\begin{aligned} u_\nu &\rightharpoonup u \quad \text{in } L^p(0,T;V_p),\\ u_\nu &\overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;H). \end{aligned}$$

With the growth condition (3.2) of the operator A, the growth condition in Remark 4.3 with $s = r$ and $\Phi_\varepsilon \le 1$ for B_ν, one can show that the sequence of time derivatives $\{u'_\nu\}_{\nu\in\mathbb{N}}$ is bounded in $L^{r'}(0,T;V'_r)$. Hence, by reflexivity one can extract another subsequence with

$$u'_\nu \rightharpoonup u' \quad \text{in } L^{r'}(0,T;V'_r).$$

Then, application of the Lions-Aubin compactness lemma (see Lemma 2.9) yields the strong convergence of $\{u_\nu\}_{\nu\in\mathbb{N}}$ in $L^p(0,T;H)$. The boundedness in $L^\infty(0,T;H)$ and Lemma 3.8 then show

$$u_\nu \to u \quad \text{in } L^{2r'}(0,T;H_{2r'}).$$

With the help of the growth conditions on the operators A and B_ν and Lebesgue's theorem on dominated convergence, it is possible to prove the existence of a function $a \in L^{p'}(0,T;V'_p)$ such that

$$Au_\nu \rightharpoonup a \quad \text{in } L^{p'}(0,T;V'_p) \quad \text{and} \quad B_\nu u_\nu \to Bu \quad \text{in } L^{r'}(0,T;V'_r)$$

for a suitable subsequence. Then, the equation

$$u' + a + Bu = f \tag{4.20}$$

holds in $L^{r'}(0,T;V'_r)$. Thus, it is only left to show $a = Au$. For this, one usually employs Minty's monotonicity trick. But since (4.20) only holds in $L^{r'}(0,T;V'_r)$, we are in general not able to test it with the solution u itself. To circumvent this problem, a parabolic Lipschitz truncation method was introduced in [22, Theorem 3.21] providing truncations $\mathcal{T}u_\nu$ which possess essentially bounded gradients on every compact subset of the time-space cylinder (see Lemma 6.4 for an adapted version for time discretizations).

However, the truncations $\mathcal{T}u_\nu$ is not divergence-free and hence, it is necessary to derive a suitable representation of (4.8) in $L^{r'}(0,T;(W_0^{1,r}(\Omega)^d)')$. This is done by introducing suitable pressure functions for the system (see [22, Section 2] or Chapter 5).

Finally, a localized version of Minty's trick (see [67, Lemma A.2]) gives $a = Au$.

5

Pressure representation

5.1 Preliminaries

The following arguments require more smoothness of the boundary, than only Lipschitz continuity. This will force us to localize our arguments in the proof of convergence of a temporal semi-discretization in Chapter 8. Thus, throughout this section let $G \subset \mathbb{R}^d$ be a bounded domain with $\partial G \in C^2$.

Let $k \in \mathbb{N}$ and $1 < s_i < \infty$, $i = 1, \ldots, k$. We consider abstract functions $u \in C_w([0,T]; H)$ and $f_i \in L^{s_i'}(0,T;(W_0^{1,s_i}(G))')$, for which the equation

$$\langle u', \varphi \rangle = \sum_{i=1}^{k} \langle f_i, \varphi \rangle \text{ for } \varphi \in L^s(0,T; V_s(G)) \tag{5.1}$$

holds with $s = \max(s_i)$.

In Chapter 8, it will be of importance to test (5.1) with a truncated version of u. Since the Lipschitz truncation method, which will be studied in Chapter 6, only provides a test function that is not necessarily divergence-free, it is essential to find a representation of (5.1) in $L^{s'}(0,T;(W_0^{1,s}(G)^d)')$. This requires to introduce a pressure function which was annihilated in the weak formulation of the problem in the solenoidal setting.

The reconstruction of the pressure function goes back to a famous theorem of de Rham: Let f be a distribution. Then $f = \nabla \pi$ for some distrbution π holds if and only if

$$\langle f, \varphi \rangle = 0 \quad \text{for } \varphi \in \mathcal{V}.$$

This result can be found in [62, Chapter 1, Paragraph 1, Proposition 1.1] or [65].

We follow the proofs of Diening, Růžička and Wolf in [22, Section 2]. The first step will be to consider only $f = \sum_i f_i$ and integrate the equation $u' = f$ in time. In this way, we can reconstruct a pressure for u pointwise in time. The function we obtain this way is an antiderivative (in time) of the pressure function we are looking for in the end. We will then decompose this function into several parts corresponding to the terms f_i and one (weakly) harmonic term, that is bounded by the function u, and derive the equation again.

Since the pressure is given as a gradient, it is natural that we will only be able to determine it up to a constant. Thus, the appropriate space for the pressure functions should be some function space divided by $\mathbb{R}$, e.g.

$$L^s(G)/\mathbb{R}$$

for some $s \in (1, \infty)$. One can show, that this quotient space is isomorphic to the Banach space

$$L_0^s(G) = \left\{ \pi \in L^s(G) : \int_G \pi(x)\, \mathrm{d}x \ = 0 \right\},$$

equipped with the norm

$$\|\pi\|_{L_0^s(G)} = \|\pi\|_{L^s(G)}.$$

5.2 Recovering the pressure in the spatial domain

Let us first consider only the spatial domain. The following lemma provides the existence of a pressure function in G. Later, we will employ this result for abstract functions pointwise in time.

Lemma 5.1. *Let $1 < s < \infty$. Let $f \in (W_0^{1,s}(G)^d)'$ with $\langle f, v\rangle = 0$ for all $v \in V_s$. Then there exists a unique $\pi \in L_0^{s'}(G)$, such that*

$$\langle f, w\rangle = \int_G \pi \ \mathrm{div}\, w \, \mathrm{d}x$$

for all $w \in W_0^{1,s}(G)$. Furthermore, the estimate

$$\|\pi\|_{L^{s'}(G)} \leq c\, \|f\|_{(W_0^{1,s}(G)^d)'}$$

holds.

Proof. From [33, Theorem III.5.2] we receive a unique function $\pi \in L_0^{s'}(G)$. By the definition of the dual norm, we have for $L_0^{s'}(G)^d \cong (L_0^s(G)^d)'$ (see e.g. [60, pp. 68f.])

$$\|\pi\|_{L_0^{s'}(G)} = \sup_{\substack{q \in L_0^s(G) \\ q \neq 0}} \frac{\int_G \pi\, q \, \mathrm{d}x}{\|q\|_{L_0^s(G)}}.$$

Now according to [2, Theorem 4.1], to any $q \in L_0^s(G)$ there exists a $w \in W_0^{1,s}(G)^d$ such that

$$q = \ \mathrm{div}\, w \quad \text{ and } \quad \|w\|_{W_0^{1,s}(G)^d} \leq c\, \|q\|_{L_0^s(G)}.$$

This result goes back to Bogovskiĭ [15] who introduced a linear and bounded operator $\mathcal{B} : L_0^s(G) \to W_0^{1,s}(G)^d$ with this property (the same result can be obtained for the more general John domains, see e.g. [2] or for a simplified proof [21]). We can estimate

$$\begin{aligned}
\|\pi\|_{L_0^{s'}(G)} &\leq c \sup_{\substack{w \in W_0^{1,s}(G)^d \\ w \neq 0}} \frac{\int_G \pi \ \mathrm{div}\, w \, \mathrm{d}x}{\|w\|_{W_0^{1,s}(G)^d}} \\
&= c \sup_{\substack{w \in W_0^{1,s}(G)^d \\ w \neq 0}} \frac{\langle f, w\rangle}{\|w\|_{W_0^{1,s}(G)^d}} \\
&= c\, \|f\|_{(W_0^{1,s}(G)^d)'}.
\end{aligned}$$

□

5.3 Recovering the pressure pointwise in time

We will now construct a function $\hat{\pi}$ for $f = \sum_i f_i$ that represents an antiderivative in time of the sought-after pressure function. Since we want to assume $u \in C_w([0,T];H)$, let $s > \frac{2d}{d+2}$ to ensure the embedding $H \hookrightarrow (W_0^{1,s}(G)^d)'$.

Lemma 5.2. *Let $\frac{2d}{d+2} < s < \infty$. For a function $u \in C_w([0,T];H)$ with a time derivative $u' \in L^{s'}(0,T;(W_0^{1,s}(G)^d)')$ and $f \in L^{s'}(0,T;(W_0^{1,s}(G)^d)')$, suppose that*

$$u' = f \qquad \text{in } L^{s'}(0,T;V_s'). \tag{5.2}$$

Then there exists a unique function $\hat{\pi} \in C_w([0,T];L_0^{s'}(G))$, such that

$$\langle u(t) - u(0), w\rangle = \left\langle \int_0^t f(\tau)\,\mathrm{d}\tau\,, w\right\rangle + \int_G \hat{\pi}(t)\operatorname{div} w \,\mathrm{d}x$$

for all $t \in [0,T]$ and for all $w \in W_0^{1,s}(G)^d$. In addition, we have the a priori estimate

$$\|\hat{\pi}\|_{L^\infty(0,T;L_0^{s'}(G))} \le c\left(\|u\|_{L^\infty(0,T;H)} + \|f\|_{L^1(0,T;(W_0^{1,s}(G)^d)')}\right)$$

with a constant c depending only on d, G and s.

Proof. We follow the first part of the proof of [67, Theorem 2.6]. By [25, Theorem 8.1.5], u is absolutely continuous as a function taking values in V_s' and we have for all $t \in [0,T]$

$$u(t) = u(0) + \int_0^t f(\tau)\,\mathrm{d}\tau \qquad \text{in } V_s'.$$

Thus, by Lemma 5.1 for every $t \in [0,T]$ there exists a unique function $\hat{\pi}(t) \in L_0^{s'}(G)$ such that

$$\int_G \hat{\pi}(t)\operatorname{div} w\,\mathrm{d}x = \left\langle u(t) - u(0) - \int_0^t f(\tau)\,\mathrm{d}\tau\,, w\right\rangle \tag{5.3}$$

for all $w \in W_0^{1,s}(G)^d$ and

$$\|\hat{\pi}(t)\|_{L_0^{s'}(G)} \le c\left(\|u(t) - u(0)\|_H + \left\|\int_0^t f(\tau)\,\mathrm{d}\tau\right\|_{(W_0^{1,s}(G)^d)'}\right), \tag{5.4}$$

since $H \hookrightarrow (W_0^{1,s}(G)^d)'$.

We now have to show that $t \mapsto \hat{\pi}(t)$ is a Bochner-measurable function. In fact, we can show that $\hat{\pi} \in C_w([0,T];L_0^{s'}(G))$.

Let $v \in L_0^s(G) \cong (L_0^{s'}(G))'$ be arbitrary. By [2, Theorem 4.1] or [21] there exists a solution $w \in W_0^{1,s}(G)^d$ to the divergence problem $\operatorname{div} w = v$. Hence, by inserting this into (5.3) we get

$$\begin{aligned}\int_G \hat{\pi}(t)\,v\,\mathrm{d}x &= \int_G \hat{\pi}(t)\,\operatorname{div} w\,\mathrm{d}x \\ &= \langle u(t) - u(0) - \int_0^t f(\tau)\,\mathrm{d}\tau\,, w\rangle.\end{aligned}$$

By the demicontinuity of u taking values in H and $H \hookrightarrow (W_0^{1,s}(G)^d)'$, we know that the mapping $t \mapsto \langle u(t), w\rangle$ is continuous for every $w \in W_0^{1,s}(G)^d$ and consequently $\hat{\pi} \in C_w([0,T]; L_0^{s'}(G))$. Taking the essential supremum in (5.4) gives

$$\begin{aligned}\|\hat{\pi}\|_{L^\infty(0,T;L_0^{s'}(G))} &= \operatorname*{ess\,sup}_{t\in[0,T]} \|\hat{\pi}(t)\|_{L_0^{s'}(G)} \\ &\leq \operatorname*{ess\,sup}_{t\in[0,T]} c \left(\|u(t)-u(0)\|_H + \int_0^t \|f(\tau)\|_{(W_0^{1,s}(G)^d)'} \,\mathrm{d}\tau \right) \\ &\leq c\left(\|u\|_{L^\infty(0,T;H)} + \|f\|_{L^1(0,T;(W_0^{1,s}(G)^d)')}\right),\end{aligned}$$

which concludes the proof. □

5.4 Decomposition of the pressure

The main result in this chapter reads as follows.

Theorem 5.3. *Let $k \in \mathbb{N}$, $\frac{2d}{d+2} < s_i < \infty$, $i = 1, 2, \ldots, k$, and $s = \max_i(s_i)$. For a function $u \in C_w([0,T]; H)$ with $u' \in L^{s'}(0,T; (W_0^{1,s}(G)^d)')$ and $f_i \in L^{s_i'}(0,T; (W_0^{1,s_i}(G)^d)')$ suppose that*

$$u' = \sum_{i=1}^k f_i \quad \text{in } L^{s'}(0,T; V_s') \tag{5.5}$$

holds. Then there exist unique functions $\pi_i \in L^{s_i'}(0,T; L_0^{s_i'}(G))$, $i = 1, 2, \ldots, k$, and $\hat{\pi}_h \in C_w([0,T]; W^{1,2}(G)) \cap C_w([0,T]; W^{2,\infty}_{\mathrm{loc}}(G))$ with $\int_G \hat{\pi}_h(t)\,\mathrm{d}x = 0$ for every $t \in [0,T]$, such that

$$\begin{aligned}\langle u', \varphi\rangle = &\sum_{i=1}^k \langle f_i, \varphi\rangle \\ &+ \int_0^T \int_G \left(\sum_{i=1}^k \pi_i\right) \operatorname{div}\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_G \nabla\hat{\pi}_h \cdot \partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all $\varphi \in C_0^\infty(G \times (0,T))$. In addition, we have $-\Delta\hat{\pi}_h(t) = 0$ for every $t \in [0,T]$, $\hat{\pi}_h(0) = 0$ and the a priori estimates

$$\begin{aligned}\|\pi_i\|_{L_0^{s_i'}(0,T;L^{s_i'}(G))} &\leq c_i \|f_i\|_{L^{s_i'}(0,T;(W_0^{1,s_i}(G)^d)')}, \quad i = 1,\ldots,k, \\ \|\hat{\pi}_h(t_1) - \hat{\pi}_h(t_2)\|_{W^{1,2}(G)} &\leq c_{h,1} \|u(t_1) - u(t_2)\|_H, \quad t_1, t_2 \in [0,T],\end{aligned}$$

and for $G' \subset\subset G$

$$\|\hat{\pi}_h(t_1) - \hat{\pi}_h(t_2)\|_{W^{2,\infty}(G')} \leq c_{h,2} \|u(t_1) - u(t_2)\|_H, \quad t_1, t_2 \in [0,T],$$

with constants c_i depending only on d, G and s_i, $i = 1, \ldots, k$, a constant $c_{h,1}$ depending only on d and G and $c_{h,2}$ depending on d, G and G'.

Proof. We follow the proof of [22, Theorem 2.3]. By applying Lemma 5.2 to the right-hand side

$$f = \sum_{i=1}^{k} f_i \in \sum_{i=1}^{k} L^{s_i'}(0,T;(W_0^{1,s_i}(G)^d)') \\ \subset L^{s'}(0,T;(W_0^{1,s}(G)^d)'),$$

we obtain a pressure $\hat{\pi} \in C_w([0,T];L_0^{s'}(G))$ for which

$$\int_G \hat{\pi}(t) \operatorname{div} w \, \mathrm{d}x = \left\langle u(t) - u(0) - \int_0^t \sum_{i=1}^{k} f_i(\tau) \, \mathrm{d}\tau \, , w \right\rangle$$

holds for all $t \in [0,T]$ for any $w \in W_0^{1,s}(G)^d$. We define the gradient $\nabla\hat{\pi} \in (W_0^{1,s}(G)^d)'$ through

$$\langle \nabla\hat{\pi}, w\rangle = -\int_G \hat{\pi}(t) \operatorname{div} w \, \mathrm{d}x \, , \quad w \in W_0^{1,s}(G)^d.$$

and therefore

$$\nabla\hat{\pi} = -u(t) + u(0) + \int_0^t \sum_{i=1}^{k} f_i(\tau) \, \mathrm{d}\tau \tag{5.6}$$

holds in $(W_0^{1,s}(G))'$.

We are now going to decompose $\hat{\pi}$ into the parts $\hat{\pi}_i$ corrensponding to f_i, $i = 1, \ldots, k$, and a weakly harmonic term $\hat{\pi}_h$ dominated by the function u.

Idea for the decomposition. The idea for this is to use the Stokes equation, its linearity and the estimates for its solution. From [33, Lemma IV.6.2] we know that the solution and the pressure for the Stokes problem are equal to zero for zero data. Thus, we want to find $\hat{\pi}_i(t)$, $\hat{\pi}_h(t)$ such that

$$-\Delta v + \nabla\big(\sum \hat{\pi}_i(t) + \hat{\pi}_h(t) - \hat{\pi}(t)\big) = 0,$$

which would imply $\sum \hat{\pi}_i(t) + \hat{\pi}_h(t) = \hat{\pi}(t)$. This equation is equivalent to

$$-\Delta v + \nabla\big(\sum \hat{\pi}_i(t) + \hat{\pi}_h(t)\big) = -u(t) + u(0) + \int_0^t \sum f_i(\tau) \, \mathrm{d}\tau \, .$$

With the linearity of the Stokes operator, we can split this equation into several parts

$$-\Delta v_i + \nabla\hat{\pi}_i(t) = \int_0^t f_i(\tau) \, \mathrm{d}\tau \, , \\ -\Delta v_h + \nabla\hat{\pi}_h(t) = -u(t) + u(0),$$

and solve these separately to obtain estimates for each pressure term.

Pressure corresponding to the terms f_i. Let us consider the energetic extension of the Stokes operator

$$C_{s_i} : V_{s_i'}(G) \to (V_{s_i}(G))', \quad \langle C_{s_i} v, w\rangle = \int_G \nabla v : \nabla w \,\mathrm{d}x\,,$$

for $v \in V_{s_i'}$ and $w \in V_{s_i}$. For $t \in [0,T]$, by [33, Theorem IV.6.1] we know that the Stokes problem

$$C_{s_i} v = \int_0^t f_i(\tau)\,\mathrm{d}\tau \quad \text{in } V_{s_i}'$$

has a unique solution $v_i(t) \in V_{s_i'}$ and a pressure $\hat{\pi}_i(t) \in L_0^{s_i'}(G)$, which satisfy the inequality

$$\|v_i(t)\|_{V_{s_i'}} + \|\hat{\pi}_i(t)\|_{L_0^{s_i'}(G)} \le c \left\| \int_0^t f_i(\tau)\,\mathrm{d}\tau \right\|_{(W_0^{1,s_i}(G)^d)'} \tag{5.7}$$

and

$$\langle C_{s_i} v_i(t), w\rangle = \left\langle \int_0^t f_i(\tau)\,\mathrm{d}\tau\,, w \right\rangle + \int_G \hat{\pi}_i(t)\,\mathrm{div}\, w\,\mathrm{d}x \quad \text{for all } w \in W_0^{1,s_i}(G)^d.$$

We now show that the mapping $t \mapsto \hat{\pi}_i(t)$ is absolutely continuous which is basically due to the stability of the Stokes problem and the regularity of f_i.

For arbitrary $\epsilon > 0$ and pairwise disjoint intervals $(x_j, y_j) \subset [0,T]$, $j = 1,\dots,m$, $m \in \mathbb{N}$, we follow the above arguments with y_j and x_j instead of t. Then by linearity and uniqueness, $\hat{\pi}_i(y_j) - \hat{\pi}_i(x_j)$ is the unique pressure function corresponding to the unique solution $v_i(y_j) - v_i(x_j)$ of the Stokes problem with the right-hand side $\int_{x_j}^{y_j} f_i(\tau)\,\mathrm{d}\tau$ and the estimate

$$\|\hat{\pi}_i(y_j) - \hat{\pi}_i(x_j)\|_{L_0^{s_i'}(G)} \le c \left\| \int_{x_j}^{y_j} f_i(\tau)\,\mathrm{d}\tau \right\|_{(W_0^{1,s_i}(G)^d)'}$$

holds.

Taking the sum over all $j = 1,\dots,m$ on both sides and considering the absolute continuity of the mapping $t \mapsto \int_0^t f_i(\tau)\,\mathrm{d}\tau$ we obtain $\delta > 0$ such that

$$\sum_{j=1}^m \|\hat{\pi}_i(y_j) - \hat{\pi}_i(x_j)\|_{L_0^{s_i'}(G)} \le c \sum_{j=1}^m \left\| \int_{x_j}^{y_j} f_i(\tau)\,\mathrm{d}\tau \right\|_{(W_0^{1,s_i}(G)^d)'} < \epsilon,$$

if $\sum_{j=1}^m |y_j - x_j| < \delta$. This shows that $\hat{\pi}_i$ is an absolutely continuous function taking values in $L_0^{s_i'}(G)$.

By the theorem of Kōmura (cf. [16, Corollary A.2]), $\hat{\pi}_i$ is almost everywhere differentiable in the classical sense, since $L_0^{s_i}(G)$ is reflexive for $1 < s_i < \infty$. By estimating the difference quotient of $\hat{\pi}_i$ we will be able to show that $\hat{\pi}_i \in W^{1,s_i'}(0,T; L_0^{s_i}(G))$.

For $0 < h < T$, X a Banach space and $g \in L^1_{\mathrm{loc}}(0,T;X)$ we denote by

$$D_h g(t) = \frac{g(t+h) - g(t)}{h}, \quad t \in (0, T-h),$$

the difference quotient of g of size h.

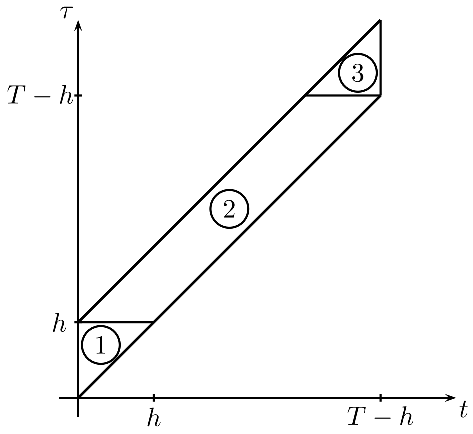

Figure 5.1: Integration domains

Again following the same arguments as above with $t, t+h$ instead of x_j and y_j, taking the s_i'-th power and dividing both sides by $h^{s_i'}$ gives after integrating over the interval $(0, T-h)$

$$\begin{aligned}\int_0^{T-h} \|D_h\hat{\pi}_i(t)\|^{s_i'}_{L_0^{s_i'}(G)} \,\mathrm{d}t &= \int_0^{T-h} \frac{\|\hat{\pi}_i(t+h) - \hat{\pi}_i(t)\|^{s_i'}_{L_0^{s_i'}(G)}}{h^{s_i'}} \,\mathrm{d}t \\ &\leq c\int_0^{T-h} \left\| \frac{1}{h}\int_t^{t+h} f_i(\tau)\,\mathrm{d}\tau \right\|^{s_i'}_{(W_0^{1,s_i}(G)^d)'} \,\mathrm{d}t \\ &\leq c\int_0^{T-h} \left(\frac{1}{h}\int_t^{t+h} \|f_i(\tau)\|_{(W_0^{1,s_i}(G)^d)'}\,\mathrm{d}\tau \right)^{s_i'} \mathrm{d}t \\ &\leq c\int_0^{T-h} \frac{1}{h}\int_t^{t+h} \|f_i(\tau)\|^{s_i'}_{(W_0^{1,s_i}(G)^d)'}\,\mathrm{d}\tau\,\mathrm{d}t\end{aligned}$$

As illustrated in Figure 5.1 we divide the integral into three parts. The first part then is with Fubini's theorem

$$\begin{aligned}\int_0^h\int_0^\tau \frac{1}{h}\|f_i(\tau)\|^{s_i'}_{(W_0^{1,s_i}(G)^d)'}\,\mathrm{d}t\,\mathrm{d}\tau &= \int_0^h \frac{1}{h}\|f_i(\tau)\|^{s_i'}_{(W_0^{1,s_i}(G)^d)'}\tau\,\mathrm{d}\tau \\ &\leq \int_0^h \|f_i(\tau)\|^{s_i'}_{(W_0^{1,s_i}(G)^d)'}\,\mathrm{d}\tau\,.\end{aligned}$$

The second integral is

$$\begin{aligned}\int_h^{T-h}\int_{\tau-h}^\tau \frac{1}{h}\|f_i(\tau)\|^{s_i'}_{(W_0^{1,s_i}(G)^d)'}\,\mathrm{d}t\,\mathrm{d}\tau &= \int_h^{T-h} \frac{1}{h}\|f_i(\tau)\|^{s_i'}_{(W_0^{1,s_i}(G)^d)'}h\,\mathrm{d}\tau \\ &= \int_h^{T-h} \|f_i(\tau)\|^{s_i'}_{(W_0^{1,s_i}(G)^d)'}\,\mathrm{d}\tau\,.\end{aligned}$$

And finally the third integral reads

$$\int_{T-h}^{T}\int_{\tau-h}^{T-h}\frac{1}{h}\|f_i(\tau)\|^{s_i'}_{(W_0^{1,s_i}(G)^d)'}\,\mathrm{d}t\,\mathrm{d}\tau = \int_{h}^{T-h}\frac{1}{h}\|f_i(\tau)\|^{s_i'}_{(W_0^{1,s_i}(G)^d)'}(T-\tau)\,\mathrm{d}\tau \leq \int_{T-h}^{T}\|f_i(\tau)\|^{s_i'}_{(W_0^{1,s_i}(G)^d)'}\,\mathrm{d}\tau\,.$$

The sum of those parts then equals the norm of f_i and thus

$$\int_0^{T-h}\|D_h\hat{\pi}_i(t)\|^{s_i'}_{L_0^{s_i'}(G)}\,\mathrm{d}t \leq c\,\|f_i\|^{s_i'}_{L^{s_i'}(0,T;(W_0^{1,s_i}(G)^d)')}.$$

Hence, with [27, Section 5.8.2, Theorem 3] or [23, Theorem 5.22], we have

$$\hat{\pi}_i \in W^{1,s_i'}(0,T;L_0^{s_i'}(G))$$

and

$$\|\partial_t\hat{\pi}_i\|_{L^{s_i'}(0,T;L_0^{s_i'}(G))} \leq c\,\|f_i\|_{L^{s_i'}(0,T;(W_0^{1,s_i}(G)^d)')}.$$

Let us now define $\pi_i = \partial_t\hat{\pi}_i$.

Pressure corresponding to the term u. Let us now consider the Stokes problem with the right-hand side $u(t)-u(0)$ for $t \in [0,T]$. Since $u(t)-u(0) \in H$, we have more regularity on the right-hand side than in the foregoing case. Thus, [33, Lemma IV.6.1] provides a unique strong solution $v_h(t)$ in $V_2(G) \cap W^{2,2}(G)$ and a pressure $\hat{\pi}_h(t) \in W^{1,2}(G)/\mathbb{R}$. In addition to that, we have the inequality

$$\|\hat{\pi}_h(t)\|_{W^{1,2}(G)/\mathbb{R}} \leq c\,\|u(t)-u(0)\|_H.$$

Since $W^{1,2}(G)/\mathbb{R}$ is isomorphic to the space of all functions $v \in W^{1,2}(G)$ with vanishing integral mean equipped with the usual $W^{1,2}(G)$-norm, we can consider $\hat{\pi}_h(t)$ as a function in $W^{1,2}(G)$ with $\int_G \hat{\pi}_h(t)\,\mathrm{d}x = 0$. Because, $v_h(t)$ is a strong solution, the equation

$$-\Delta v_h(t) + \nabla\hat{\pi}_h(t) = u(t) - u(0) \tag{5.8}$$

holds true in $L^2(G)^d$ for all $t \in [0,T]$.

Similarly to the procedure above, we can take differences of the Stokes equations for times t_1 and t_2. Linearity and uniqueness then yield

$$\|\hat{\pi}_h(t_1) - \hat{\pi}_h(t_2)\|_{W^{1,2}(G)} \leq c\,\|u(t_1) - u(t_2)\|_H.$$

In order to show $-\Delta\hat{\pi}_h(t) = 0$ in a weak sense, we start with the observation that the right-hand side in the Stokes equation is solenoidal. To exploit this, we take the weak divergence of (5.8), i.e. we test (5.8) with $\nabla\varphi$ for a function $\varphi \in C_0^\infty(G)^d$. This gives

$$\int_G \nabla\hat{\pi}_h \cdot \nabla\varphi\,\mathrm{d}x = \int_G \Delta v_h \cdot \nabla\varphi\,\mathrm{d}x + \int_G (u(t)-u(0))\cdot\nabla\varphi\,\mathrm{d}x\,.$$

Since $u(t) - u(0) \in H$, we know that

$$-\int_G (u(t)-u(0))\cdot\nabla\varphi\,\mathrm{d}x = \langle\,\mathrm{div}\,(u(t)-u(0)),\varphi\rangle = 0,$$

see (2.1).

For smooth φ holds with Schwarz' theorem

$$\operatorname{div} \nabla\nabla\varphi = \nabla \operatorname{div} \nabla\varphi.$$

Hence, because $v_h(t)$ is solenoidal

$$\begin{aligned}
\int_G \big(- \Delta v_h(t) \cdot \nabla\varphi\big) \,\mathrm{d}x &= -\int_G \nabla v_h(t) : \nabla\nabla\varphi \,\mathrm{d}x \\
&= \int_G v_h(t) \cdot \operatorname{div} \nabla\nabla\varphi \,\mathrm{d}x \\
&= \int_G v_h(t) \cdot \nabla \operatorname{div} \nabla\varphi \,\mathrm{d}x \\
&= -\int_G \operatorname{div} v_h(t) \operatorname{div} \nabla\varphi \,\mathrm{d}x \\
&= 0.
\end{aligned}$$

Thus, together with a density argument follows

$$\langle -\Delta\hat{\pi}_h, w\rangle = \int_G \nabla\hat{\pi}_h \nabla w \,\mathrm{d}x = \int_G \Delta v_h \nabla w \,\mathrm{d}x + \int_G (u(t) - u(0)) \cdot \nabla w \,\mathrm{d}x = 0$$

for all $w \in W_0^{1,2}(G)$.

Unfortunately, it is not possible to apply the well known theory of Agmon, Douglis and Nirenberg (see e.g. [4]) on G to prove more regularity for $\hat{\pi}_h$, since $\hat{\pi}_h$ has no regularity on the boundary of G. Thus we have to restrict ourselves to local regularity arguments. By [34, Theorem 8.24] and $-\Delta\hat{\pi}_h = 0$, we have $\hat{\pi}_h(t) \in C^\alpha(\overline{G'})$ for some $0 < \alpha < 1$ and $G' \subset\subset G$ and in particular $\hat{\pi}_h(t) \in C(\overline{G'})$. Thus, $\hat{\pi}_h(t)$ is continuous on the boundary of G' and by [34, Corollary 9.18] and uniqueness follows $\hat{\pi}_h(t) \in W^{2,\infty}_{\mathrm{loc}}(G')$. Furthermore, with [34, Theorem 9.11] follows

$$\|\hat{\pi}_h(t)\|_{W^{2,\infty}(G'')} \le c\, \|\hat{\pi}_h(t)\|_{L^\infty(G')}$$

for $G'' \subset\subset G'$. The embedding $C^\alpha(\overline{G'}) \hookrightarrow L^\infty(G')$ and the estimate in [34, Theorem 8.24] then imply

$$\|\hat{\pi}_h(t)\|_{W^{2,\infty}(G'')} \le \|\hat{\pi}_h(t)\|_{L^2(G)}.$$

Since $G'' \subset\subset G' \subset\subset G$ were arbitrary, we also have $\hat{\pi}_h(t) \in W^{2,\infty}_{\mathrm{loc}}(G)$.

Putting the equations together. It is now mandatory to show that these pressure terms fulfil the original equation. By the linearity of the Stokes operator, we have for the function $v(t) = \sum_i v_i(t) + v_h(t)$ and every $\varphi \in C_0^\infty(G)^d$

$$\begin{aligned}
\langle C_s v(t), \varphi\rangle &+ \left\langle \nabla \left(\sum_{i=1}^k \hat{\pi}_i(t) + \hat{\pi}_h(t) \right), \varphi \right\rangle \\
&= \left\langle \int_0^t \sum_{i=1}^k f_i(\tau) \,\mathrm{d}\tau - u(t) + u(0), \varphi \right\rangle.
\end{aligned}$$

Inserting (5.6) yields

$$\langle C_s v(t), \varphi\rangle + \left\langle \nabla \left(\sum_{i=1}^{k} \hat{\pi}_i(t) + \hat{\pi}_h(t) - \hat{\pi}(t) \right), \varphi \right\rangle = 0$$

and thus, the pair $\left(v(t), \sum_{i=1}^{k} \hat{\pi}_i(t) + \hat{\pi}_h(t) - \hat{\pi}(t)\right)$ is a weak solution to the Stokes problem with zero data. By [33, Lemma IV.6.2] we know, that velocity and pressure are zero for zero data, which implies $v(t) = 0$ and $\hat{\pi}(t) = \sum_i \hat{\pi}_i(t) + \hat{\pi}_h(t)$.

It is now left to restore the original equation by differentiating (5.6). For this, we test (5.6) with $\partial_t \psi(t) = \eta\varphi'(t)$, for $\varphi \in C_0^\infty(0,T)$, $\eta \in V_s$ and receive

$$\langle u(t), \eta\varphi'(t)\rangle - \langle u(0), \eta\varphi'(t)\rangle = -\langle \nabla\hat{\pi}, \eta\varphi'(t)\rangle + \left\langle \int_0^t \sum_{i=1}^{k} f_i(\tau)\, \mathrm{d}\tau\, , \eta\varphi'(t) \right\rangle .$$

Integrating over $(0,T)$, applying the definition of the weak derivative and [25, Theorem 8.1.5 (iii)] gives

$$\int_0^T \langle u(t), \eta\rangle \varphi'(t)\, \mathrm{d}t \;= -\int_0^T \langle u'(t), \eta\rangle \varphi(t)\, \mathrm{d}t \;= -\langle u', \psi(t)\rangle$$

and

$$\begin{aligned} \int_0^T \left\langle \int_0^t \sum_{i=1}^{k} f_i(\tau)\, \mathrm{d}\tau\, , \eta \right\rangle \varphi'(t)\, \mathrm{d}t \;&= -\int_0^T \left\langle \sum_{i=1}^{k} f_i(t), \eta \right\rangle \varphi(t)\, \mathrm{d}t \\ &= -\left\langle \sum_{i=1}^{k} f_i, \psi \right\rangle . \end{aligned}$$

For the pressure term we remind $\pi_i = \partial_t \hat{\pi}_i$ and calculate

$$\begin{aligned} &\int_0^T \langle \nabla\hat{\pi}(t), \eta\rangle \varphi'(t)\, \mathrm{d}t \\ &\quad = \int_0^T \left\langle \nabla \left(\sum_{i=1}^{k} \hat{\pi}_i(t) \right), \eta \right\rangle \varphi'(t)\, \mathrm{d}t + \int_0^T \langle \nabla\hat{\pi}_h(t), \eta\rangle \varphi'(t)\, \mathrm{d}t \\ &\quad = -\int_0^T \int_G \left(\sum_{i=1}^{k} \hat{\pi}_i(t) \right) \operatorname{div} \eta\, \mathrm{d}x \;\varphi'(t)\, \mathrm{d}t + \int_0^T \langle \nabla\hat{\pi}_h(t), \eta\rangle \varphi'(t)\, \mathrm{d}t \\ &\quad = \int_0^T \int_G \left(\sum_{i=1}^{k} \pi_i(t) \right) \operatorname{div} \psi(t)\, \mathrm{d}x \;\mathrm{d}t + \int_0^T \int_G \nabla\hat{\pi}_h(t) \cdot \partial_t\psi(t)\, \mathrm{d}x \;\mathrm{d}t\, . \end{aligned}$$

Altogether, this gives

$$\begin{aligned}\langle u', \psi \rangle &= \int_0^T \langle \nabla \hat{\pi}(t), \eta \rangle \varphi'(t) \, \mathrm{d}t \; + \left\langle \sum_{i=1}^k f_i, \psi \right\rangle \\ &= \int_0^T \int_G \left(\sum_{i=1}^k \pi_i(t) \right) \operatorname{div} \psi(t) \, \mathrm{d}x \; \mathrm{d}t \\ &\quad + \int_0^T \int_G \nabla \hat{\pi}_h(t) \cdot \partial_t \psi(t) \, \mathrm{d}x \; \mathrm{d}t \; + \left\langle \sum_{i=1}^k f_i, \psi \right\rangle .\end{aligned}$$

Finally Corollary A.2 ensures, that it is sufficient to test with $\psi = \varphi \, \eta$. □

6

Parabolic Lipschitz truncation

6.1 Preliminaries

In this chapter, we introduce a Lipschitz truncation method for non-stationary problems. We follow the ideas of Diening, Růžička and Wolf [22, Section 3]. In the stationary case, the Lipschitz truncation technique has been successfully employed in our context by Diening, Málek and Steinhauer in [19] and Frehse, Málek and Steinhauer in [31].

For an open, bounded set G and a given function $u \in L^p(0,T;W^{1,p}(G)^d)$ we try to find an approximation $\mathcal{T}u$ that equals u on a large set and has an essentially bounded gradient on every compact subset of $G \times (0,T)$. For this, we first determine a set E of irregularity where u is to be cut off. The set E will then be covered by a Whitney-type covering so that one can define a suitable partition of unity on it. In this way, we can smoothly extend the cut of u to the whole domain again.

Definition of an anisotropic metric. In order to show regularity of the extension of u, we will employ Poincaré-type inequalities in the space-time cylinders of the Whitney covering. For these inequalities, it is crucial to have a suitable representation of a time derivative of u.

In [22] the Lipschitz truncation relies on the existence of a full time derivative of u belonging to some dual space $L^{s'}(0,T;(W_0^{1,s}(G)^d)')$, $1 < s < \infty$, and the representation

$$\langle u', \varphi \rangle = \int_{G\times(0,T)} H : \nabla\varphi \, \mathrm{d}(x,t)$$

for some $H \in L^{s'}(0,T;L^{s'}(G)^{d\times d})$. The constant in Poincaré's inequality then comes with an exponent $1/2$ in time (see [22, Theorem B.1] for Poincaré's inequality). Thus, Diening, Růžička and Wolf introduce an anisotropic metric on $\mathbb{R}^{d+1}$, where the time-component is scaled with the exponent $1/2$.

In our case, we want to employ the truncation theorem to functions that possess fractional regularity (i.e. belong to some Nikolskii or Sobolev-Slobodeckii space) of order $\sigma > 0$ taking values in H (see Section 8.7). The constant in the corresponding Poincaré inequality (see Lemma 2.11) then has the exponent σ in time. Therefore, we will introduce a metric that is scaled with the exponent σ in time.[1]

[1] To be precise: The functions will be of order $\bar{\sigma} > \sigma$, but for compactness arguments we have to decrease the order somewhat.

We will often work with subsets of the space-time cylinder $G\times(0,T)\subset\mathbb{R}^{d+1}$. To avoid unnecessary complexity, we stick to the notation of [22] and denote typical points (x,t) and (y,s) in $\mathbb{R}^{d+1}$ by X and Y, respectively.

For a given $\sigma>0$, we introduce the metric

$$\varrho_\sigma(X,Y)=\max\{|y-x|,|s-t|^\sigma\},\quad X,Y\in\mathbb{R}^{d+1},$$

where $|\cdot|$ denotes the Euklidean norm or the absolute value respectively.

Let us write $\varrho=\varrho_\sigma$ for notational simplicity. The balls with respect to this metric are space-time cylinders of the form

$$Q_r(X)=B_r(x)\times(t-r^{1/\sigma},t+r^{1/\sigma}),$$

for an Euklidean ball $B_r(x)$. Let us denote scaled balls by $\gamma Q_r=Q_{\gamma r}$.

Definition of the Hardy-Littlewood maximal function. The determination of the cut-off sets will be done with the help of the Hardy-Littlewood maximal function. Following [22, Appendix A], we define for $1<s<\infty$, $g\in L^s(\mathbb{R}^{d+1})$ and $(x,t)\in\mathbb{R}^{d+1}$

$$\mathcal{M}_x(g)(x,t)=\sup_{0<\gamma<\infty}\fint_{B_\gamma(x)}|g(y,t)|\,\mathrm{d}y\,,$$
$$\mathcal{M}_t(g)(x,t)=\sup_{0<\gamma<\infty}\fint_{I_\gamma(t)}|g(x,\tau)|\,\mathrm{d}\tau\,,$$

where $B_\gamma(x)$ denotes an Euklidean ball in $\mathbb{R}^d$ with radius γ centered in x and $I_\gamma(t)$ denotes the interval $(t-\gamma,t+\gamma)$.

Next, we define

$$\mathcal{M}(g)=\mathcal{M}_t(\mathcal{M}_x(g)).$$

In the following, let us abbreviate sets $\{X\in\mathbb{R}^{d+1}:f(X)>\lambda\}$ with $\{f>\lambda\}$. One can show (see [22, Appendix A], [61] or [49, Theorem 1.4.2]) the strong type estimate

$$\|\mathcal{M}(g)\|_{L^s(\mathbb{R}^{d+1})}\le c\,\|g\|_{L^s(\mathbb{R}^{d+1})},\tag{6.1}$$

and the weak type estimate

$$\mu_{d+1}\big(\{\mathcal{M}(g)>\lambda\}\big)\le c\,\lambda^{-s}\|g\|^s_{L^s(\mathbb{R}^{d+1})},\quad\lambda>0.\tag{6.2}$$

Furthermore,

$$g(X)\le\mathcal{M}(g)(X)\tag{6.3}$$

holds for almost all $X\in\mathbb{R}^{d+1}$.

Basic idea for the truncation. Assuming that we had a full time derivative in some L^p-space, we can examplify the idea of the proof with the following observation: One can consider the space of all functions u for which there is a function g with

$$|u(X)-u(Y)|\le\varrho(X,Y)(g(X)+g(Y))\quad\text{for almost every }X,Y\in\mathbb{R}^{d+1}.$$

The boundedness of g would then imply the Lipschitz continuity of u. These spaces have been studied for example by Hajłasz [7]. It turns out that for a

Sobolev-space $W^{1,p}(G \times (0,T))$, this holds true for the function $g = c\,\mathcal{M}(|\nabla u|)$, where the gradient is to be understood in spatial as well as in time direction and $\mathcal{M}$ is the Hardy-Littlewood maximal function. This motivates the name "generalized gradient" for g. Thus, cutting off the maximal function of the gradient of u results in a Lipschitz-contiuous function $\tilde{u}$.

Important in this context is, that the measure fulfils the homogeneity property

$$\mu_{d+1}(Q_{2r}(X)) \leq c(d)\,\mu_{d+1}(Q_r(X))$$

for balls with respect to the metric ϱ and $r > 0$. One then calls the considered measure "doubling" and the metric measure space of "homogeneous type" (see [57]). Of course, this is true in our case.

Unfortunately, we do not have a full time derivative at hand and thus cannot directly apply this method like in [1] or [57]. But still, we will determine the set E of irregularity, on which u will be regularized, with the maximal function of the gradient of u and some quantity related to the time derivative.

6.2 Covering of the cut-off set E

We will now introduce the Whitney covering for an open, bounded set. The actual set E will be determined in Section 6.6.

Lemma 6.1 (Whitney covering)**.** *Let $E \subset \mathbb{R}^{d+1}$ be a non-empty, open and bounded set. Then there exists a countable covering of E consisting of balls $\{Q_i\}_{i\in\mathbb{N}} = \{Q_{r_i}(X_i)\}_{i\in\mathbb{N}}$ centered in $X_i \in \mathbb{R}^{d+1}$ and with radii $r_i > 0$, which depends on the metric ϱ_σ, such that*

(W1) $\bigcup_{i\in\mathbb{N}} \frac{1}{2}Q_i = E$,

(W2) for all $i \in \mathbb{N}$ we have $8Q_i \subset E$ and $16Q_i \cap E^c \neq \emptyset$,

(W3) for all $i, j \in \mathbb{N}$ there holds: if $Q_i \cap Q_j \neq \emptyset$ then $\frac{1}{2}r_j \leq r_i \leq 2r_j$,

(W4) each $X \in E$ belongs to at most 120^{d+2} of the sets $4Q_i$,

(W5) $\sum_{i\in\mathbb{N}} \mu_{d+1}(4Q_i) \leq c\,\mu_{d+1}(E)$.

Defining the set $A_i = \left\{j \in \mathbb{N} : \frac{2}{3}Q_i \cap \frac{2}{3}Q_j \neq \emptyset\right\}$ we have $\operatorname{card}(A_i) \leq 120^{d+2}$ *and*

(W6) $Q_j \subset 4Q_i \subset E$ for all $j \in A_i$, $i \in \mathbb{N}$.

Proof. A proof of this lemma can be found in [22, Appendix C], [61] or [36]. □

6.3 Partition of unity

For each of the balls Q_i let $\eta_i \in C_0^\infty(\mathbb{R}^{d+1})$ be a cut-off function with $0 \leq \eta_i \leq 1$, that vanishes on $\mathbb{R}^{d+1} \setminus \frac{2}{3}Q_i$ and equals one on $\frac{1}{2}Q_i$. Let us denote by $\operatorname{Lip}_\varrho(\eta_i)$ the smallest Lipschitz-constant of η_i with respect to the metric ϱ_σ. It is no restriction to assume that for all $i \in \mathbb{N}$ and $X \in Q_i$ holds

$$\operatorname{Lip}_\varrho(\eta_i) + |\partial_t \eta_i(X)|^\sigma \leq c\,r_i^{-1}. \tag{6.4}$$

If we sum up all functions η_i, then the properities of A_i imply that the non-trivial part of the sum is actually finite. This means that on a ball Q_i, we know

$$\sum_{j\in\mathbb{N}} \eta_j = \sum_{j\in A_i} \eta_j \in C^\infty(\mathbb{R}^{d+1}).$$

Due to the property (W1) of the Whitney covering lemma it is clear, that $\sum_{j\in\mathbb{N}} \eta_j(X) \neq 0$ holds for every $X \in E$. This justifies the definition

$$\psi_i(X) = \frac{\eta_i(X)}{\sum_{j\in\mathbb{N}} \eta_j(X)}, \quad X \in E, \quad i \in \mathbb{N}.$$

Of course, it is $\psi_i \in C^\infty(\mathbb{R}^{d+1})$ and we have $\psi_i \equiv 0$ in $\mathbb{R}^{d+1} \setminus \frac{2}{3}Q_i$.

Lemma 6.2. *The set of functions $\{\psi_i\}_{i\in\mathbb{N}}$ forms a partition of unity on E, such that*

$$\sum_{j\in\mathbb{N}} \psi_j \equiv 1 \quad \text{in } E$$

and we have the estimate

$$Lip_\varrho(\psi_i) + |\partial_t \psi_i(X)|^\sigma \leq c\, r_i^{-1} \quad \text{in } \mathbb{R}^{d+1},$$

for every $i \in \mathbb{N}$ and $X \in Q_i$.

Proof. See also [22, pp. 10f.]. Let $i \in \mathbb{N}$, $X, Y \in Q_i$. Then (W1) implies $\sum_{j\in\mathbb{N}} \eta_j \geq 1$ and hence

$$\begin{aligned}
&|\psi_i(X) - \psi_i(Y)| \\
&\leq \left| \frac{\eta_i(X)}{\sum_j \eta_j(X)} - \frac{\eta_i(Y)}{\sum_j \eta_j(X)} \right| + \left| \frac{\eta_i(Y)}{\sum_j \eta_j(X)} - \frac{\eta_i(Y)}{\sum_j \eta_j(Y)} \right| \\
&\leq \frac{1}{\sum_j \eta_j(X)} |\eta_i(X) - \eta_i(Y)| + \eta_i(Y) \left| \frac{\sum_j (\eta_j(X) - \eta_j(Y))}{\sum_j \eta_j(X) \sum_j \eta_j(Y)} \right| \\
&\leq |\eta_i(X) - \eta_i(Y)| + \sum_{j\in A_i} |\eta_j(X) - \eta_j(Y)|,
\end{aligned}$$

where each sum is taken over all $j \in A_i$.

We can now estimate the several terms using (6.4). We receive

$$|\psi_i(X) - \psi_i(Y)| \leq c \left(r_i^{-1} + \sum_{j\in A_i} r_j^{-1} \right) \varrho(X, Y).$$

With property (W3) we can estimate $r_j^{-1} \leq 2r_i^{-1}$ and with $\operatorname{card}(A_i) \leq 120^{d+2}$ follows

$$|\psi_i(X) - \psi_i(Y)| \leq c\, r_i^{-1} \varrho(X, Y).$$

One similarly shows for arbitrary $X \in Q_i$

$$|\partial_t \psi_i(X)| \leq c\, r_i^{-1/\sigma}.$$

□

6.4 Definition and properties of the truncation operator

For $E \subset G \times (0,T)$ let $\{\psi_i\}_{i\in\mathbb{N}}$ be the partition of unity corresponding to E and ϱ_σ from above. For a function $u \in L^1_{\mathrm{loc}}(G \times (0,T))^d$ we define the linear truncation operator $\mathcal{T} = \mathcal{T}_{E,\sigma}$ by

$$(\mathcal{T}u)(X) = \begin{cases} u(X) & \text{if} \quad X \in E^c, \\ \sum_i \psi_i(X) u_{Q_i} & \text{if} \quad X \in E. \end{cases}$$

We remind that

$$u_{Q_i} = \fint_{Q_i} u(X)\,\mathrm{d}X = \frac{1}{\mu_{d+1}(Q_i)} \int_{Q_i} u(X)\,\mathrm{d}X$$

is the integral mean of u on Q_i.

We will see that on a Whitney ball Q_i in E, most of the properties of the functions ψ_i carry over to the truncation $\mathcal{T}u$ with the cost of an additional term

$$\fint_{4Q_i} |u - u_{4Q_i}|\,\mathrm{d}X\,.$$

So one can say, that controlling this term means controlling the regularity of the truncation. It is therefore natural, that a crucial point will be Poincaré's inequality.

Lemma 6.3 (Properties of the truncation operator)**.** *For all $p \in [1,\infty]$, the operator*

$$\mathcal{T} : L^p(G \times (0,T))^d \to L^p(G \times (0,T))^d$$

is well-defined, linear and bounded, i.e., there exists a constant $c > 0$ depending on the dimension d such that

$$\|\mathcal{T}u\|_{L^p(G\times(0,T))^d} \le c\,\|u\|_{L^p(G\times(0,T))^d}$$

for all functions $u \in L^p(G \times (0,T))^d$.

For any function $u \in L^1_{\mathrm{loc}}(G \times (0,T))^d$ and for all $Y, Z \in Q_i$, $i \in \mathbb{N}$, we have

$$|(\mathcal{T}u)(Y) - (\mathcal{T}u)(Z)| \le c\,r_i^{-1}\varrho(Y,Z) \fint_{4Q_i} |u - u_{4Q_i}|\,\mathrm{d}X\,, \tag{6.5}$$

$$|(\partial_t \mathcal{T}u)(Y)| \le c\,r_i^{-1/\sigma} \fint_{4Q_i} |u - u_{4Q_i}|\,\mathrm{d}X \tag{6.6}$$

and

$$\fint_{Q_i} |\mathcal{T}u - u|\,\mathrm{d}X \le c \fint_{4Q_i} |u - u_{4Q_i}|\,\mathrm{d}X\,. \tag{6.7}$$

Proof. The proof is straightforward calculation using the properties (W1)–(W6) of the Whitney covering and Lemma 6.2. For more details, see [22, Lemma 3.11, Lemma 3.13, Lemma 3.16, Lemma 3.19]. ☐

6.5 Lipschitz truncation

Let us now state the main theorem of this section. For the sake of better readability, we will abbreviate the notation for sequences $\{u_{(\Delta t)_\nu}\}_{\nu\in\mathbb{N}}$ with $\{u_{\Delta t}\}$.

Theorem 6.4. *Let $\{u_{\Delta t}\}$ be a bounded sequence in*

$$L^\infty(0,T;L^2(G)^d)\cap L^p(0,T;W^{1,p}(G)^d)\cap W^{\bar\sigma,q}(0,T;L^2(G)^d)$$

for $0<\bar\sigma<1$, $1<q<\infty$, such that

$$u_{\Delta t}\rightharpoonup 0 \quad in \quad L^p(0,T;W^{1,p}(G)^d) \text{ and } W^{\bar\sigma,q}(0,T;L^2(G)^d).$$

Lemma 2.4 then implies

$$u_{\Delta t}\to 0 \quad in \quad L^2(G\times(0,T))^d.$$

Let $\{\theta_{\Delta t}\}\subset(0,\infty)$, such that

$$\theta_{\Delta t}\to 0 \quad and \quad \frac{\|u_{\Delta t}\|_{L^2(G\times(0,T))^d}}{\theta_{\Delta t}}\to 0.$$

Let $0<\sigma<\bar\sigma$ be arbitrary. Then, for any $k\in\mathbb{N}$ and Δt, there exists a number $\lambda_{k,\Delta t}\in[2^{2^k},2^{2^{k+1}}]$ and a set $E_{k,\Delta t}\subset G\times(0,T)$ with

$$\limsup_{\Delta t\to 0}\lambda^p_{k,\Delta t}\mu_{d+1}(E_{k,\Delta t})\le c\,2^{-k}. \tag{6.8}$$

Corresponding to the metric $\varrho=\varrho_\sigma$, there exists a truncation operator $\mathcal{T}=\mathcal{T}_{E_{k,\Delta t},\varrho}$ such that

$$\{\mathcal{T}u_{\Delta t}\neq u_{\Delta t}\}\subset E_{k,\Delta t}$$

and for any compact set $K\subset G\times(0,T)$ the Lipschitz truncation $\mathcal{T}u_{\Delta t}$ is Lipschitz continuous with respect to the metric $\varrho=\varrho_\sigma$. We have

$$\|\nabla\mathcal{T}u_{\Delta t}\|_{L^\infty(K)^{d\times d}}\le c\left(\lambda_{k,\Delta t}+\delta^{-d-1-\frac{1}{\sigma}}\|u_{\Delta t}\|_{L^1(E_{k,\Delta t})^d}\right), \tag{6.9}$$

$$\|\mathcal{T}u_{\Delta t}\|_{L^\infty(K)^d}\le c\left(\theta_{\Delta t}+\delta^{-d-\frac{1}{\sigma}}\|u_{\Delta t}\|_{L^1(E_{k,\Delta t})^d}\right), \tag{6.10}$$

where $\delta=\delta_{K,\varrho}=dist_\varrho(K,\partial(G\times(0,T)))$ denotes the distance between K and the boundary of $G\times(0,T)$ with respect to the metric ϱ_σ.

Additionally, the size of the gradient of $\mathcal{T}u_{\Delta t}$ gets small on $E_{k,\Delta t}$, i.e.

$$\limsup_{\Delta t\to 0}\|\nabla\mathcal{T}u_{\Delta t}\|_{L^p(E_{k,\Delta t}\cap K)^{d\times d}}\le c\,2^{-k/p}.$$

For a function $\zeta\in C_0^\infty(G\times(0,T))$ with supp $\zeta\subset K$, we have for any fixed $k\in\mathbb{N}$

$$\zeta\mathcal{T}u_{\Delta t}\to 0 \quad in\ L^s(0,T;L^s(G)^d) \text{ for all } s\in[1,\infty],$$

$$\zeta\mathcal{T}u_{\Delta t}\rightharpoonup 0 \quad in\ L^s(0,T;W_0^{1,s}(G)^d) \text{ for all } s\in[1,\infty),$$

as $\Delta t\to 0$.

Remark 6.5. The set $E_{k,\Delta t}$ is determined by the maximal function of derivatives of $u_{\Delta t}$. To be precise, it is given by

$$E_{k,\Delta t} = \{\mathcal{M}(|\nabla u_{\Delta t}|) > \lambda_{k,\Delta t}\} \\ \cup \{\mathcal{M}(|\partial_t^\sigma u_{\Delta t}|) > \lambda_{k,\Delta t}\} \cup \{\mathcal{M}(|u_{\Delta t}|) > \theta_{\Delta t}\},$$

where $\partial_t^\sigma u$ is a quantity related to a fractional time derivative of u defined in (6.11).

We split the proof into several parts.

6.6 Definition of the set $E_{k,\Delta t}$

We introduce the following quantity which reminds of a fractional time derivative of u. For $0 < \sigma < \bar{\sigma}$ we define

$$\partial_t^\sigma u_{\Delta t}(x,t) = \int_0^T \frac{|u_{\Delta t}(x,t) - u_{\Delta t}(x,s)|}{|t-s|^{1+\sigma}} \, \mathrm{d}s \,. \tag{6.11}$$

Lemma 6.6. *Let the assumptions of Theorem 6.4 be satisfied. Then for* $\tilde{q} = \min(q,2)$ *holds*

$$\|\partial_t^\sigma u_{\Delta t}\|_{L^{\tilde{q}}(G\times(0,T))} \to 0 \quad \textit{as} \quad \Delta t \to 0.$$

Proof. Let $\varepsilon > 0$ be sufficiently small such that $\sigma + \varepsilon < \bar{\sigma}$. We omit the subscript Δt for better readability. Employing Hölder's inequality and Fubini's theorem yields

$$\begin{aligned}
&\|\partial_t^\sigma u\|_{L^{\tilde{q}}(G\times(0,T))}^{\tilde{q}} \\
&= \int_0^T \int_G \left(\int_0^T \frac{|u(x,t) - u(x,s)|}{|t-s|^{1+\sigma}} \, \mathrm{d}s \right)^{\tilde{q}} \mathrm{d}x \; \mathrm{d}t \\
&= \int_0^T \int_G \left(\int_0^T \frac{1}{|t-s|^{1-\frac{1}{\tilde{q}}-\varepsilon}} \frac{|u(x,t) - u(x,s)|}{|t-s|^{\frac{1}{\tilde{q}}+\sigma+\varepsilon}} \, \mathrm{d}s \right)^{\tilde{q}} \mathrm{d}x \; \mathrm{d}t \\
&\le \int_0^T \int_G \left(\int_0^T \frac{1}{|t-s|^{1-\frac{\tilde{q}}{\tilde{q}-1}\varepsilon}} \, \mathrm{d}s \right)^{\tilde{q}-1} \int_0^T \frac{|u(x,t) - u(x,s)|^{\tilde{q}}}{|t-s|^{1+\tilde{q}(\sigma+\varepsilon)}} \, \mathrm{d}s \; \mathrm{d}x \; \mathrm{d}t \\
&\le c \int_0^T \int_0^T \frac{\|u(t) - u(s)\|_{L^{\tilde{q}}(G)^d}^{\tilde{q}}}{|t-s|^{1+\tilde{q}(\sigma+\varepsilon)}} \, \mathrm{d}s \; \mathrm{d}t \\
&\le c \, |u|_{W^{\sigma+\varepsilon,\tilde{q}}(0,T;L^{\tilde{q}}(G)^d)}^{\tilde{q}} \\
&\le c \, |u|_{W^{\sigma+\varepsilon,q}(0,T;L^2(G)^d)}^{\tilde{q}}
\end{aligned}$$

From the assumptions of Theorem 6.4 we know, that

$$u_{\Delta t} \rightharpoonup 0 \quad \text{in} \quad W^{\bar{\sigma},q}(0,T;L^2(G)^d).$$

The compactness result Lemma 2.5 implies the strong convergence of $\{u_{\Delta t}\}$ in $W^{\sigma+\varepsilon,q}(0,T;L^2(G)^d)$. □

The next step will be to find suitable sets where our function will be cut off. For this purpose we take advantage of the Hardy-Littlewood maximal operator $\mathcal{M}$ defined in the beginning of this chapter and some weak-type inequalities.

The idea is to use the Chebyshev (weak-type) inequality

$$\mu_{d+1}(\{|g| > \lambda\}) \leq c^p \, \lambda^{-p} \|g\|^p_{L^p(\mathbb{R}^{d+1})}$$

to estimate the largeness of a function g. We could take $g = \mathcal{M}(|\nabla u_{\Delta t}|)$ for example. But if g converges weakly, we only know that the right-hand side is bounded. The next lemma gives a finer estimate in which we choose the parameter λ conveniently, such that the constant in the inequality behaves like 2^{-k}.

Lemma 6.7. *For a given function $g \in L^p(\mathbb{R}^{d+1})$ and any $k \in \mathbb{N}$ there exists a number $\lambda_k \in [2^{2^k}, 2^{2^{k+1}}]$ such that*

$$\lambda_k^p \, \mu_{d+1}(\{|g| > \lambda_k\}) \leq (\ln 2)^{-1} \, 2^{-k} \, \|g\|^p_{L^p(\mathbb{R}^{d+1})}.$$

Proof. See also [22, pp. 28f.] or [49, Lemma 1.1.2]. With Fubini's theorem we calculate

$$\begin{aligned}
\|g\|^p_{L^p(\mathbb{R}^{d+1})} &= \int_{\mathbb{R}^{d+1}} \int_0^{|g(x,t)|} (\lambda^p)' \, \mathrm{d}\lambda \; \mathrm{d}(x,t) \\
&= p \int_{\mathbb{R}^{d+1}} \int_0^{\infty} \lambda^{p-1} \chi_{[0,|g(x,t)|)}(\lambda) \, \mathrm{d}\lambda \; \mathrm{d}(x,t) \\
&= p \int_0^{\infty} \lambda^{p-1} \int_{\mathbb{R}^{d+1}} \chi_{[0,|g(x,t)|)}(\lambda) \, \mathrm{d}(x,t) \; \mathrm{d}\lambda \\
&= p \int_0^{\infty} \lambda^{p-1} \mu_{d+1}(\{|g(x,t)| > \lambda\}) \, \mathrm{d}\lambda \\
&\geq p \int_{2^{2^k}}^{2^{2^{k+1}}} \lambda^{-1} \inf_{\gamma \in [2^{2^k}, 2^{2^{k+1}}]} \gamma^p \mu_{d+1}(\{|g(x,t)| > \gamma\}) \, \mathrm{d}\lambda \\
&= p \left(2^{k+1} \ln(2) - 2^k \ln 2\right) \inf_{\gamma \in [2^{2^k}, 2^{2^{k+1}}]} \gamma^p \mu_{d+1}(\{|g(x,t)| > \gamma\}).
\end{aligned}$$

This gives the inequality

$$\inf_{\gamma \in [2^{2^k}, 2^{2^{k+1}}]} \gamma^p \mu_{d+1}(\{|g(x,t)| > \gamma\}) \leq p^{-1} (\ln 2)^{-1} 2^{-k} \|g\|^p_{L^p(\mathbb{R}^{d+1})}.$$

Thus, there exists a $\lambda_k \in [2^{2^k}, 2^{2^{k+1}}]$ such that

$$\lambda_k^p \mu_{d+1}(\{|g(x,t)| > \gamma\}) \leq (\ln(2))^{-1} \, 2^{-k} \, \|g\|^p_{L^p(\mathbb{R}^{d+1})}.$$

□

We extend $u_{\Delta t}$, $\nabla u_{\Delta t}$ and $\partial_t^\sigma u_{\Delta t}$ with zero outside of $G \times (0,T)$. From Lemma 6.6 and the assumptions of Theorem 6.4, it is clear that in this case, we have $u_{\Delta t} \to 0$ in $L^2(\mathbb{R}^{d+1})^d$, $\nabla u_{\Delta t}$ is bounded in $L^p(\mathbb{R}^{d+1})^{d \times d}$ and $\partial_t^\sigma u_{\Delta t} \to 0$ in $L^{\tilde{q}}(\mathbb{R}^{d+1})$ for $\tilde{q} = \min(q, 2)$.

For $k \in \mathbb{N}$ let $\lambda_{k,\Delta t}$ be as in Lemma 6.7 for $g = \mathcal{M}(|\nabla u_{\Delta t}|)$. We define the sets

$$M^{\nabla u}_{k,\Delta t} = \{\mathcal{M}(|\nabla u_{\Delta t}|) > \lambda_{k,\Delta t}\},$$
$$M^{\partial_t^\sigma u}_{k,\Delta t} = \{\mathcal{M}(|\partial_t^\sigma u_{\Delta t}|) > \lambda_{k,\Delta t}\},$$
$$M^{u}_{k,\Delta t} = \{\mathcal{M}(|u_{\Delta t}|) > \theta_{\Delta t}\}.$$

Lemma 6.8. *Let* $\{u_{\Delta t}\}$ *satisfy the assumptions of Theorem 6.4. Then for every* Δt *and every* $k \in \mathbb{N}$

$$\lambda^p_{k,\Delta t}\mu_{d+1}(M^{\nabla u}_{k,\Delta t}) \le c\,2^{-k},$$
$$\lim_{\Delta t \to 0} \mu_{d+1}(M^{\partial_t^\sigma u}_{k,\Delta t}) = 0,$$
$$\lim_{\Delta t \to 0} \mu_{d+1}(M^{u}_{k,\Delta t}) = 0.$$

Proof. See also [22, p. 29] for a similar proof in the case of a full time derivative. For the first estimate we note that $\mathcal{M}$ is a bounded operator (see (6.1)) and hence $\mathcal{M}(|\nabla u_{\Delta t}|) \in L^p(\mathbb{R}^{d+1})$. This gives us the possibility to apply Lemma 6.7 with $g = \mathcal{M}(|\nabla u_{\Delta t}|)$. Finally, using the boundedness of $\{\nabla u_{\Delta t}\}$ in $L^p(\mathbb{R}^{d+1})^{d\times d}$ yields the desired estimate.

Since the maximal operator is of weak type $(\tilde q, \tilde q)$ (see (6.2) with $s = \tilde q$), we can show for $\tilde q = \min(q, 2)$ using Lemma 6.6

$$\begin{aligned}\mu_{d+1}(M^{\partial_t^\sigma u}_{k,\Delta t}) &= \mu_{d+1}\left(\{\mathcal{M}(|\partial_t^\sigma u_{\Delta t}|) > \lambda_{k,\Delta t}\}\right)\\ &\le \left(\frac{c}{\lambda_{k,\Delta t}}\right)^{\tilde q} \|\partial_t^\sigma u_{\Delta t}\|^{\tilde q}_{L^{\tilde q}(\mathbb{R}^{d+1})} \longrightarrow 0\end{aligned}$$

as $\Delta t \to 0$ because $\lambda_{k,\Delta t} \in [2^{2^k}, 2^{2^{k+1}}]$.

Analogously, we exploit the strong convergence of $\{u_{\Delta t}\}$ in $L^2(\mathbb{R}^{d+1})^d$ and the choice of $\{\theta_{\Delta t}\}$ to show

$$\begin{aligned}\mu_{d+1}(M^{u}_{k,\Delta t}) &= \mu_{d+1}\left(\{\mathcal{M}(|u_{\Delta t}|) > \theta_{\Delta t}\}\right)\\ &\le \left(\frac{c\,\|u_{\Delta t}\|_{L^2(\mathbb{R}^{d+1})^d}}{\theta_{\Delta t}}\right)^2 \longrightarrow 0.\end{aligned}$$

□

We now define the set

$$E_{k,\Delta t} = \left(M^{\nabla u}_{k,\Delta t} \cup M^{\partial_t^\sigma u}_{k,\Delta t} \cup M^{u}_{k,\Delta t}\right) \cap (G \times (0,T)).$$

From Lemma 6.8 it immediately follows (6.8), i.e.

$$\limsup_{\Delta t \to 0} \lambda^p_{k,\Delta t}\mu_{d+1}(E_{k,\Delta t}) \le c\,2^{-k}. \tag{6.12}$$

6.7 Properties of the truncation operator and of $u_{\Delta t}$ on $E_{k,\Delta t}$

We consider the metric $\varrho = \varrho_\sigma$. Let $\{Q_i\}_{i\in\mathbb{N}} = \{Q_{i,E_{k,\Delta t},\varrho}\}_{i\in\mathbb{N}}$ be the Whitney covering of the set $E_{k,\Delta t}$, $\mathcal{T} = \mathcal{T}_{E_{k,\Delta t},\varrho}$ the associated truncation operator and $K \subset G \times (0,T)$ compact. This compact subset is needed, because we assume no boundary regularity for the functions $u_{\Delta t}$. For the sake of simplicity, we will omit the subscript Δt of $u_{\Delta t}$ and the argument X whenever integrating.

As indicated in Lemma 6.3, it is crucial to bound the term $⨍_{4Q_i} |u - u_{4Q_i}| \,\mathrm{d}X$.

Lemma 6.9. *Under the assumptions of Theorem 6.4 we have for all Q_i belonging to the Whitney covering of $E_{k,\Delta t}$ with $Q_i \cap K \neq \emptyset$*

$$⨍_{4Q_i} |u - u_{4Q_i}| \,\mathrm{d}X \le c\, r_i \left(\lambda_{k,\Delta t} + \delta^{-d-1-\frac{1}{\sigma}} \|u\|_{L^1(E_{k,\Delta t})^d}\right),$$

where again $\delta = \delta_{K,\varrho}$ is the distance between K and the boundary of $G \times (0,T)$ with respect to ϱ_σ.

Proof. See also [22, Lemma 3.29] for the case of a full time derivative. Consider $Q_i \subset E_{k,\Delta t}$ belonging to the Whitney covering such that $Q_i \cap K \neq \emptyset$. Property (W2) of the Whitney covering tells us, that $16Q_i \cap E^c \neq \emptyset$. This means that one of the following holds

$$\text{(i)} \qquad 16Q_i \cap (G \times (0,T))^c \neq \emptyset,$$

$$\text{(ii)} \qquad 16Q_i \cap (M^{\nabla u}_{k,\Delta t})^c \cap (M^{\partial_t^\sigma u}_{k,\Delta t})^c \cap (M^{u}_{k,\Delta t})^c \neq \emptyset.$$

In the first case (i), the condition $Q_i \cap K \neq \emptyset$ ensures that Q_i lies sufficiently far away from the boundary of $G \times (0,T)$ and thus, the radius r_i of the Whitney ball Q_i cannot be very small. We know, that there exists a point $\hat{X} \in Q_i \cap K$ and thus, the triangle inequality for $\varrho = \varrho_\sigma$ and δ as in the lemma gives

$$\begin{aligned}\delta &\le \varrho(\hat{X}, (G \times (0,T))^c)\\ &\le \varrho(\hat{X}, X_i) + \varrho(X_i, (G \times (0,T))^c)\\ &\le r_i + 17 r_i\end{aligned}$$

and hence

$$r_i \ge c\,\delta.$$

With $\mu_{d+1}(4Q_i) = c\, r_i^{d+1/\sigma}$ we can estimate

$$\begin{aligned}⨍_{4Q_i} |u - u_{4Q_i}| \,\mathrm{d}X &\le 2 ⨍_{4Q_i} |u| \,\mathrm{d}X\\ &\le c\, r_i^{-d-1-\frac{1}{\sigma}} r_i \int_{Q_i} |u| \,\mathrm{d}X\\ &\le c\, r_i\, \delta^{-d-1-\frac{1}{\sigma}} \|u\|_{L^1(E_{k,\Delta t})^d}.\end{aligned}$$

In the case (ii), we do not know how small r_i can become. But since we know that Q_i is somehow near the set $(M^{\nabla u}_{k,\Delta t})^c \cap (M^{\partial_t^\sigma u}_{k,\Delta t})^c \cap (M^{u}_{k,\Delta t})^c$, we can use the smallness of the maximal functions there. Employing the Poincaré-type inequality from Lemma 2.11 on $4Q_i$ with $r = 4r_i$ and $|b-a| = 2\,(4r_i)^{1/\sigma}$ yields

$$\begin{aligned}\fint_{4Q_i} |u - u_{4Q_i}|\,\mathrm{d}X &\le c\,r_i \fint_{4Q_i} |\nabla u|\,\mathrm{d}X + \left|2\,r_i^{1/\sigma}\right|^{\sigma} \fint_{4Q_i} |\partial_t^{\sigma} u|\,\mathrm{d}X \\ &= c\,r_i \left(\fint_{4Q_i} |\nabla u|\,\mathrm{d}X + \fint_{4Q_i} |\partial_t^{\sigma} u|\,\mathrm{d}X \right).\end{aligned}$$

Here we see the reason for the scaling of the metric. Without the exponent σ, we had the factor r_i^{σ} instead of r_i.

Since $16Q_i \cap (M_{k,\Delta t}^{\nabla u})^c \cap (M_{k,\Delta t}^{\partial_t^{\sigma} u})^c \neq \emptyset$, there exists a point $\hat{X}$ in it, which means that the maximal functions of both $|\nabla u|$ and $|\partial_t^{\sigma} u|$ are bounded by $\lambda_{k,\Delta t}$ in $\hat{X}$. It is

$$4Q_i \subset 20Q_{r_i}(\hat{X})$$

and hence, with the definition of the maximal function we obtain

$$\begin{aligned}\fint_{4Q_i} |u - u_{4Q_i}|\,\mathrm{d}X &\le c\,r_i \left(\fint_{20Q_{r_i}(\hat{X})} |\nabla u|\,\mathrm{d}X + \fint_{20Q_{r_i}(\hat{X})} |\partial_t^{\sigma} u|\,\mathrm{d}X \right) \\ &\le c\,r_i \left(\mathcal{M}(|\nabla u|)(\hat{X}) + \mathcal{M}(|\partial_t^{\sigma} u|)(\hat{X}) \right) \\ &\le c\,r_i\,\lambda_{k,\Delta t}.\end{aligned}$$

This finishes the proof. □

Corollary 6.10. *Under the assumptions of Theorem 6.4 we know for all Q_i belonging to the Whitney covering of $E_{k,\Delta t}$ with $Q_i \cap K \neq \emptyset$ that*

$$\fint_{Q_i} |\mathcal{T}u - u|\,\mathrm{d}X \le c\,r_i \left(\lambda_{k,\Delta t} + \delta^{-d-1-\frac{1}{\sigma}} \|u\|_{L^1(E_{k,\Delta t})^d} \right).$$

Proof. This is an immediate implication of (6.7) and the above lemma. □

Lemma 6.11. *Under the assumptions of Theorem 6.4 we know for all Q_i belonging to the Whitney covering of $E_{k,\Delta t}$ with $Q_i \cap K \neq \emptyset$ that*

$$|(\mathcal{T}u)(X)| \le c \left(\theta_{\Delta t} + \delta^{-d-\frac{1}{\sigma}} \|u\|_{L^1(E_{k,\Delta t})^d} \right)$$

for $X \in Q_i \cap K$.

Proof. The method of proof is analogous to the proof of Lemma 6.7. Instead of using Poincaré's inequality, one estimates

$$|(\mathcal{T}u)(X)| \le \sum_{j \in A_i} |u_{Q_j}| \psi_j(X) \le c \fint_{4Q_i} |u|\,\mathrm{d}Y \le c \fint_{20Q_{r_i}(\hat{X})} |u|\,\mathrm{d}Y \le c\,\theta_{\Delta t}$$

in case (i). In (ii), using (W2) we estimate

$$\begin{aligned}|(\mathcal{T}u)(X)| &\le c \fint_{4Q_i} |u|\,\mathrm{d}Y \\ &\le c\,r_i^{-d-\frac{1}{\sigma}} \int_{E_{k,\Delta t}} |u|\,\mathrm{d}Y \\ &\le c\,\delta^{-d-\frac{1}{\sigma}} \|u\|_{L^1(E_{k,\Delta t})^d}.\end{aligned}$$

□

6.8 Lipschitz-continuity of $\mathcal{T}u$ w.r.t. ϱ_σ

Lemma 6.12. *Under the assumptions of Theorem 6.4, we have for any compact set $K \subset G \times (0,T)$, every Δt and $k \in \mathbb{N}$, that $\mathcal{T}u_{\Delta t} \in C^{0,1}_{\varrho_\sigma}(K)^d$. In particular, $\mathcal{T}u_{\Delta t}$ is differentiable almost everywhere in $G \times (0,T)$.*

Proof. The proof is not straightforward and can be found in [22, pp. 17ff.]. □

6.9 Proof of L^∞-bounds

Lemma 6.13. *Under the assumptions of Theorem 6.4, we have the following bound on the truncation $\mathcal{T}u_{\Delta t}$ and its gradient for almost all $X \in K$:*

$$|(\nabla \mathcal{T}u_{\Delta t})(X)| \le c\left(\lambda_{k,\Delta t} + \delta^{-d-\frac{1}{\sigma}} \|u_{\Delta t}\|_{L^1(E_{k,\Delta t})^d}\right),$$
$$|(\mathcal{T}u_{\Delta t})(X)| \le c\left(\theta_{\Delta t} + \delta^{-d-\frac{1}{\sigma}} \|u_{\Delta t}\|_{L^1(E_{k,\Delta t})^d}\right).$$

Proof. See also [22, pp. 22f.]. We split the gradient $\nabla\mathcal{T}u_{\Delta t}$ into two parts. Since $\mathcal{T}u_{\Delta t} = u_{\Delta t}$ on $E^c_{k,\Delta t}$, we have

$$\nabla\mathcal{T}u_{\Delta t} = \chi_{E_{k,\Delta t}} \nabla\mathcal{T}u_{\Delta t} + \chi_{(G\times(0,T))\setminus E_{k,\Delta t}} \nabla u_{\Delta t}$$

almost everywhere in K.

Let us again omit the subscript Δt of $u_{\Delta t}$ to keep things simple. For a point $X \in \big(K \cap (G \times (0,T))\big) \setminus E_{k,\Delta t}$, the property (6.3) of the maximal function and the definition of $E_{k,\Delta t}$ immediately give

$$|\nabla u(X)| \le \mathcal{M}(|\nabla u|)(X) \le \lambda_{k,\Delta t}.$$

On the other hand, a point $X \in K \cap E_{k,\Delta t}$ lies in some ball Q_i of the Whitney covering. Using the property

$$\sum_{j\in A_i} \psi_j \equiv 1 \quad \text{on } Q_i$$

of the partition of unity, we have

$$\begin{aligned}
|(\nabla\mathcal{T}u)(X)| &= \left|\nabla\Big(\sum_{j\in A_i} \psi_j(X) u_{Q_j}\Big)\right| \\
&= \left|\nabla\Big(\sum_{j\in A_i} \psi_j(X)(u_{Q_j} - u_{Q_i})\Big)\right| \\
&= \left|\Big(\sum_{j\in A_i} \nabla\psi_j(X)(u_{Q_j} - u_{Q_i})\Big)\right|.
\end{aligned}$$

From the construction of the partition of unity it follows $|\nabla\psi_j(X)| \le c\, r_j^{-1}$ for $X \in Q_j$. Together with the sum over $j \in A_i$ being finite and (W3), we obtain

$$\begin{aligned}
|(\nabla\mathcal{T}u)(X)| &\le c \sum_{j\in A_i} r_j^{-1} \fint_{4Q_i} |u - u_{4Q_i}|\,\mathrm{d}Y \\
&\le c\, r_i^{-1} \fint_{4Q_i} |u - u_{4Q_i}|\,\mathrm{d}Y\,.
\end{aligned}$$

Here, we also used (W6) and (W3) to calculate

$$\begin{aligned}|u_{Q_j} - u_{Q_i}| &= \left| \fint_{Q_j} u \,\mathrm{d}Y - \fint_{Q_i} u \,\mathrm{d}Y + u_{4Q_i} - u_{4Q_i} \right| \\ &\le \fint_{Q_j} |u - u_{4Q_i}| \,\mathrm{d}Y + \fint_{Q_i} |u - u_{4Q_i}| \,\mathrm{d}Y \\ &\le c \fint_{4Q_i} |u - u_{4Q_i}| \,\mathrm{d}Y \,.\end{aligned}$$

Finally, Lemma 6.9 gives

$$|(\nabla \mathcal{T}u)(X)| \le c \left(\lambda_{k,\Delta t} + \delta^{-d-1-\frac{1}{\sigma}} \|u\|_{L^1(E_{k,\Delta t})} \right).$$

The proof of the second estimate is analogous. For a point $X \in \big(K \cap (G \times (0,T))\big) \setminus E_{k,\Delta t}$ we obtain from the definition of $E_{k,\Delta t}$ and the property (6.3) of the maximal function

$$|\mathcal{T}u(X)| = |u(X)| \le \mathcal{M}(|u|)(X) \le \theta_{\Delta t}.$$

On the other hand, for all $X \in K \cap E_{k,\Delta t}$ there exists $i \in \mathbb{N}$ such that $X \in Q_i$, where Q_i is a space-time cylinder belonging to the Whitney covering. Then Lemma 6.11 yields

$$|\mathcal{T}u(X)| \le c \left(\theta_{\Delta t} + \delta^{-d-\frac{1}{\sigma}} \|u\|_{L^1(E_{k,\Delta t})^d} \right).$$

□

6.10 Smallness of the gradient of $\mathcal{T}u_{\Delta t}$ on $E_{k,\Delta t}$

Lemma 6.14. *Under the assumptions of Theorem 6.4 for any $k \in \mathbb{N}$, there exists a constant $c > 0$ independent of k such that*

$$\limsup_{\Delta t \to 0} \|\nabla \mathcal{T}u_{\Delta t}\|_{L^p(E_{k,\Delta t} \cap K)^{d \times d}} \le c\, 2^{-k/p}.$$

Proof. See also [22, p. 32]. Using the L^∞-bound on the truncation from Lemma 6.13 and the strong convergence of $\{u_{\Delta t}\}$, it is easy to calculate

$$\begin{aligned}&\limsup_{\Delta t \to 0} \|\nabla \mathcal{T}u_{\Delta t}\|_{L^p(E_{k,\Delta t} \cap K)^{d \times d}} \\ &\quad \le \limsup_{\Delta t \to 0} \mu_{n+1}(E_{k,\Delta t})^{1/p} \|\nabla \mathcal{T}u_{\Delta t}\|_{L^\infty(K)^{d \times d}} \\ &\quad \le \limsup_{\Delta t \to 0} \mu_{n+1}(E_{k,\Delta t})^{1/p}\, c \left(\lambda_{k,\Delta t} + \delta_{k,\Delta t}^{-d-1-\frac{1}{\sigma}} \|u_{\Delta t}\|_{L^1(E_{k,\Delta t})^d} \right) \\ &\quad \le c \limsup_{\Delta t \to 0} \lambda_{k,\Delta t} \mu_{n+1}(E_{k,\Delta t})^{1/p} \\ &\quad \le c \limsup_{\Delta t \to 0} \left(\lambda_{k,\Delta t}^p \mu_{n+1}(E_{k,\Delta t}) \right)^{1/p}.\end{aligned}$$

Finally, employing (6.12) delivers the desired estimate. □

6.11 Convergence

Lemma 6.15. *Let the assumptions of Theorem 6.4 hold and let* $\zeta \in C_0^\infty(G \times (0,T))$. *Then for fixed* $k \in \mathbb{N}$ *holds*

$$\begin{aligned} \zeta \mathcal{T} u_{\Delta t} &\to 0 \quad \text{in } L^s(0,T;L^s(G)^d) \text{ for all } s \in [1,\infty], \\ \zeta \mathcal{T} u_{\Delta t} &\rightharpoonup 0 \quad \text{in } L^s(0,T;W_0^{1,s}(G)^d) \text{ for all } s \in [1,\infty), \end{aligned}$$

as $\Delta t \to 0$.

Proof. Let $K \subset G \times (0,T)$ be compact and let $\zeta \in C_0^\infty(G \times (0,T))$ with compact support in K. The first convergence is clear since

$$\|\mathcal{T} u_{\Delta t}\|_{L^\infty(K)^d} \leq c \left(\theta_{\Delta t} + \delta^{-d-\frac{1}{\sigma}} \|u\|_{L^1(E_{k,\Delta t})^d} \right) \to 0$$

as $\Delta t \to 0$. For the other results let $\varphi \in C_0^\infty(G \times (0,T))^{d \times d}$. Then with (6.9), (6.10) and Lemma 6.12 we know that $\nabla(\zeta \mathcal{T} u_{\Delta t})$ exists and belongs to $L^s(G \times (0,T))^{d\times d}$ for any $s \in [1,\infty]$ and hence

$$\begin{aligned} \Big| \int_{G\times(0,T)} \nabla(\zeta \mathcal{T} u_{\Delta t}) : \varphi \, \mathrm{d}X \Big| &\leq \int_K |\zeta \mathcal{T} u_{\Delta t} \cdot \operatorname{div} \varphi| \, \mathrm{d}X \\ &\leq \|\mathcal{T} u_{\Delta t}\|_{L^\infty(K)^d} \int_K |\zeta \operatorname{div} \varphi| \, \mathrm{d}X \\ &\to 0 \end{aligned}$$

as $\Delta t \to 0$. The dense embedding $C_0^\infty(G \times (0,T))^{d\times d} \overset{d}{\hookrightarrow} L^s(0,T;L^s(G)^{d\times d})$ gives the weak convergence of $\nabla(\zeta \mathcal{T} u_{\Delta t})$ in $L^s(0,T;L^s(G)^{d\times d})$ and thus the weak convergence of $\zeta \mathcal{T} u_{\Delta t}$ in $L^s(0,T;W_0^{1,s}(G)^d)$ for $s \in [1,\infty)$. ☐

7

Stationary Problem

In the course of showing the existence of a solution to (3.8) in Chapter 8, we require the existence of a solution $u^n_{\Delta t} \in V_p$ to a stationary problem of the form

$$\frac{1}{\Delta t}(u^n_{\Delta t} - u^{n-1}_{\Delta t}) + Au^n_{\Delta t} + Bu^n_{\Delta t} = f^n_{\Delta t},$$

for given $u^{n-1}_{\Delta t} \in H$, $f^n_{\Delta t} \in V'_p$, $\Delta t > 0$ and $n \in \mathbb{N}$. This chapter is dedicated to show the solvability of this problem in the general case $p > \frac{2d}{d+2}$ and the derivation of suitable estimates on the solution $u^n_{\Delta t}$. We follow the proof in [19] and add a regularizing term. As we will see, many ideas and obstacles in this proof are similar to the ones in the evolutionary case.

Remember that for $p > \frac{3d}{d+2}$ the solvability already follows from the classical theory of monotone operators since with $V_p \overset{c}{\hookrightarrow} H_{2p'}$ we can regard $B : V_p \to V'_p$.

7.1 Solution to an approximate system and a priori estimates

For $q = 2p'$ and $\nu \in \mathbb{N}$ we define the operator

$$Q_\nu : H_q \to H_{q'} \quad \text{with} \quad \langle Q_\nu v, w\rangle = \frac{1}{\nu}\int_\Omega |v|^{q-2}v \cdot w \,\mathrm{d}x$$

for all $v, w \in H_q$.

Lemma 7.1. *For every $\nu \in \mathbb{N}$, the operator Q_ν defined above is well-defined, continuous, strictly monotone, bounded such that*

$$\|Q_\nu v\|_{H_{q'}} \le \frac{1}{\nu}\|v\|^{q-1}_{H_q},$$

and coercive such that

$$\langle Q_\nu v, v\rangle = \frac{1}{\nu}\|v\|^q_{H_q}$$

for any $v \in H_q$.

Proof. This follows from simple calculations and [68, Proposition 26.7] since Q_ν is the Nemyckii operator of the function $v \mapsto (1/\nu)\,|v|^{p-2}v$, $v \in \mathbb{R}$. □

This operator ensures that, when added to the equation, solutions belong to $H_q = H_{2p'}$. Thus, it makes sense to employ the operator

$$B_p : H_{2p'} \to V_p' \tag{7.1}$$

(which means we take $r = p$ in (3.5)). This justifies $\langle B_p u, u\rangle = 0$ for $u \in V_p \cap H_q$.

For $\nu \in \mathbb{N}$ we consider the approximate problem of finding a function $u_\nu \in V_p \cap H_q$ such that for given $v \in H$, $f \in V_p'$ and $\Delta t > 0$, there holds

$$\left\langle \frac{u_\nu - v}{\Delta t}, \varphi \right\rangle + \langle A u_\nu, \varphi\rangle + \langle B_p u_\nu, \varphi\rangle + \langle Q_\nu u_\nu, \varphi\rangle = \langle f, \varphi\rangle \tag{7.2}$$

for all $\varphi \in V_p \cap H_q$.

Lemma 7.2. *For each $\nu \in \mathbb{N}$, there exists a solution $u_\nu \in V_p \cap H_q$ to* (7.2) *that satisfies the a priori estimate*

$$\frac{1}{2\Delta t}\left(\|u_\nu\|_H^2 - \|v\|_H^2 + \|u_\nu - v\|_H^2\right) + \frac{c_0}{2}\|u_\nu\|_{V_p}^p + \frac{1}{\nu}\|u_\nu\|_{H_q}^q \le c\|f\|_{V_p'}^{p'},$$

where $c > 0$ does not depend on ν.

Proof. We follow the first steps of the proof of [19, Theorem 3.1]. The existence of a solution $u_\nu \in V_p \cap H_q$ can be shown with Brézis' theorem on pseudomonotone operators, see e.g. [25, Theorem 3.6.2], since $Q_\nu + A : V_p \cap H_q \to (V_p \cap H_q)'$ is monotone and hemicontinuous and $B_p : V_p \cap H_q \to (V_p + H_q)'$ is strongly continuous. We test the equation with the solution itself and use the fact $u_\nu \in H_q = H_{2p'}$ to show $\langle B_p u_\nu, u_\nu\rangle = 0$. This leads to

$$\frac{1}{\Delta t}(u_\nu - v, u_\nu) + \langle A u_\nu, u_\nu\rangle + \frac{1}{\nu}\langle Q_\nu u_\nu, u_\nu\rangle = \langle f, u_\nu\rangle.$$

Now, the identity $(a - b, a) = \frac{1}{2}(|a|^2 - |b|^2 + |a - b|^2)$, $a, b \in H$, Young's inequality, the coercivity (3.2) of A and of Q_ν yield the desired estimate. □

The goal is to prove existence of a solution to an equation involving $B : H_{2r'} \to V_r'$ (for the definition of B and r, see Section 3.2.4) instead of B_p. But since

$$\langle B_p u_\nu, \varphi\rangle = \int_\Omega u_\nu \otimes u_\nu : \nabla\varphi \,\mathrm{d}x = \langle B u_\nu, \varphi\rangle$$

for $\varphi \in \mathcal{V}$, the validity of (7.2) implies, that

$$\left\langle \frac{u_\nu - v}{\Delta t}, \varphi \right\rangle + \langle A u_\nu, \varphi\rangle + \langle B u_\nu, \varphi\rangle + \langle Q_\nu u_\nu, \varphi\rangle = \langle f, \varphi\rangle \tag{7.3}$$

holds for all $\varphi \in \mathcal{V}$. A density argument extends this to all $\varphi \in V_r \cap H_q$.

7.2 Convergence

Let $f \in V_p'$ and $v \in H$ be as in (7.2). Applying Lemma 7.2 to every $\nu \in \mathbb{N}$, we obtain a sequence $\{u_\nu\}_{\nu\in\mathbb{N}}$ of solutions. We remind, that $r \ge p$ determines the regularity of the convection term $B : H_{2r'} \to V_r'$ and was chosen in Section 3.2.4.

Lemma 7.3. *There exists a subsequence of $\{u_\nu\}_{\nu\in\mathbb{N}}$, which we still denote by $\{u_\nu\}_{\nu\in\mathbb{N}}$, a function $u \in V_p$ and a functional $a \in V_p'$ such that*

$$\begin{aligned} u_\nu &\rightharpoonup u && \text{in } V_p, \\ u_\nu &\to u && \text{in } H_{2r'}, \\ Bu_\nu &\to Bu && \text{in } V_r', \\ Q_\nu u_\nu &\to 0 && \text{in } H_{q'}, \\ Au_\nu &\rightharpoonup a && \text{in } V_p'. \end{aligned}$$

The function u solves

$$\frac{u-v}{\Delta t} + a + Bu = f \quad \text{in } V_r'. \tag{7.4}$$

Proof. See also [19, Theorem 3.1]. The a priori estimate in Lemma 7.2 shows, that $\{u_\nu\}_{\nu\in\mathbb{N}}$ is bounded in V_p. The reflexivity of V_p implies the existence of a weakly convergent subsequence of $\{u_\nu\}_{\nu\in\mathbb{N}}$. For brevity of notation, we may always write $\{u_\nu\}_{\nu\in\mathbb{N}}$ for its subsequences. Let us denote the limit of this subsequence by $u \in V_p$. From the compact embedding $V_p \overset{c}{\hookrightarrow} H_{2r'}$ follows the strong convergence of $\{u_\nu\}_{\nu\in\mathbb{N}}$ towards u in $H_{2r'}$ and H. This immediately shows

$$\frac{u_\nu - v}{\Delta t} \to \frac{u-v}{\Delta t} \quad \text{in } H.$$

The continuity of the operator $B : H_{2r'} \to V_r'$ gives

$$Bu_\nu \to Bu \quad \text{in } V_r'.$$

For the additional term Q_ν, we employ of the a priori estimate in Lemma 7.2 to calculate

$$\begin{aligned} \|Q_\nu(u_\nu)\|_{H_{q'}} &\le \frac{1}{\nu}\|u_\nu\|_{H_q}^{q-1} \\ &= \left(\frac{1}{\nu}\right)^{\frac{1}{q}} \left(\frac{1}{\nu}\|u_\nu\|_{H_q}^{q}\right)^{\frac{q-1}{q}} \\ &\le \left(\frac{1}{\nu}\right)^{\frac{1}{q}} \left(c\|f\|_{V_p'}^{p'}\right)^{\frac{q-1}{q}}, \end{aligned}$$

which tends to zero as $\nu \to \infty$. Thus,

$$Q_\nu(u_\nu) \to 0 \quad \text{in } H_{q'}.$$

The growth condition of A together with the bound on $\{u_\nu\}_{\nu\in\mathbb{N}}$ shows the boundedness of the set $\{Au_\nu\}_{\nu\in\mathbb{N}}$ in V_p'. Hence, there exists $a \in V_p'$ and a subsequence of $\{u_\nu\}_{\nu\in\mathbb{N}}$ such that

$$Au_\nu \rightharpoonup a \quad \text{in } V_p'.$$

Finally, taking the limit $\nu \to \infty$ in (7.3) gives for any $\varphi \in \mathcal{V}$

$$\left\langle \frac{u-v}{\Delta t}, \varphi \right\rangle + \langle a, \varphi\rangle + \langle Bu, \varphi\rangle = \langle f, \varphi\rangle.$$

By density, we conclude

$$\frac{u-v}{\Delta t} + a + Bu = f \quad \text{in } V_r'.$$

□

7.3 Lipschitz truncation and solenoidal test functions

The next goal is to prove $a = Au$. Here, the stationary Lipschitz truncation comes into play as we are not able to test (7.4) with the solution u itself. With the help of Lipschitz truncations of u_ν, $\nu \in \mathbb{N}$, we will be able to show almost everywhere convergence of the symmetric parts of the gradients $\{Du_\nu\}_{\nu\in\mathbb{N}}$.

For every $\nu \in \mathbb{N}$, let us define

$$w_\nu = u_\nu - u \in V_p,$$

which fulfil the assumptions of the theorem on the stationary Lipschitz truncation [19, Theorem 2.5]. An application of this theorem provides for every $j \in \mathbb{N}$ a bounded sequence $\{w_{\nu,j}\}_{\nu\in\mathbb{N}} \subset W_0^{1,\infty}(\Omega)^d$ of truncations of $\{w_\nu\}_{\nu\in\mathbb{N}}$ such that $\|\nabla w_{\nu,j}\|_{L^\infty(G)^{d\times d}} \leq c\, 2^{2^{j+1}}$ and for fixed $j \in \mathbb{N}$ holds

$$w_{\nu,j} \to 0 \qquad \text{in } L^s(\Omega)^d \text{ for all } s \in [1,\infty], \tag{7.5}$$

$$w_{\nu,j} \rightharpoonup 0 \qquad \text{in } W_0^{1,s}(\Omega)^d \text{ for all } s \in [1,\infty), \tag{7.6}$$

as $\nu \to \infty$. Moreover, it is

$$\limsup_{\nu\to\infty} \|\nabla w_{\nu,j}\|_{L^p(\{w_{\nu,j}\neq w_\nu\})^{d\times d}} \leq c\, 2^{-j/p}, \tag{7.7}$$

where c depends on Ω and $\sup_{\nu\in\mathbb{N}} \|w_\nu\|_{W_0^{1,p}(\Omega)^d}$. Here, we wrote $\{w_{\nu,j} \neq w_\nu\}$ for the set $\{x \in \Omega : w_{\nu,j}(x) \neq w_\nu(x)\}$.

The problem is, that we still cannot test (7.4) with the functions $w_{\nu,j}$ since they are not divergence-free on the set $\{w_{\nu,j} \neq w_\nu\}$. However, we have

Lemma 7.4. *Let $1 \leq s < \infty$. For every $j \in \mathbb{N}$ there exists a sequence $\{\psi_{\nu,j}\}_{\nu\in\mathbb{N}} \subset W_0^{1,s}(\Omega)^d$, such that*

$$\varphi_{\nu,j} = w_{\nu,j} - \psi_{\nu,j} \in V_s. \tag{7.8}$$

and

$$\varphi_{\nu,j}, \psi_{\nu,j} \to 0 \qquad \text{in } L^s(\Omega)^d,$$

$$\varphi_{\nu,j}, \psi_{\nu,j} \rightharpoonup 0 \qquad \text{in } W_0^{1,s}(\Omega)^d,$$

as ν tends to infinity. Furthermore, the gradients of $\psi_{\nu,j}$ become small in $L^p(\Omega)^{d\times d}$ with large ν, in the sense that

$$\limsup_{\nu\to\infty} \|\psi_{\nu,j}\|_{W_0^{1,p}(\Omega)^d} \leq c\, 2^{-j/p}. \tag{7.9}$$

Proof. See also [19, Theorem 3.1]. Similarly to Section 5.1, we consider the space

$$L_0^s(\Omega) = \left\{ g \in L^s(\Omega) : \int_\Omega g \,\mathrm{d}x = 0 \right\}.$$

Since our domain is sufficiently smooth (i.e. Lipschitz), there exists a linear bounded operator $\mathcal{B} : L_0^s(\Omega) \to W_0^{1,s}(\Omega)^d$ with

$$\operatorname{div} \mathcal{B}g = g,$$

$$\|\mathcal{B}g\|_{W_0^{1,s}(\Omega)^d} \leq c\, \|g\|_{L_0^s(\Omega)}.$$

This operator was first introduced by Bogovskiĭ [15]. Later, the existence of these kinds of operators could be generalized for so-called John domains, see [21] and [2]. See also [29, Section 10.5].

For every $\nu, j \in \mathbb{N}$, we want to apply $\mathcal{B}$ to $\operatorname{div} w_{\nu,j}$ in order to find a function $\psi_{\nu,j}$ which has a gradient that is small on the whole domain Ω (instead of only on $\{w_{\nu,j} \neq w_\nu\}$) and has the same divergence as $w_{\nu,j}$.

Taking the integral of $\operatorname{div} w_{\nu,j}$ gives with Gauß' divergence theorem and $w_{\nu,j}$ vanishing on $\partial\Omega$

$$\int_\Omega \operatorname{div} w_{\nu,j}\,\mathrm{d}x = \int_{\partial\Omega} w_{\nu,j} \cdot \eta\,\mathrm{d}x = 0,$$

where η is the outward normal on $\partial\Omega$. Thus, $\operatorname{div} w_{\nu,j} \in L^s_0(\Omega)$ and we are able to apply $\mathcal{B}$ to $\operatorname{div} w_{\nu,j}$ to receive a function

$$\psi_{\nu,j} = \mathcal{B} \operatorname{div} w_{\nu,j}$$

with

$$\begin{aligned} \|\psi_{\nu,j}\|_{W^{1,s}_0(\Omega)^d} &\le c\,\|\operatorname{div} w_{\nu,j}\|_{L^s(\Omega)} \\ &= c\,\|\chi_{\{w_{\nu,j} \neq w_\nu\}} \operatorname{div} w_{\nu,j}\|_{L^s(\Omega)} \\ &\le c\,\|\nabla w_{\nu,j}\|_{L^p(\{w_{\nu,j} \neq w_\nu\})^{d\times d}}, \end{aligned}$$

since $\operatorname{div} w_{\nu,j} = \operatorname{div} w_\nu = 0$, where $w_{\nu,j} = w_\nu$. From (7.7), it follows

$$\limsup_{\nu\to\infty} \|\psi_{\nu,j}\|_{W^{1,p}_0(\Omega)^d} \le c\,2^{-j/p}.$$

The operator $\mathcal{B}$ is linear and bounded and thus preserves weak convergence, see e.g. [17, Theorem III.9]. Hence, (7.6) with the compact embedding $W^{1,s}_0(\Omega)^d \overset{c}{\hookrightarrow} L^s(\Omega)^d$ implies for $j \in \mathbb{N}$

$$\begin{aligned} \psi_{\nu,j} &\rightharpoonup 0 && \text{in } W^{1,s}_0(\Omega)^d \text{ for all } s \in [1,\infty), \\ \psi_{\nu,j} &\to 0 && \text{in } L^s(\Omega)^d \text{ for all } s \in [1,\infty). \end{aligned}$$

We conclude by defining

$$\varphi_{\nu,j} = w_{\nu,j} - \psi_{\nu,j},$$

for which obviously hold $\operatorname{div} \varphi_{\nu,j} = 0$. The convergence results for $\psi_{\nu,j}$ and $w_{\nu,j}$ carry over to $\varphi_{\nu,j}$. □

Lemma 7.5. *There holds*

$$a = Au \quad \textit{in } V_p'.$$

Proof. See also [19, Theorem 3.1]. The functions $\varphi_{\nu,j}$, $\nu, j \in \mathbb{N}$, introduced in (7.8) are suitable test functions for (7.3). Testing results in the equation

$$\langle Au_\nu, \varphi_{\nu,j}\rangle = -\left\langle \frac{u_\nu - v}{\Delta t}, \varphi_{\nu,j} \right\rangle - \langle Bu_\nu, \varphi_{\nu,j}\rangle - \langle Q_\nu u_\nu, \varphi_{\nu,j}\rangle + \langle f, \varphi_{\nu,j}\rangle.$$

We now use the definition of the operator A (see Section 3.1) to split the term $\langle Au_\nu, \varphi_{\nu,j}\rangle$ into

$$\begin{aligned} \langle Au_\nu, \varphi_{\nu,j}\rangle = &\int_\Omega \big(S(Du_\nu) - S(Du)\big) : Dw_{\nu,j}\,\mathrm{d}x \\ &- \int_\Omega S(Du_\nu) : D\psi_{\nu,j}\,\mathrm{d}x + \int_\Omega S(Du) : Dw_{\nu,j}\,\mathrm{d}x\,. \end{aligned}$$

Here we implicitly use the fact, that the operator A can be regarded as an operator mapping into $(W_0^{1,p}(\Omega)^d)'$ instead of V_p'.

This gives the equation

$$\begin{aligned}\int_\Omega &\big(S(Du_\nu) - S(Du)\big) : Dw_{\nu,j}\,\mathrm{d}x \\ &= \int_\Omega S(Du_\nu) : D\psi_{\nu,j}\,\mathrm{d}x - \int_\Omega S(Du) : Dw_{\nu,j}\,\mathrm{d}x \\ &\quad - \left\langle \frac{u_\nu - v}{\Delta t}, \varphi_{\nu,j} \right\rangle - \langle Bu_\nu, \varphi_{\nu,j}\rangle - \langle Q_\nu u_\nu, \varphi_{\nu,j}\rangle + \langle f, \varphi_{\nu,j}\rangle.\end{aligned}$$

For fixed $j \in \mathbb{N}$, the last five terms on the right-hand side of the foregoing equation converge to zero due to the results in Lemma 7.3 and since $\{\varphi_{\nu,j}\}_{\nu\in\mathbb{N}}$ and $\{w_{\nu,j}\}_{\nu\in\mathbb{N}}$ converge to zero weakly in $W_0^{1,s}(\Omega)^d$ for any $1 \le s < \infty$. The first term becomes small since the gradient of $\psi_{\nu,j}$ becomes small in $L^p(\Omega)^{d\times d}$. Indeed, with Lemma 7.2, the growth condition (1.8) for S and (7.9), there holds

$$\begin{aligned}\limsup_{\nu\to\infty} \int_\Omega S(Du_\nu) : D\psi_{\nu,j}\,\mathrm{d}x &\le \limsup_{\nu\to\infty} \|S(Du_\nu)\|_{L^{p'}(\Omega)^{d\times d}} \|D\psi_{\nu,j}\|_{L^p(\Omega)^{d\times d}} \\ &\le c\, 2^{-j/p}.\end{aligned}$$

Altogether, this means

$$\limsup_{\nu\to\infty} \int_\Omega \big(S(Du_\nu) - S(Du)\big) : Dw_{\nu,j}\,\mathrm{d}x \le c\, 2^{-j/p}.$$

Application of [19, Lemma 2.6], gives

$$\limsup_{\nu\to\infty} \int_\Omega \Big((S(Du_\nu) - S(Du)) : (Du_\nu - Du) \Big)^\theta \mathrm{d}x = 0$$

for a number $\theta \in (0,1)$. Following the steps in the proof of Lemma 8.19 or more precisely [14, Step 2 of the proof of Theorem 2.1], one can show $Du_\nu \to Du$ almost everywhere in $\Omega\times(0,T)$. Then the continuity of S in the second argument ensures, that $\{S(Du_\nu)\}_{\nu\in\mathbb{N}}$ also converges almost everywhere towards $S(Du)$. Finally, the growth condition (1.8) implies, that the sequence $\{S(Du_\nu)\}_{\nu\in\mathbb{N}}$ is bounded in $L^p(\Omega)^{d\times d}$, which allows us to apply [41, Lemma 1.3, pp. 12f.] to show

$$Au_\nu \rightharpoonup Au \quad \text{in} \quad V_p'.$$

This means Au=a. □

7.4 A priori estimate

We compile the above results to the main result of this section. We already employ the notation of Chapter 8 where this theorem will be applied.

Theorem 7.6. *For given $u_{\Delta t}^{n-1} \in H$, $f_{\Delta t}^n \in V_p'$ and $\Delta t > 0$ there exists a solution $u_{\Delta t}^n \in V_p$ of*

$$\frac{u_{\Delta t}^n - u_{\Delta t}^{n-1}}{\Delta t} + Au_{\Delta t}^n + Bu_{\Delta t}^n = f_{\Delta t}^n \quad \textit{in } V_r', \tag{7.10}$$

that satisfies the estimate

$$\frac{1}{2\Delta t}\left(\|u^n_{\Delta t}\|^2_H - \|u^{n-1}_{\Delta t}\|^2_H + \|u^n_{\Delta t} - u^{n-1}_{\Delta t}\|^2_H\right) + \frac{c_0}{2}\|u^n_{\Delta t}\|^p_{V_p} \le c\|f^n_{\Delta t}\|^{p'}_{V'_p}, \quad (7.11)$$

with $c > 0$ independent of Δt.

The "discrete time-derivative" satisfies

$$\left\|\frac{u^n_{\Delta t} - u^{n-1}_{\Delta t}}{\Delta t}\right\|^{r'}_{V'_r} \le c\,\|u^n_{\Delta t}\|^p_{V_p} + c\,\|u^n_{\Delta t}\|^{2r'}_{H_{2r'}} + c\,\|f^n_{\Delta t}\|^{r'}_{V'_p}. \quad (7.12)$$

Proof. The existence of a solution $u^n_{\Delta t} \in V_p$ to (7.10) follows from the above lemmas, where we used the regularizing term Q_ν to obtain approximate solutions $u_\nu \in V_p$, $\nu \in \mathbb{N}$. Since $u_\nu \rightharpoonup u^n_{\Delta t}$ in V_p, we have the estimate

$$\|u^n_{\Delta t}\|_{V_p} \le \liminf_{\nu\to\infty} \|u_\nu\|_{V_p}.$$

Due to the compact embedding $V_p \overset{c}{\hookrightarrow} H$, we further have

$$\|u^n_{\Delta t}\|_H = \lim_{\nu\to\infty} \|u_\nu\|_H.$$

Hence, taking the limit $\nu \to \infty$ in Lemma 7.2 gives (7.11).

For the second estimate, we employ the differential equation (7.10). Due to $r \ge p$, we have $V_r \overset{d}{\hookrightarrow} V_p$ and hence by reflexivity $V'_p \overset{d}{\hookrightarrow} V'_r$. Therefore, it is

$$\begin{aligned}\left\|\frac{u^n_{\Delta t} - u^{n-1}_{\Delta t}}{\Delta t}\right\|_{V'_r} &\le \|Au^n_{\Delta t}\|_{V'_r} + \|Bu^n_{\Delta t}\|_{V'_r} + \|f\|_{V'_r} \\ &\le c\,\|Au^n_{\Delta t}\|_{V'_p} + c\,\|Bu^n_{\Delta t}\|_{V'_r} + c\,\|f\|_{V'_p} \\ &\le c\,(1 + \|u^n_{\Delta t}\|_{V_p})^{p-1} + c\,\|u^n_{\Delta t}\|^2_{H_{2r'}} + \|f\|_{V'_p}.\end{aligned}$$

Taking the power r' then gives with $(1 + \|u^n_{\Delta t}\|_{V_p}) \ge 1$ and $r' \le p'$

$$\left\|\frac{u^n_{\Delta t} - u^{n-1}_{\Delta t}}{\Delta t}\right\|^{r'}_{V'_r} \le c\,(1 + \|u^n_{\Delta t}\|_{V_p})^p + \|u^n_{\Delta t}\|^{2r'}_{H_{2r'}} + \|f\|^{r'}_{V'_p}.$$

□

8

Time discretization

In this chapter, we will present an alternative approach for constructing approximate solutions to (3.8). In Chapter 4, the approximation was achieved by regularizing the convection term (for some parameter $\varepsilon > 0$) so that the resulting problem could be easily solved employing results from the standard theory of monotone operators and a fixed point argument. This approach is of theoretical nature, since one still has to numerically solve the time-dependent approximate problem for every considered ε.

Thus, we want to establish a temporal semi-discretization scheme for (3.8) without relying on an alteration of the equation itself. In this way, we wish prove the existence of a weak solution to (3.8) and show the convergence of a numerical scheme at the same time. For now, we only consider the question of convergence and neglect the study of error estimates or convergence orders, since these always require regularity of the exact solution, which is not known so far.

8.1 Temporal semi-discretization and a priori estimates

Let us now discretize the problem (3.8) in time: For $N \in \mathbb{N}$ we consider an equidistant time-grid $\{t_n\}_{n=0}^{N}$ on $[0,T]$ with step-size $\Delta t = \frac{T}{N}$ and $t_n = n\Delta t$. Let us shortly identify the corresponding sequence of time-grids with $\{(\Delta t)_N\}_{N\in\mathbb{N}}$. We will denote all quantities related to such time-grids with the subscript Δt, omitting the declaration of N.

The right-hand side f is approximated by the natural restriction

$$f_{\Delta t}^{n} = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} f(\tau)\,\mathrm{d}\tau\,, \quad n = 1,\dots,N.$$

For the definition of suitable prolongations of the discrete problem, we will see that the starting value $u_{\Delta t}^{0}$ has to belong to V_p. The actual initial value $u_0 \in H$ is then approximated by a sequence $\{u_{\Delta t}^{0}\}$ that satisfies

$$\{u_{\Delta t}^{0}\} \subset V_p, \quad u_{\Delta t}^{0} \to u_0 \text{ in } H, \quad \Delta t\|u_{\Delta t}^{0}\|_{V_p}^{p} \le c, \tag{8.1}$$

where $c > 0$ is independent of Δt. The existence of such a sequence is assured for any $u_0 \in H$ since V_p lies dense in H.

Let us now consider the fully implicit numerical scheme

$$
\begin{aligned}
&u^0_{\Delta t} \in V_p \text{ given (with } u^0_{\Delta t} \approx u_0 \text{ in the above sense)},\\
&\frac{1}{\Delta t}(u^n_{\Delta t} - u^{n-1}_{\Delta t}) + Au^n_{\Delta t} + Bu^n_{\Delta t} = f^n_{\Delta t}, \quad n = 1, \dots, N.
\end{aligned} \tag{8.2}
$$

Remark 8.1. If we would assume the viscosity term S to be time-dependent, we would additionally have to approximate the operator $A = A(x, t, z)$ in time. A possible approach would be the natural restriction

$$
A_{\Delta t}(t) = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} A(\tau)\, \mathrm{d}\tau\,, \quad \text{for } t \in (t_{n-1}, t_n],\, n = 1, \dots, N.
$$

Lemma 8.2. *There exists a solution* $\{u^n_{\Delta t}\}_{n=0}^N \subset V_p$ *to* (8.2) *that satisfies the a priori estimate*

$$
\|u^n_{\Delta t}\|_H^2 + \sum_{k=1}^n \|u^k_{\Delta t} - u^{k-1}_{\Delta t}\|_H^2 + c_0 \Delta t \sum_{k=1}^n \|u^k_{\Delta t}\|_{V_p}^p \le c \tag{8.3}
$$

for all $n = 1, 2, \dots, N$*, where* c *is independent of* Δt*.*

Proof. For a given $u^{k-1}_{\Delta t} \in H$, the existence of a solution $u^k_{\Delta t} \in V_p$ to (8.2) follows from Theorem 7.6. For the a priori estimate, summing up (7.11) over $k = 1, \dots, n$ gives

$$
\|u^n_{\Delta t}\|_H^2 + \sum_{k=1}^n \|u^k_{\Delta t} - u^{k-1}_{\Delta t}\|_H^2 + c_0 \Delta t \sum_{k=1}^n \|u^k_{\Delta t}\|_{V_p}^p \le \|u^0_{\Delta t}\|_H^2 + c\, \Delta t \sum_{j=1}^n \|f^j_{\Delta t}\|_{V_p'}^{r'}
$$

for all $n = 1, 2, \dots, N$. With the choice of the initial data in (8.1) we have

$$
\|u^0_{\Delta t}\|_H^2 \le c.
$$

Moreover, with Hölder's inequality and $r' \le p'$, we find

$$
\Delta t \sum_{j=1}^n \|f^j_{\Delta t}\|_{V_p'}^{r'} \le \int_0^{t_n} \|f(t)\|_{V_p'}^{r'}\, \mathrm{d}t \le \|f\|_{L^{r'}(0,t;V_p')}^{r'} \le c\, \|f\|_{L^{p'}(0,T;V_p')}^{r'}. \tag{8.4}
$$

□

8.2 Definition of corresponding prolongations

From the discrete solutions $\{u^n_{\Delta t}\}_{n=0}^N$ we construct piecewise polynomial prolongations $u_{\Delta t}$ and $v_{\Delta t}$ for any stepsize Δt, which are defined on the whole time interval $[0, T]$. For $n = 1, \dots, N$, we define

$$
u_{\Delta t}(0) = u^1_{\Delta t}, \quad u_{\Delta t}(t) = u^n_{\Delta t} \text{ if } t \in (t_{n-1}, t_n],
$$

and

$$
v_{\Delta t}(0) = u^0_{\Delta t}, \quad v_{\Delta t}(t) = \frac{u^n_{\Delta t} - u^{n-1}_{\Delta t}}{\Delta t}(t - t_{n-1}) + u^{n-1}_{\Delta t} \text{ if } t \in (t_{n-1}, t_n].
$$

We remark that in contrast to $u_{\Delta t}$, the function $v_{\Delta t}$ is continuous and weakly differentiable in time. Both functions attain the value $u^n_{\Delta t}$ for $t = t_n$. This construction allows us to use the validity of the discrete equation (8.2) and $v'_{\Delta t}(t) = \frac{u^n_{\Delta t} - u^{n-1}_{\Delta t}}{\Delta t}$ in $(t_{n-1}, t_n]$ to derive

$$v'_{\Delta t}(t) + Au_{\Delta t}(t) + Bu_{\Delta t}(t) = f_{\Delta t}(t) \quad \text{in } V'_r,$$

for every $t \in [0, T]$ and $f_{\Delta t}(t) = f^n_{\Delta t}$ in $(t_{n-1}, t_n]$. Integrating over the interval $[0, T]$ then gives

$$v'_{\Delta t} + Au_{\Delta t} + Bu_{\Delta t} = f_{\Delta t} \quad \text{in } L^{r'}(0, T; V'_r). \tag{8.5}$$

We will now consider null-sequences of time-steps. It sufficient to consider an arbitrary subsequence $\{(\Delta t)_{N_j}\}_{j\in\mathbb{N}}$ of the sequence $\{(\Delta t)_N\}_{N\in\mathbb{N}}$. Nevertheless, we will write down the original sequence for notational brevity.

Lemma 8.3. *Let $\{(\Delta t)_N\}_{N\in\mathbb{N}}$ be a null sequence of time-steps (i.e. let N tend to infinity). Then the sequences of prolongations $\{u_{\Delta t}\} = \{u_{(\Delta t)_N}\}_{N\in\mathbb{N}}$ and $\{v_{\Delta t}\} = \{v_{(\Delta t)_N}\}_{N\in\mathbb{N}}$ are bounded in the spaces $L^\infty(0, T; H)$, $L^p(0, T; V_p)$ and $L^{2r'}(0, T; H_{2r'})$. The sequence of derivatives $\{v'_{\Delta t}\}$ is bounded in $L^{r'}(0, T; V'_r)$.*

Proof. For simplicity, we omit the subscript N. Using the a priori estimate (8.3), the bound on the initial data (8.1) and the estimates

$$\begin{aligned}
\int_0^T \|u_{\Delta t}(t)\|^p_{V_p}\,\mathrm{d}t &= \sum_{k=1}^N \int_{t_k-1}^{t_k} \|u^k_{\Delta t}\|^p_{V_p}\,\mathrm{d}t = \Delta t \sum_{k=1}^N \|u^k_{\Delta t}\|^p_{V_p}, \\
\int_0^T \|v_{\Delta t}(t)\|^p_{V_p}\,\mathrm{d}t &\le c\,\Delta t \sum_{k=1}^N \Big(\|u^k_{\Delta t}\|^p_{V_p} + \|u^{k-1}_{\Delta t}\|^p_{V_p}\Big) \\
&\le c\bigg(\Delta t \|u^0_{\Delta t}\|^p_{V_p} + \Delta t \sum_{k=1}^N \|u^k_{\Delta t}\|^p_{V_p}\bigg),
\end{aligned}$$

it is clear that the sequences of prolongations $\{u_{\Delta t}\}$, $\{v_{\Delta t}\}$ are bounded in $L^p(0, T; V_p)$. The boundedness in $L^\infty(0, T; H)$ is obvious, since

$$\max_{t\in[0,T]} \|u_{\Delta t}\|_H \le \max_{t\in[0,T]} \|v_{\Delta t}\|_H = \max_{k=0,\dots,N} \|u^k\|_H.$$

The derivative can be handled with (7.12) by estimating

$$\begin{aligned}
&\int_0^T \|v_{\Delta t}\|^{r'}_{V'_r}\,\mathrm{d}t \\
&\quad = \Delta t \sum_{k=1}^N \left\| \frac{u^k_{\Delta t} - u^{k-1}_{\Delta t}}{\Delta t} \right\|^{r'}_{V'_r} \\
&\quad \le \Delta t \sum_{k=1}^N c\,(1 + \|u^k_{\Delta t}\|_{V_p})^p + \Delta t \sum_{k=1}^N c\,\|u^k_{\Delta t}\|^{2r'}_{H_{2r'}} + \Delta t \sum_{k=1}^N c\,\|f^k_{\Delta t}\|^{r'}_{V'_p} \\
&\quad \le c\,(T + \|u_{\Delta t}\|^p_{L^p(0,T;V_p)} + \|u_{\Delta t}\|^{2r'}_{L^{2r'}(0,T;H_{2r'})} + \|f_{\Delta t}\|^{r'}_{L^{r'}(0,T;V'_p)}) \\
&\quad \le c\,(T + \|u_{\Delta t}\|^p_{L^p(0,T;V_p)} + \|u_{\Delta t}\|^{2r'}_{L^{2r'}(0,T;H_{2r'})} + \|f_{\Delta t}\|^{r'}_{L^{p'}(0,T;V'_p)}).
\end{aligned}$$

Since $\{u_{\Delta t}\}$ is bounded in $L^p(0,T;V_p)$ and $L^\infty(0,T;H)$, the parabolic interpolation (3.4) yields the boundedness of $\{u_{\Delta t}\}$ in $L^{2r'}(0,T;H_{2r'})$. The growth of $f_{\Delta t}$ can be estimated as in (8.4). □

8.3 Boundedness in fractional order spaces

In order to employ the Lipschitz truncation technique discussed in Chapter 6, it is necessary that the sequence $\{u_{\Delta t}\}$ is bounded in some Sobolev-Slobodeckii space $W^{\bar{\sigma},q}(0,T;H)$. When working with piecewise constant functions in this space one has to carefully handle the limiting case $\bar{\sigma} = 1/q$, which cannot be attained. To circumvent this difficulties, we will instead show boundedness in the corresponding Nikolskii space $N^{\bar{\sigma},q}(0,T;H)$. Here, we do not have to consider singular integrals, which makes life a little easier.

Nikolskii and Slobodeckii spaces measure local regularity as well as global regularity. For a piecewise constant function, the local regularity is responsible for the condition $\bar{\sigma} < 1/q$ (or $\bar{\sigma} \le 1/q$ in Nikolskii spaces). For the global regularity, one has to bound terms similar to

$$\sum_{n=1}^{N-k} \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^2.$$

Our discretization scheme admits a bound on the term $\sum_{n=1}^N \|u^n - u^{n-1}\|_H^2$. It appears natural to employ this bound to estimate the above term. Unfortunately, this has the price of a factor k. Indeed,

$$\begin{aligned}\|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^2 &\le \left(\sum_{j=n+1}^{n+k} \|u_{\Delta t}^j - u_{\Delta t}^{j-1}\|_H\right)^2 \\ &\le k \sum_{j=n+1}^{n+k} \|u_{\Delta t}^j - u_{\Delta t}^{j-1}\|_H^2.\end{aligned}$$

This factor prevents us from acquiring estimates on the fractional order norms with the aid of this stabilizing term.

Another possibility is to employ the differential equation itself to directly derive an estimate on

$$\begin{aligned}\|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^2 &= (u_{\Delta t}^{n+k} - u_{\Delta t}^n, u_{\Delta t}^{n+k} - u_{\Delta t}^n) \\ &= \langle u_{\Delta t}^{n+k} - u_{\Delta t}^n, u_{\Delta t}^{n+k} - u_{\Delta t}^n\rangle \le \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_{V_p} \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_{V_p'}\end{aligned}$$

Unfortunately, this method only works under more restrictive conditions on the parameter p. We will start with deriving an estimate in the simple case $p > 1 + 2d/(d+2)$. After this, we present a weaker estimate for $p > \max((d + \sqrt{d^2 + 2d(d+2)})/(d+2), 3d/(d+2))$.

8.3.1 Simple case

Lemma 8.4. *Let* $p > 1 + \frac{2d}{d+2}$. *Then for the discrete solutions* $\{u_{\Delta t}^n\}_{n=0}^N$ *of* (8.2) *holds*

$$\sum_{n=1}^{N-k} \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^2 \le c\, k \tag{8.6}$$

for every $0 \leq k \leq N$.

Proof. Let $0 \leq k \leq N$. By assumption, $u_{\Delta t}^n \in V_p$ is a solution of

$$\frac{u_{\Delta t}^n - u_{\Delta t}^{n-1}}{\Delta t} + Au_{\Delta t}^n + Bu_{\Delta t}^n = f_{\Delta t}^n$$

for every $1 \leq n \leq N$.

Since $p > 1 + \frac{2d}{d+2}$, we have $r = p$ in the definition of the convection term (3.5) and thus $B : H_{2p'} \to V_p'$, such that $Bu_{\Delta t}^n \in V_p'$. Then for $n \leq N - k$ holds

$$\frac{u_{\Delta t}^{n+k} - u_{\Delta t}^n}{\Delta t} + \sum_{j=n+1}^{n+k} Au_{\Delta t}^j + \sum_{j=n+1}^{n+k} Bu_{\Delta t}^j = \sum_{j=n+1}^{n+k} f_{\Delta t}^j \quad \text{in} \quad V_p'.$$

With this, we estimate

$$\begin{aligned}
&\|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^2 \\
&\quad = (u_{\Delta t}^{n+k} - u_{\Delta t}^n, u_{\Delta t}^{n+k} - u_{\Delta t}^n) \\
&\quad = \Delta t \Big\langle \sum_{j=n+1}^{n+k} \Big(f_{\Delta t}^j - Au_{\Delta t}^j - Bu_{\Delta t}^j \Big), u_{\Delta t}^{n+k} - u_{\Delta t}^n \Big\rangle \\
&\quad \leq \Delta t \Big(\sum_{j=n+1}^{n+k} \Big(\|f_{\Delta t}^j\|_{V_p'} + \|Au_{\Delta t}^j\|_{V_p'} + \|Bu_{\Delta t}^j\|_{V_p'} \Big) \Big) \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_{V_p}.
\end{aligned}$$

Hölder's inequality and the boundedness of $\{u_{\Delta t}\}$ in $L^p(0, T; V_p)$ from Lemma 8.3 yield

$$\begin{aligned}
&\sum_{n=1}^{N-k} \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^2 \\
&\quad \leq \Delta t \Big(\sum_{n=1}^{N-k} \Big(\sum_{j=n+1}^{n+k} \Big(\|f_{\Delta t}^j\|_{V_p'} + \|Au_{\Delta t}^j\|_{V_p'} + \|Bu_{\Delta t}^j\|_{V_p'} \Big) \Big)^{p'} \Big)^{1/p'} \times \\
&\qquad \times \Big(\sum_{n=1}^{N-k} \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_{V_p}^p \Big)^{1/p} \\
&\quad \leq c \Big(\Delta t \sum_{n=1}^{N-k} k^{p'/p} \sum_{j=n+1}^{n+k} \Big(\|f_{\Delta t}^j\|_{V_p'}^{p'} + \|Au_{\Delta t}^j\|_{V_p'}^{p'} + \|Bu_{\Delta t}^j\|_{V_p'}^{p'} \Big) \Big)^{1/p'} \times \\
&\qquad \times 2 \Big(\Delta t \sum_{n=1}^{N} \|u_{\Delta t}^n\|_{V_p}^p \Big)^{1/p} \\
&\quad \leq c\, k^{1/p} \Big(\Delta t \sum_{n=1}^{N-k} \sum_{j=n+1}^{n+k} \Big(\|f_{\Delta t}^j\|_{V_p'}^{p'} + \|Au_{\Delta t}^j\|_{V_p'}^{p'} + \|Bu_{\Delta t}^j\|_{V_p'}^{p'} \Big) \Big)^{1/p'}.
\end{aligned}$$

Let us exemplarily consider the term

$$\Delta t \sum_{n=1}^{N-k} \sum_{j=n+1}^{n+k} \|Bu_{\Delta t}^j\|_{V_p'}^{p'}.$$

Changing the order of summation gives

$$\begin{aligned}\Delta t \sum_{n=1}^{N-k} \sum_{j=n+1}^{n+k} \|Bu_{\Delta t}^j\|_{V_p'}^{p'} &\le \Delta t \sum_{j=1}^{N} \sum_{n=j-k}^{j-1} \|Bu_{\Delta t}^j\|_{V_p'}^{p'} \\ &= k\,\Delta t \sum_{j=1}^{N} \|Bu_{\Delta t}^j\|_{V_p'}^{p'} \\ &\le k\,\|Bu_{\Delta t}\|_{L^{p'}(0,T;V_p')}^{p'} \\ &\le k\,\|u_{\Delta t}\|_{L^{2p'}(0,T;H_{2p'})}^{2p'},\end{aligned}$$

which is bounded after Lemma 8.3 with $r' = p'$.

The other terms can be handled analogously estimating

$$\begin{aligned}\Delta t \sum_{n=1}^{N-k} \sum_{j=n+1}^{n+k} \|Au_{\Delta t}^j\|_{V_p'}^{p'} &\le k\,\|Au_{\Delta t}\|_{L^{p'}(0,T;V_p')}^{p'} \\ &\le k\,(1 + \|u_{\Delta t}\|_{L^p(0,T;V_p)})^p\end{aligned}$$

and

$$\Delta t \sum_{n=1}^{N-k} \sum_{j=n+1}^{n+k} \|f_{\Delta t}^j\|_{V_p'}^{p'} \le k\,\|f_{\Delta t}\|_{L^{p'}(0,T;V_p')}^{p'}.$$

By means of Lemma 8.3, we find

$$\sum_{n=1}^{N-k} \|u^{n+k} - u^n\|_H^2 \le c\,k^{1/p} k^{1/p'} = c\,k.$$

□

Lemma 8.5. *Suppose that* (8.6) *holds. Then* $\{u_{\Delta t}\}$ *is bounded in the Nikolskii space* $N^{\bar\sigma,q}(0,T;H)$ *for* $\bar\sigma \le \min(\frac{1}{q},\frac{1}{2})$, $1 < q < \infty$.

Proof. We have to show

$$\int_0^{T-h} \|u_{\Delta t}(t+h) - u_{\Delta t}(t)\|_H^q \,\mathrm{d}t \le c\,h^{\bar\sigma q}.$$

for every $h \in [0,T]$.

Let $h \in [0,T]$ and $k \in \mathbb{N}$ such that $k\Delta t \le h \le (k+1)\Delta t$. The integral on the left can be written as

$$\begin{aligned}
&\int_0^{T-h} \|u_{\Delta t}(t+h) - u_{\Delta t}(t)\|_H^q \,\mathrm{d}t \\
&\quad = \sum_{n=1}^{N-k-1} \left(\int_{t_{n-1}}^{t_{n+k}-h} \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^q \,\mathrm{d}t + \int_{t_{n+k}-h}^{t_n} \|u_{\Delta t}^{n+k+1} - u_{\Delta t}^n\|_H^q \,\mathrm{d}t \right) \\
&\qquad + \int_{t_{N-k-1}}^{T-h} \|u_{\Delta t}^N - u_{\Delta t}^{N-k}\|_H^q \\
&\quad = \sum_{n=1}^{N-k-1} \Big(\big((k+1)\Delta t - h\big) \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^q + \big(h - k\Delta t\big) \|u_{\Delta t}^{n+k+1} - u_{\Delta t}^n\|_H^q \Big) \\
&\qquad + \big((k+1)\Delta t - h\big) \|u_{\Delta t}^N - u_{\Delta t}^{N-k}\|_H^q \\
&\quad = \big((k+1)\Delta t - h\big) \sum_{n=1}^{N-k} \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^q + \big(h - k\Delta t\big) \sum_{n=1}^{N-k-1} \|u_{\Delta t}^{n+k+1} - u_{\Delta t}^n\|_H^q.
\end{aligned}$$

For $q \geq 2$, the boundedness of $\{u_{\Delta t}\}$ in $L^\infty(0,T;H)$ provides exemplarily for the first term

$$\begin{aligned}
\sum_{n=1}^{N-k} \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^q &\leq \max_n \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^{q-2} \sum_{n=1}^{N-k} \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^2 \\
&\leq c \sum_{n=1}^{N-k} \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^2.
\end{aligned}$$

Employing (8.6) then altogether gives

$$\begin{aligned}
&\int_0^{T-h} \|u_{\Delta t}(t+h) - u_{\Delta t}(t)\|_H^q \,\mathrm{d}t \\
&\quad \leq c \big((k+1)\Delta t - h\big) k + c \big(h - k\Delta t\big)(k+1) \\
&\quad \leq c\,\Delta t\,(k+1) \\
&\quad \leq c\,h.
\end{aligned}$$

This gives boundedness of $\{u_{\Delta t}\}$ in $N^{\bar{\sigma},q}(0,T;H)$ for $\bar{\sigma} \leq 1/q$.

If $q < 2$, we use Hölder's inequality to show (again exemplarily for the first term)

$$\sum_{n=1}^{N-k} \|u^{n+k} - u^n\|_H^q \leq (N-k)^{1-q/2} \left(\sum_{n=1}^{N-k} \|u^{n+k} - u^n\|_H^2 \right)^{q/2}.$$

From (8.6) it then follows

$$\begin{aligned}
&\int_0^{T-h} \|u_{\Delta t}(t+h) - u_{\Delta t}(t)\|_H^q \,\mathrm{d}t \\
&\quad \leq c \big((k+1)\Delta t - h\big)(N-k)^{1-q/2} k^{q/2} \\
&\qquad\quad + c \big(h - k\Delta t\big)(N-k-1)^{1-q/2}(k+1)^{q/2} \\
&\quad \leq c\,\Delta t\,(N-k)^{1-q/2}(k+1)^{q/2} \\
&\quad \leq c\,(T-h)^{1-q/2} h^{q/2} \\
&\quad \leq c\,T\,h^{q/2}.
\end{aligned}$$

In that case, $\{u_{\Delta t}\}$ is bounded in $N^{\bar\sigma,q}(0,T;H)$ if $\bar\sigma q \le q/2$, which is true for $\bar\sigma \le 1/2$. □

Corollary 8.6. *Suppose that* (8.6) *holds. Then* $\{u_{\Delta t}\}$ *is bounded in the Sobolev-Slobodeckii space* $W^{\bar\sigma,q}(0,T;H)$ *for* $\bar\sigma < \min(\frac{1}{q},\frac{1}{2})$, $1<q<\infty$.

Proof. This result immediately follows from the foregoing lemma and Lemma 2.3, but can also be calculated directly by estimating the $W^{\bar\sigma,q}(0,T;H)$-seminorm. □

8.3.2 Difficult case

If we weaken the assumptions on p, there is still a suitable estimate for this term. We make use of the interpolation (2.3) and the embedding $V_p \hookrightarrow H$ to show

$$\|Bv\|_{V_p'} \le c\,\|v\|^2_{H_{2p'}} \le c\,\|v\|_H^{2(1-\vartheta)}\|v\|_{V_p}^{2\vartheta}, \quad v\in V_p, \tag{8.7}$$

which is possible for $p > \frac{3d}{d+2}$ as then we can choose $r=p$ in the definition of B (see Section 3.2.4). The conditions on ϑ are

$$\vartheta \in [1/p,1] \quad \text{if} \quad p \ge d,$$

and

$$\vartheta = \frac{1}{2p}\left(\frac{1}{2}+\frac{1}{d}-\frac{1}{p}\right)^{-1} \quad \text{if} \quad p<d,$$

see also [26, Lemma 2.1]. Of course, it is $\vartheta\in[1/p,1]$ in the case $p<d$. Thus, we can always choose ϑ as in the case $p<d$.

Lemma 8.7. *Let* $p>\frac{3d}{d+2}$. *Then for the discrete solutions* $\{u^n_{\Delta t}\}_{n=0}^N$ *of* (8.2) *there holds*

$$\sum_{n=1}^{N-k}\|u^{n+k}_{\Delta t}-u^n_{\Delta t}\|_H^2 \le c\,k^{1-\frac{2\vartheta}{p}}(\Delta t)^{-\frac{2\vartheta}{p}}, \tag{8.8}$$

if in addition

$$p \ge \frac{d+\sqrt{d^2+2d(d+2)}}{d+2}, \tag{8.9}$$

i.e.

$$p\ge\frac{1+\sqrt5}{2} \quad \textit{if} \quad d=2, \qquad p\ge \frac{3+\sqrt{39}}{5} \quad \textit{if} \quad d=3.$$

Proof. The first part of the proof is similar to the one in the simple case. We have

$$\begin{aligned}
&\sum_{n=1}^{N-k}\|u^{n+k}_{\Delta t}-u^n_{\Delta t}\|_H^2\\
&\le \left(\Delta t\sum_{n=1}^{N-k}\left(\sum_{j=n+1}^{n+k}\left(\|f^j_{\Delta t}\|_{V_p'}+\|Au^j_{\Delta t}\|_{V_p'}+\|B_p u^j_{\Delta t}\|_{V_p'}\right)\right)^{p'}\right)^{1/p'}\times\\
&\quad\times\left(\Delta t\sum_{n=1}^{N-k}\|u^{n+k}_{\Delta t}-u^n_{\Delta t}\|^p_{V_p}\right)^{1/p},
\end{aligned}$$

where the second factor can again be estimated by the $L^p(0,T;V_p)$-norm of $u_{\Delta t}$. For the first factor, let us estimate

$$\begin{aligned}
&\Big(\sum_{j=n+1}^{n+k} \Big(\|f_{\Delta t}^j\|_{V_p'} + \|Au_{\Delta t}^j\|_{V_p'} + \|B_p u_{\Delta t}^j\|_{V_p'} \Big) \Big)^{p'} \\
&\le c \Big(\sum_{j=n+1}^{n+k} \|f_{\Delta t}^j\|_{V_p'} \Big)^{p'} + c \Big(\sum_{j=n+1}^{n+k} \|Au_{\Delta t}^j\|_{V_p'} \Big)^{p'} + c \Big(\sum_{j=n+1}^{n+k} \|Bu_{\Delta t}^j\|_{V_p'} \Big)^{p'} .
\end{aligned}$$

Considering the convection term, we employ the interpolation (8.7) and the boundedness of $\{u_{\Delta t}\}$ in $L^\infty(0,T;H)$. Then, if $p \ge 2\vartheta$, we find with Hölder's inequality

$$\begin{aligned}
c\,\Delta t \sum_{n=1}^{N-k} \Big(\sum_{j=n+1}^{n+k} \|Bu_{\Delta t}^j\|_{V_p'} \Big)^{p'} &\le c\,\Delta t \sum_{n=1}^{N-k} \Big(\sum_{j=n+1}^{n+k} \|u_{\Delta t}^j\|_{V_p}^{2\vartheta} \Big)^{p'} \\
&\le c\,\Delta t \sum_{n=1}^{N-k} k^{p'\left(1-\frac{2\vartheta}{p}\right)} \Big(\sum_{j=n+1}^{n+k} \|u_{\Delta t}^j\|_{V_p}^{p} \Big)^{p'\frac{2\vartheta}{p}} \\
&\le c\,(\Delta t)^{1-p'\frac{2\vartheta}{p}} k^{p'\left(1-\frac{2\vartheta}{p}\right)} \sum_{n=1}^{N-k} \|u_{\Delta t}\|_{L^p(0,T;V_p)}^{p'\frac{2\vartheta}{p}} \\
&\le c\,(\Delta t)^{1-p'\frac{2\vartheta}{p}} k^{p'\left(1-\frac{2\vartheta}{p}\right)} (N-k) \\
&\le c\,(T-h)\,(\Delta t)^{-p'\frac{2\vartheta}{p}} k^{p'\left(1-\frac{2\vartheta}{p}\right)} .
\end{aligned}$$

Concerning the terms with $Au_{\Delta t}^j$ and $f_{\Delta t}^j$, we proceed analogously to the proof in the simple case and receive

$$c\,\Delta t \sum_{n=1}^{N-k} \Big(\sum_{j=n+1}^{n+k} \|Au_{\Delta t}^j\|_{V_p'} \Big)^{p'} \le c\,k^{1/p} \Delta t \sum_{n=1}^{N-k} \sum_{j=n+1}^{n+k} \|Au_{\Delta t}^j\|_{V_p'}^{p'} \le c\,k^{p'}$$

and

$$c\,\Delta t \sum_{n=1}^{N-k} \Big(\sum_{j=n+1}^{n+k} \|f_{\Delta t}^j\|_{V_p'} \Big)^{p'} \le c\,k^{1/p} \Delta t \sum_{n=1}^{N-k} \sum_{j=n+1}^{n+k} \|f_{\Delta t}^j\|_{V_p'}^{p'} \le c\,k^{p'} .$$

Altogether, we have

$$\begin{aligned}
\sum_{n=1}^{N-k} \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^2 &\le c\,(\Delta t)^{-\frac{2\vartheta}{p}} k^{1-\frac{2\vartheta}{p}} + c\,k \\
&\le c\,(\Delta t)^{-\frac{2\vartheta}{p}} k^{1-\frac{2\vartheta}{p}} + c\,k^{1-\frac{2\vartheta}{p}} N^{\frac{2\vartheta}{p}} \\
&\le c\,(\Delta t)^{-\frac{2\vartheta}{p}} k^{1-\frac{2\vartheta}{p}} .
\end{aligned}$$

We still have to analyse, under which assumptions the condition $p \ge 2\vartheta$ is valid. One verifies, that

$$\frac{1}{2p} \left(\frac{1}{2} + \frac{1}{d} - \frac{1}{p} \right)^{-1} \le \frac{p}{2}$$

is true if and only if

$$p^2 - \frac{2d}{d+2}p - \frac{2d}{d+2} \geq 0.$$

This gives the condition

$$p \geq \frac{d + \sqrt{d^2 + 2d(d+2)}}{d+2}.$$

□

Lemma 8.8. *Suppose that* (8.8) *holds. Then* $\{u_{\Delta t}\}$ *is bounded in the Nikolskii-space* $N^{\bar{\sigma},q}(0,T;H)$ *for* $\bar{\sigma} \leq \min(\frac{1}{q} - \frac{2\vartheta}{pq}, \frac{1}{2} - \frac{\vartheta}{p})$, $1 < q < \infty$.

Proof. Similarly to the proof in the simple case, we estimate

$$\begin{aligned}
&\int_0^{T-h} \|u_{\Delta t}(t+h) - u_{\Delta t}(t)\|_H^q \, \mathrm{d}t \\
&\quad \leq \big((k+1)\Delta t - h\big) \sum_{n=1}^{N-k} \|u_{\Delta t}^{n+k} - u_{\Delta t}^n\|_H^q \\
&\qquad + \big(h - k\Delta t\big) \sum_{n=1}^{N-k-1} \|u_{\Delta t}^{n+k+1} - u_{\Delta t}^n\|_H^q.
\end{aligned}$$

For $q \geq 2$, with (8.8) and the boundedness of $\{u_{\Delta t}\}$ in $L^\infty(0,T;H)$ we receive

$$\begin{aligned}
&\int_0^{T-h} \|u_{\Delta t}(t+h) - u_{\Delta t}(t)\|_H^q \, \mathrm{d}t \\
&\leq c\,\big((k+1)\Delta t - h\big) k^{1-\frac{2\vartheta}{p}} (\Delta t)^{-\frac{2\vartheta}{p}} + c\,\big(h - k\Delta t\big)(k+1)^{1-\frac{2\vartheta}{p}} (\Delta t)^{-\frac{2\vartheta}{p}} \\
&\leq c\,(k+1)^{1-\frac{2\vartheta}{p}} (\Delta t)^{1-\frac{2\vartheta}{p}} \\
&\leq c\,h^{1-\frac{2\vartheta}{p}}.
\end{aligned}$$

This gives boundedness of $\{u_{\Delta t}\}$ in $N^{\bar{\sigma},q}(0,T;H)$ for $\bar{\sigma} \leq \frac{1}{q} - \frac{2\vartheta}{pq}$.

For $q < 2$, we have with Hölder's inequality analogously to the simple case

$$\begin{aligned}
&\int_0^{T-h} \|u_{\Delta t}(t+h) - u_{\Delta t}(t)\|_H^q \, \mathrm{d}t \\
&\leq c\,\big((k+1)\Delta t - h\big)(N-k)^{1-\frac{q}{2}} \left(k^{1-\frac{2\vartheta}{p}} (\Delta t)^{-\frac{2\vartheta}{p}}\right)^{\frac{q}{2}} \\
&\quad + c\,\big(h - k\Delta t\big)(N-k-1)^{1-\frac{q}{2}} \left((k+1)^{1-\frac{2\vartheta}{p}} (\Delta t)^{-\frac{2\vartheta}{p}}\right)^{\frac{q}{2}} \\
&\leq c\,(\Delta t)\,(N-k)^{1-\frac{q}{2}} (k+1)^{\frac{q}{2}-\frac{\vartheta q}{p}} (\Delta t)^{-\frac{\vartheta q}{p}} \\
&\leq c\,(T-h)\,h^{\frac{q}{2}-\frac{\vartheta q}{p}}.
\end{aligned}$$

This means boundedness of $u_{\Delta t}$ in $N^{\bar{\sigma},q}(0,T;H)$ for $\bar{\sigma} \leq \frac{1}{2} - \frac{\vartheta}{p}$. □

Corollary 8.9. *Suppose,* (8.8) *holds. Then* $\{u_{\Delta t}\}$ *is bounded in the Sobolev-Slobodeckii-space* $W^{\bar{\sigma},q}(0,T;H)$ *for* $\bar{\sigma} < \min(\frac{1}{q} - \frac{2\vartheta}{pq}, \frac{1}{2} - \frac{\vartheta}{p})$, $1 < q < \infty$.

Proof. This result immediately follows from the foregoing lemma and Lemma 2.3.

□

8.4 Convergence

Let $p > 3d/(d+2)$ and (8.9) hold, i.e.

$$p > \max\left(\frac{d+\sqrt{d^2+2d(d+2)}}{d+2}, \frac{3d}{d+2}\right)$$

and

$$\vartheta = \frac{1}{2p}\left(\frac{1}{2}+\frac{1}{d}-\frac{1}{p}\right)^{-1}.$$

Then it is $\frac{1}{2}-\frac{\vartheta}{p} > 0$. We remind, that ϑ is the interpolation exponent we chose in (8.7).

Theorem 8.10. *Assume $p > 3d/(d+2)$ and (8.9). Let $\{(\Delta t)_N\}_{N\in\mathbb{N}}$ be a null sequence of time steps and let $\{u_{(\Delta t)_N}\}_{N\in\mathbb{N}}$, $\{v_{(\Delta t)_N}\}_{N\in\mathbb{N}}$ be the corresponding prolongations defined in Section 8.2 of the discrete solutions to the implicit Euler-scheme (8.2) with initial values (8.1).*

Then there is a subsequence of time steps $\{(\Delta t)_{N'}\}$ such that $\{u_{(\Delta t)_{N'}}\}$ and $\{v_{(\Delta t)_{N'}}\}$ converge weakly∗ in $L^\infty(0,T;H)$ and $\{u_{(\Delta t)_{N'}}\}$ and $\{v_{(\Delta t)_{N'}}\}$ converge weakly in $L^p(0,T;V_p)$ towards a weak solution $u \in L^\infty(0,T;H)\cap L^p(0,T;V_p)$ to (3.8). The sequence of time derivatives $\{v'_{(\Delta t)_{N'}}\}$ converges weakly in $L^{r'}(0,T;V_r')$ towards the $u' \in L^{r'}(0,T;V_r')$.

The proof of Theorem 8.10 will be split into several lemmas.

Lemma 8.11. *There is a subsequence $\{(\Delta t)_{k'}\}$ and a limit $u \in L^\infty(0,T;H)\cap L^p(0,T;V_p)\cap W^{\bar\sigma,2}(0,T;H)$ for $\bar\sigma < \frac{1}{2}-\frac{\vartheta}{p}$, with $u' \in L^{r'}(0,T;V_r')$ such that*

$$\begin{aligned}
&u_{(\Delta t)_{k'}} \rightharpoonup u \text{ in } L^p(0,T;V_p), && v_{(\Delta t)_{k'}} \rightharpoonup u \text{ in } L^p(0,T;V_p),\\
&u_{(\Delta t)_{k'}} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H), && v_{(\Delta t)_{k'}} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;H),\\
&u_{(\Delta t)_{k'}} \to u \text{ in } L^{2r'}(0,T;H_{2r'}), && v_{(\Delta t)_{k'}} \to u \text{ in } L^{2r'}(0,T;H_{2r'}).
\end{aligned}$$

Furthermore,

$$u_{(\Delta t)_{k'}} \rightharpoonup u \quad \text{in} \quad W^{\bar\sigma,2}(0,T;H)$$

and

$$v'_{(\Delta t)_{k'}} \rightharpoonup u' \quad \text{in} \quad L^{r'}(0,T;V_r').$$

Proof. Let us, for simplicity, omit the subscript k and denote subsequences again by their original identifiers. By Lemma 8.3 the sequences $\{u_{\Delta t}\}$ and $\{v_{\Delta t}\}$ are bounded in $L^\infty(0,T;H)$ and in $L^p(0,T;V_p)$ and thus, thanks to reflexivity (see e.g. [66, Theorem III.37] or [17, Theorem III.27]), we can extract subsequences which converge weakly in $L^p(0,T;V_p)$ towards u and v respectively. By Corollary 8.9 with $q=2$ and $0<\bar\sigma<\frac{1}{2}-\frac{\vartheta}{p}$, we have the boundedness of $\{u_{\Delta t}\}$ in $W^{\bar\sigma,2}(0,T;H)$ and with the same argument, we can further extract a weakly convergent subsequence in $W^{\bar\sigma,2}(0,T;H)$. Since $L^1(0,T;H)$ is separable, we can extract weakly∗ convergent subsequences in $L^\infty(0,T;H) = (L^1(0,T;H))'$ (see e.g. [17, Corollary III.26]). By density, the limits are again u and v.

Again by reflexivity, we receive a weakly convergent subsequence $v'_{\Delta t} \rightharpoonup \chi$ in $L^{r'}(0,T;V_r')$. To show $\chi = v'$ one proceeds in the same way as for (4.14).

Hence, we can employ Lions-Aubin's lemma (see Lemma 2.9) to show

$$v_{\Delta t} \to v \quad \text{in} \quad L^p(0,T;H).$$

The boundedness of $\{v_{\Delta t}\}$ in $L^\infty(0,T;H)$ provides convergence in the space $L^2(0,T;H)$ (in fact, in every $L^s(0,T;H)$, $1 \le s < \infty$). Finally, the parabolic interpolation in Lemma 3.8 proves the convergence

$$v_{\Delta t} \to v \quad \text{in} \quad L^{2r'}(0,T;H_{2r'}).$$

There are two different ways to show the strong convergence of $\{u_{\Delta t}\}$ in $L^{2r'}(0,T;H_{2r'})$. The first one employs a weakened form of the Lions-Aubin compactness lemma, namely Lemma 2.4, and the boundedness of $\{u_{\Delta t}\}$ in $W^{\bar{\sigma},2}(0,T;H)$. The other, maybe easier, way is to use the strong convergence of $\{v_{\Delta t}\}$. With (8.2) we calculate

$$\begin{aligned}
\|u_{\Delta t} - v_{\Delta t}\|^2_{L^2(0,T;H)} &= \int_0^T \|u_{\Delta t}(t) - v_{\Delta t}(t)\|^2_H \, \mathrm{d}t \\
&= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left\| u^j_{\Delta t} - u^{j-1}_{\Delta t} - \frac{u^j_{\Delta t} - u^{j-1}_{\Delta t}}{\Delta t}(t - t_{j-1}) \right\|^2_H \, \mathrm{d}t \\
&= \sum_{j=1}^n \left\| \frac{u^j_{\Delta t} - u^{j-1}_{\Delta t}}{\Delta t} \right\|^2_H \int_{t_{j-1}}^{t_j} (\Delta t + t_{j-1} - t)^2 \, \mathrm{d}t \\
&\le \sum_{j=1}^n \|u^j_{\Delta t} - u^{j-1}_{\Delta t}\|^2_H (\Delta t)^{-2} \int_{t_{j-1}}^{t_j} (t_j - t)^2 \, \mathrm{d}t \\
&\le \frac{1}{3} \Delta t \sum_{j=1}^n \|u^j_{\Delta t} - u^{j-1}_{\Delta t}\|^2_H \to 0 \text{ as } \Delta t \to 0.
\end{aligned}$$

On the one hand, this shows $u = v$ almost everywhere in $\Omega \times (0,T)$. On the other hand, the strong convergence of $\{v_{\Delta t}\}$ now implies the strong convergence of $\{u_{\Delta t}\}$ towards u in $L^2(0,T;H)$. Once again, from Lemma 3.8 we obtain

$$u_{\Delta t} \to u \quad \text{in} \quad L^{2r'}(0,T;H_{2r'}).$$

□

We now turn to the convergence of the terms in our differential equation (3.8). From now on, for better readability, we shall write $u_{\Delta t}$ instead of $u_{(\Delta t)_{k'}}$. Whenever we speak of subsequences of some sequences indexed with Δt, we implicitly consider the underlying subsequences of the sequence of time-steps.

Lemma 8.12. *There exists a function $a \in L^{p'}(0,T;V_p')$ such that for a (not relabeled) subsequence of $\{u_{\Delta t}\}$ holds*

$$Au_{\Delta t} \rightharpoonup a \qquad \text{in } L^{p'}(0,T;V_p'), \tag{8.10}$$

$$Bu_{\Delta t} \to Bu \qquad \text{in } L^{r'}(0,T;V_r'), \tag{8.11}$$

$$f_{\Delta t} \to f \qquad \text{in } L^{p'}(0,T;V_p'), \tag{8.12}$$

as $\Delta t \to 0$.

Furthermore, we have $u \in C_w([0,T];H)$ *and*

$$u' = f + a + Bu \ \text{in} \ L^{r'}(0,T;V_r'). \tag{8.13}$$

Proof. Considering (3.2), we find that the sequence $\{Au_{\Delta t}\}$ is bounded in $L^{p'}(0,T;V_p')$. Therefore, we can extract a weakly convergent subsequence, again denoted by $\{Au_{\Delta t}\}$, with

$$Au_{\Delta t} \rightharpoonup a \text{ in } L^{p'}(0,T;V_p').$$

Furthermore, by the continuity of B we know with Lemma 8.11

$$Bu_{\Delta t} \to Bu \text{ in } L^{r'}(0,T;V_r').$$

Finally, to show $f_{\Delta t} \to f$ in $L^{p'}(0,T;V_p')$, we consider the restriction operator

$$R_{\Delta t} : L^{p'}(0,T;V_p') \to L^{p'}(0,T;V_p'), \quad f \mapsto f_{\Delta t},$$

which is a linear operator. A calculation similar to (8.4) shows that $R_{\Delta t}$ is bounded with operator-norm equal to one. Indeed, Hölders inequality provides

$$\begin{aligned}
\|R_{\Delta t} f\|^p_{L^{p'}(0,T;V_p')} &= \Delta t \sum_{j=1}^{N} \|f^j_{\Delta t}\|^{p'}_{V_p'} \\
&\le \Delta t \sum_{j=1}^{N} \left(\frac{1}{\Delta t} \int_{t_{j-1}}^{t_j} \|f(\tau)\|_{V_p'} \, \mathrm{d}\tau \right)^{p'} \\
&\le (\Delta t)^{1-p'} \sum_{j=1}^{N} (\Delta t)^{\frac{p'}{p}} \int_{t_{j-1}}^{t_j} \|f(\tau)\|^{p'}_{V_p'} \, \mathrm{d}\tau \\
&= \int_0^T \|f(\tau)\|^{p'}_{V_p'} \, \mathrm{d}\tau \, .
\end{aligned}$$

By means of the dense embedding $C^1([0,T];V_p') \overset{d}{\hookrightarrow} L^{p'}(0,T;V_p')$, for arbitrary $\varepsilon > 0$ there exists a function $\tilde{f} \in C^1([0,T];V_p')$, such that

$$\|\tilde{f} - f\|_{L^{p'}(0,T;V_p')} < \varepsilon.$$

It is easy to see, that for smooth $\tilde{f}$

$$\|R_{\Delta t}\tilde{f} - \tilde{f}\|_{L^{p'}(0,T;V_p')} \to 0 \quad \text{for } \Delta t \to 0.$$

Indeed, any $\tilde{f} \in C^1([0,T];V_p')$ is Lipschitz continuous in time and hence we have

$$\begin{aligned}\int_0^T \|R_{\Delta t}\tilde{f} - \tilde{f}\|_{V_p'}^{p'} &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left\| \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} \tilde{f}(s)\,\mathrm{d}s \, - \tilde{f}(t) \right\|_{V_p'}^{p'} \mathrm{d}t \\ &\le \frac{1}{\Delta t}\sum_{n=1}^N \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t_n} \|\tilde{f}(s) - \tilde{f}(t)\|_{V_p'}^{p'} \,\mathrm{d}s \;\mathrm{d}t \\ &\le \frac{2}{\Delta t}\sum_{n=1}^N \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} \|\tilde{f}\|_{C^1([0,T];V_p')}\,|s-t|^{p'}\,\mathrm{d}s \;\mathrm{d}t \\ &\le c\,(\Delta t)^{-1}\sum_{n=1}^N (\Delta t)^{2+p'} \\ &= c\,T\,(\Delta t)^{p'} < \varepsilon,\end{aligned}$$

for Δt sufficiently small. Note, that c depends on $\tilde{f}$. With the triangle inequality, we find

$$\begin{aligned}\|R_{\Delta t}f - f\|_{L^{p'}(0,T;V_p')} &\le \|R_{\Delta t}f - R_{\Delta t}\tilde{f}\|_{L^{p'}(0,T;V_p')} \\ &\quad + \|R_{\Delta t}\tilde{f} - \tilde{f}\|_{L^{p'}(0,T;V_p')} + \|\tilde{f} - f\|_{L^{p'}(0,T;V_p')} \\ &\le 3\varepsilon,\end{aligned}$$

and thus

$$f_{\Delta t} \to f \quad \text{in } L^{p'}(0,T;V_p'). \tag{8.14}$$

Combining (8.5), (8.10), (8.11), (8.14) and Lemma 8.11 we obtain

$$\begin{aligned}-\int_0^T \langle u(t), v\rangle \varphi'(t)\,\mathrm{d}t + \int_0^T \langle a(t), v\rangle\varphi(t)\,\mathrm{d}t + \int_0^T \langle Bu(t), v\rangle\varphi(t)\,\mathrm{d}t \\ = \int_0^T \langle f(t), v\rangle \varphi(t)\,\mathrm{d}t\,.\end{aligned}$$

for $\varphi \in C_0^\infty(0,T)$ and $v \in \mathcal{V}$.

It is $f \in L^{p'}(0,T;V_p')$, $a \in L^{p'}(0,T;V_p')$ and $Bu \in L^{r'}(0,T;V_r')$ and hence with Lemma A.2 follows

$$u' = f - a - Bu \quad \text{ in } L^{r'}(0,T;V_r').$$

Finally, Lemma 3.1 implies $u \in C_w([0,T];H)$. □

Lemma 8.13. *There holds*

$$u(0) = u_0 \text{ in } H.$$

Proof. Let $v \in V_r$ and $\varphi \in C^1([0,T])$ with $\varphi(0) = 1$ and $\varphi(T) = 0$. With Lemma A.4, we have

$$\begin{aligned}-\langle u(0), v\rangle &= \langle u(T), v\rangle\varphi(T) - \langle u(0), v\rangle\varphi(0) \\ &= \langle u', v\varphi\rangle + \langle u, v\varphi'\rangle \\ &= \langle u', v\varphi\rangle - \langle v_{\Delta t}', v\varphi\rangle + \langle v_{\Delta t}', v\varphi\rangle + \langle u, v\varphi\rangle \\ &= \langle u' - v_{\Delta t}', v\varphi\rangle + \langle v_{\Delta t}', v\varphi\rangle + \langle v_{\Delta t}, v\varphi'\rangle + \langle u - v_{\Delta t}, v\varphi'\rangle \\ &= \langle u' - v_{\Delta t}', v\varphi\rangle - \langle v_{\Delta t}(0), v\rangle\varphi(0) + \langle u - v_{\Delta t}, v\varphi'\rangle \\ &= \langle u' - v_{\Delta t}', v\varphi\rangle - \langle u_{\Delta t}^0, v\rangle + \langle u - v_{\Delta t}, v\varphi'\rangle,\end{aligned}$$

which converges towards $-\langle u_0, v\rangle$ thanks to Lemma 8.11. Hence, $u(0) = u_0$ in V_r'. Since $u \in C_w([0,T];H)$ and $H \overset{d}{\hookrightarrow} V_r'$ we also have $u(0) = u_0$ in H. □

8.5 Decisive monotonicity trick

8.5.1 Simple case

It is now left to show $a = Au$. This is usually done with aid of the decisive monotonicity trick (or Minty's monotonicity trick). Let us for a moment consider the case

$$p > 1 + \frac{2d}{d+2}$$

and with it $B : L^{2p'}(0,T;H_{2p'}) \to L^{p'}(0,T;V_p')$. One way of applying the monotonicity trick lies in showing

$$\limsup_{\Delta t\to 0}\langle Au_{\Delta t}, u_{\Delta t}\rangle \le \langle a, u\rangle, \tag{8.15}$$

since then the monotonicity of A together with Lemma 8.11 implies

$$\begin{aligned} 0 &\le \limsup_{\Delta t\to 0}\langle Au_{\Delta t} - Aw, u_{\Delta t} - w\rangle \\ &\le \limsup_{\Delta t\to 0}\langle Au_{\Delta t}, u_{\Delta t}\rangle - \lim_{\Delta t\to 0}\langle Aw, u_{\Delta t} - w\rangle - \lim_{\Delta t\to 0}\langle Au_{\Delta t}, w\rangle \\ &\le \langle a, u\rangle - \langle Aw, u - w\rangle - \langle a, w\rangle. \end{aligned}$$

for arbitrary $w \in L^p(0,T;V_p)$. This is equivalent to

$$\langle a - Aw, u - w\rangle \ge 0.$$

Now choosing $w = u \pm \tau v$ for $\tau \in [0,1]$ and arbitrary $v \in L^p(0,T;V_p)$ and using the hemicontinuity of A gives

$$\mp\langle a, v\rangle \ge \mp\langle Au, v\rangle.$$

This implies $Au = a$ in $L^{p'}(0,T;V_p')$.

Hence, we only need to show (8.15). Due to the more restrictive condition on p, we have $u' \in L^{p'}(0,T;V_p')$ and thus, equation (3.8) can be tested with u. Testing (8.5) with $u_{\Delta t}$ gives

$$\langle v_{\Delta t}', u_{\Delta t}\rangle + \langle Au_{\Delta t}, u_{\Delta t}\rangle + \langle Bu_{\Delta t}, u_{\Delta t}\rangle = \langle f_{\Delta t}, u_{\Delta t}\rangle,$$

which is equivalent to

$$\langle Au_{\Delta t}, u_{\Delta t}\rangle = -\langle v_{\Delta t}', v_{\Delta t}\rangle - \langle v_{\Delta t}', u_{\Delta t} - v_{\Delta t}\rangle + \langle f_{\Delta t}, u_{\Delta t}\rangle.$$

With the boundedness of $\{v_{\Delta t}\}$ in $C_w([0,T];H)$ one can show, that there exists a subsequence for which $v_{\Delta t}(T)$ converges weakly in H. Similarly to Lemma 8.13, one can show, that the limit of this sequence is indeed $u(T)$. With integration by parts (2.2) and $u_{\Delta t}^0 \to u_0$ in H then follows

$$\langle u', u\rangle = \|u(T)\|_H^2 - \|u(0)\|_H^2 \le \liminf_{\Delta t\to 0}\langle v_{\Delta t}', v_{\Delta t}\rangle.$$

Here we used, that for weakly convergent sequences holds

$$\|u(T)\|_H \le \liminf_{\Delta t \to 0} \|v_{\Delta t}(T)\|_H,$$

see e.g. [25, Lemma A.2.15]. One can derive from the definition of $u_{\Delta t}$ and $v_{\Delta t}$, that

$$\langle v'_{\Delta t}, u_{\Delta t} - v_{\Delta t}\rangle \ge 0.$$

Together with $\langle Bu, u\rangle = 0$, the strong convergence $f_{\Delta t} \to f$, (8.5) and (8.13), this gives

$$\begin{aligned}\limsup_{\Delta t\to 0}\langle Au_{\Delta t}, u_{\Delta t}\rangle &\le -\liminf_{\Delta t\to 0}\langle v'_{\Delta t}, v_{\Delta t}\rangle - \liminf_{\Delta t\to 0}\langle u_{\Delta t} - v_{\Delta t}, u_{\Delta t}\rangle \\ &\quad + \limsup_{\Delta t\to 0}\langle f_{\Delta t}, u_{\Delta t}\rangle \\ &\le -\big(\|u(T)\|_H^2 - \|u(0)\|_H^2\big) + \langle f, u\rangle \\ &= -\langle u', u\rangle - \langle Bu, u\rangle + \langle f, u\rangle \\ &= \langle a, u\rangle.\end{aligned}$$

Finally, the decisive monotonicity trick implies $a = Au$.

8.5.2 Difficult case

When considering the less restrictive condition (8.9) on p, we are not allowed to test equation (3.8) with u since the time derivative u' is not regular enough. This calls for sufficiently smooth test functions that preserve some of the properties of $u_{\Delta t}$ and u. We will construct these functions with the help of the Lipschitz truncation theorem introduced in Chapter 6.

Since this truncation will not be divergence-free, we have to find a representation of our differential equation (3.8) in a non-solenoidal context. For this, we recover pressure functions, which will be conveniently split into several parts corresponding to the terms $Au_{\Delta t}$, $Bu_{\Delta t}$ and $f_{\Delta t}$.

Finally, we will be employing these test functions in the non-solenoidal context in order to show almost everywhere convergence of the sequence of (symmetric parts of) gradients $\{Du_{\Delta t}\}$. In this process, we have to rely on the special form of the diffusion term, i.e. the pointwise coercivity, monotonicity and growth condition on the integrand function S.

8.6 Reconstruction of the pressure

We now fix a domain $G \subset\subset \Omega$ with $\partial G \in C^2$. Note, that on this domain we do not have properties like

$$u_{\Delta t} \in L^p(0,T; V_p(G)),$$

since $u_{\Delta t}$ does not necessarily vanish on the boundary of G.

Theorem 5.3 requires the terms $Au_{\Delta t}$, $Bu_{\Delta t}$ and $f_{\Delta t}$ to belong to spaces $L^{p'}(0,T; (W_0^{1,p}(G)^d)')$ and $L^{r'}(0,T; (W_0^{1,r}(G)^d)')$ instead of $L^{p'}(0,T; V_p(G)')$ and $L^{r'}(0,T; V_r(G)')$, respectively. In fact, in the definitions of the operators A and B in Section 3.1 and 3.2.4 it is not necessary to assume that the arguments vanish on the boundary or are divergence-free. Furthermore, Av and Bv can in fact be considered as functions in $L^{p'}(0,T; (W_0^{1,p}(G)^d)')$ and

$L^{r'}(0,T;(W_0^{1,r}(G)^d)')$ for $v \in L^p(0,T;W^{1,p}(G)^d)$ and $L^{2r'}(0,T;L^{2r'}(G)^d)$ respectively. Then of course, the concept of monotonicity is no longer available for the operator A, which finally forces us to employ the pointwise monotonicity of the function S.

Employing Theorem 5.3 to the difference of the limit equation (8.13) and the approximation (8.5), namely

$$(u - v_{\Delta t})' = (f - f_{\Delta t}) + (Au_{\Delta t} - a) + (Bu_{\Delta t} - Bu) \quad \text{in } L^{r'}(0,T;V_r(G)'),$$

provides unique functions

$$\begin{aligned}
&\pi_{f,\Delta t} \in L^{p'}(0,T;L_0^{p'}(G)),\\
&\pi_{A,\Delta t} \in L^{p'}(0,T;L_0^{p'}(G)),\\
&\pi_{B,\Delta t} \in L^{r'}(0,T;L_0^{r'}(G)),\\
&\hat{\pi}_{h,\Delta t} \in C_w([0,T];W^{1,2}(G)) \cap C_w([0,T];W^{2,\infty}_{\text{loc}}(G))
\end{aligned}$$

with $-\Delta\hat{\pi}_{h,\Delta t} = 0$, $\int_G \hat{\pi}_{h,\Delta t}\,\mathrm{d}x = 0$ and $\hat{\pi}_{h,\Delta t}(0) = 0$, such that

$$\begin{aligned}
\langle (u - v_{\Delta t})', \varphi\rangle =& \langle (f - f_{\Delta t}) + (Au_{\Delta t} - a) + (Bu_{\Delta t} - Bu), \varphi\rangle\\
&+ \int_0^T \int_G ((-\pi_{f,\Delta t}) + \pi_{A,\Delta t} + \pi_{B,\Delta t})\,\operatorname{div}\varphi\,\mathrm{d}x\;\mathrm{d}t\\
&+ \int_0^T \int_G \nabla\hat{\pi}_{h,\Delta t}\cdot\partial_t\varphi\,\mathrm{d}x\;\mathrm{d}t
\end{aligned} \tag{8.16}$$

holds for all $\varphi \in C_0^\infty(G \times (0,T))^d$ with $\operatorname{div}\varphi = 0$. Moreover, the estimates

$$\begin{aligned}
\|\pi_{f,\Delta t}\|_{L^{p'}(0,T;L_0^{p'}(G))} &\le c\,\|f - f_{\Delta t}\|_{L^{p'}(0,T;(W_0^{1,p}(G))')},\\
\|\pi_{A,\Delta t}\|_{L^{p'}(0,T;L_0^{p'}(G))} &\le c\,\|Au_{\Delta t} - a\|_{L^{p'}(0,T;(W_0^{1,p}(G))')},\\
\|\pi_{B,\Delta t}\|_{L^{r'}(0,T;L_0^{r'}(G))} &\le c\,\|Bu_{\Delta t} - Bu\|_{L^{r'}(0,T;(W_0^{1,r}(G))')},\\
\|\hat{\pi}_{h,\Delta t}(t) - \hat{\pi}_{h,\Delta t}(s)\|_{W^{1,2}(G)} &\le c\,\|u(t) - u(s) + v_{\Delta t}(t) - v_{\Delta t}(s)\|_H
\end{aligned}$$

and

$$\|\hat{\pi}_{h,\Delta t}(t) - \hat{\pi}_{h,\Delta t}(s)\|_{W^{2,\infty}(G')} \le c\,\|u(t) - u(s) + v_{\Delta t}(t) - v_{\Delta t}(s)\|_H$$

hold true for all $G' \subset\subset G$ and $t, s \in [0,T]$.

Let us now fix a domain $G' \subset\subset G$ and shortly write Q' instead of $G' \times (0,T)$. With $\nabla\pi_{f,\Delta t} \in L^{p'}(0,T;(W_0^{1,p}(G')^d)')$ (and analogously $\nabla\pi_{A,\Delta t}$ and $\nabla\pi_{B,\Delta t}$) we denote the functional defined through

$$\langle \nabla\pi_{f,\Delta t}, w\rangle = -\int_{Q'} \pi_{f,\Delta t}\,\operatorname{div} w\,\mathrm{d}(x,t)\,, \quad w \in L^p(0,T;W_0^{1,p}(G')^d).$$

Lemma 8.14. *For the pressure terms $\nabla\pi_{f,\Delta t}$, $\nabla\pi_{A,\Delta t}$, $\nabla\pi_{B,\Delta t}$ we have*

$$\begin{aligned}
\nabla\pi_{f,\Delta t} \to 0 \quad &\text{in} \quad L^{p'}(0,T;(W_0^{1,p}(G')^d)'),\\
\nabla\pi_{B,\Delta t} \to 0 \quad &\text{in} \quad L^{r'}(0,T;(W_0^{1,r}(G')^d)')
\end{aligned}$$

and

$$\|\nabla \pi_{A,\Delta t}\|_{L^{p'}(0,T;(W_0^{1,p}(G')^d)')} \le c.$$

For the remaining pressure term there holds

$$\nabla \hat{\pi}_{h,\Delta t} \to 0 \quad \textit{in} \quad L^s(0,T;W^{1,s}(G')^d), \tag{8.17}$$

for all $s \in [1,\infty)$.

Proof. We have

$$\|\hat{\pi}_{h,\Delta t}(t)\|_{W^{2,s}(G')} \le c\,(\|u(t) - v_{\Delta t}(t)\|_H + \|u(0) - u^0_{\Delta t}\|_H)$$

for any $s \in [1,\infty]$. Integrating over time leads to

$$\|\hat{\pi}_{h,\Delta t}\|_{L^s(0,T;W^{2,s}(G'))} \le c\,\|u - v_{\Delta t}\|_{L^s(0,T;H)} + c\|u_0 - u^0_{\Delta t}\|_H \to 0,$$

since $v_{\Delta t} \to u$ in $L^s(0,T;H)$ for any $s \in [1,\infty)$ (see Lemma 8.11 and boundedness in $L^\infty(0,T;H)$). This means

$$\hat{\pi}_{h,\Delta t} \to 0 \quad \text{in} \quad L^s(0,T;W^{2,s}(G'))$$

for any $s \in [1,\infty)$. In particular, the sequence of gradients $\{\nabla \hat{\pi}_{h,\Delta t}\}$ converges strongly in $L^s(0,T;W^{1,s}(G')^d)$.

Due to Lemma 8.12, we know that $\{\pi_{f,\Delta t}\}$ and $\{\pi_{B,\Delta t}\}$ converge strongly to zero in their respective spaces and $\{\pi_{A,\Delta t}\}$ is bounded in $L^{p'}(0,T;L_0^{p'}(G))$. Hence, with Hölder's inequality we find

$$\begin{aligned} |\langle \nabla \pi_{f,\Delta t}, \varphi \rangle| &\le \int_{Q'} |\pi_{f,\Delta t}||\operatorname{div} \varphi| \,\mathrm{d}(x,t) \\ &\le \|\pi_{f,\Delta t}\|_{L^{p'}(0,T;L_0^{p'}(G'))} \|\varphi\|_{L^p(0,T;W_0^{1,p}(G')^d)} \end{aligned}$$

for any $\varphi \in C_0^\infty(Q')^d$, establishing the result. For the terms $\pi_{B,\Delta t}$ and $\pi_{A,\Delta t}$ we proceed analogously. □

Equation (8.16) then reads

$$\begin{aligned} (u - v_{\Delta t} + \nabla \hat{\pi}_{h,\Delta t})' = {} & f - f_{\Delta t} + \nabla \pi_{f,\Delta t} \\ & + A u_{\Delta t} - a - \nabla \pi_{A,\Delta t} \\ & + B u_{\Delta t} - B u - \nabla \pi_{B,\Delta t} \end{aligned} \tag{8.18}$$

in $L^{r'}(0,T;(W_0^{1,r}(G')^d)')$.

8.7 Application of the Lipschitz truncation theorem

We remind that $Q' = G' \times (0,T)$. Let us define the functions

$$w_{\Delta t} = u - u_{\Delta t} \quad \text{in} \quad Q'$$

for every step-size Δt. Note, that these functions do not vanish on the boundary of G'.

Since the results of the Lipschitz truncation are restricted to compact subsets of the time-space cylinder, we have to localize several arguments, i.e. we choose an arbitrary smooth function with compact support in Q' which will be multiplied to our test function.

Let $\zeta \in C_0^\infty(Q')$ with $K = \operatorname{supp}(\zeta)$. It is no restriction to assume, that $0 \le \zeta \le 1$.

Lemma 8.11 ensures the boundedness of the sequence $\{w_{\Delta t}\}$ in the spaces $L^\infty(0,T;L^2(G')^d)$, $L^p(0,T;W^{1,p}(G')^d)$ and $W^{\bar\sigma,2}(0,T;L^2(G')^d)$ and the weak convergence towards zero in $L^p(0,T;W^{1,p}(G')^d)$ and $W^{\bar\sigma,2}(0,T;L^2(G')^d)$.

Thus, the premises for the Lipschitz truncation theorem (see Theorem 6.4) for $\{w_{\Delta t}\}$ are fulfilled. Let $\theta_{\Delta t} = \sqrt{\|w_{\Delta t}\|_{L^2(Q')^d}}$. For any $k \in \mathbb{N}$, we obtain a sequence of numbers $\{\lambda_{k,\Delta t}\} \subset [2^{2^k}, 2^{2^{k+1}}]$ and sets $\{E_{k,\Delta t}\}$ with

$$\limsup_{\Delta t \to 0} \lambda_{k,\Delta t}^p \mu_{d+1}(E_{k,\Delta t}) \le c\, 2^{-k} \tag{8.19}$$

and corresponding truncations $\mathcal{T} w_{\Delta t} = \mathcal{T}_{E_{k,\Delta t}} w_{\Delta t}$ with

$$\begin{aligned}
\|\mathcal{T} w_{\Delta t}\|_{L^\infty(K)^d} &\le c\left(\theta_{\Delta t} + \delta^{-d-\frac{1}{\sigma}} \|w_{\Delta t}\|_{L^1(E_{k,\Delta t})^d}\right),\\
\|\nabla \mathcal{T} w_{\Delta t}\|_{L^\infty(K)^{d\times d}} &\le c\left(\lambda_{k,\Delta t} + \delta^{-d-1-\frac{1}{\sigma}} \|w_{\Delta t}\|_{L^1(E_{k,\Delta t})^d}\right),
\end{aligned}$$

where $\delta = \operatorname{dist}_{\varrho_\sigma}(K, \partial Q')$. In particular, this means

$$\lim_{\Delta t \to 0} \|\mathcal{T} w_{\Delta t}\|_{L^\infty(K)^d} = 0, \tag{8.20}$$

$$\limsup_{\Delta t \to 0} \|\nabla \mathcal{T} w_{\Delta t}\|_{L^\infty(K)^{d\times d}} \le 2^{2^{k+1}}, \tag{8.21}$$

since $\|w_{\Delta t}\|_{L^1(E_{k,\Delta t})^d} \to 0$.

Furthermore, we have

$$\limsup_{\Delta t \to 0} \|\nabla \mathcal{T} w_{\Delta t}\|_{L^p(E_{k,\Delta t}\cap K)^{d\times d}} \le c\, 2^{-k/p}. \tag{8.22}$$

8.8 Almost everywhere convergence of the gradients

In order to efficiently deal with the diffusion term, we want to employ the identity $\mathcal{T} w_{\Delta t} = w_{\Delta t}$ on $Q' \setminus E_{k,\Delta t}$. Therefore, it is necessary to have a local representation of it. Moreover, we want to show the almost everywhere convergence of the (symmetric parts of the) gradients of $\{u_{\Delta t}\}$, which requires the coercivity, strict monotonicity and growth condition to be given pointwise. Let us employ the special integral form of the operator A, namely

$$\langle Av, w\rangle = \int_{Q'} S(Dv) : Dw \,\mathrm{d}(x,t),$$

for any $v \in L^p(0,T;W^{1,p}(G)^d)$ and $w \in L^p(0,T;W_0^{1,p}(G)^d)$.

The weak convergence of $\{Au_{\Delta t}\}$ in $L^{p'}(0,T;(W_0^{1,p}(G')^d)')$ corresponds to the weak convergence of $\{S(Du_{\Delta t})\}$ in $L^{p'}(Q')^{d\times d}$. Indeed, with the growth condition (1.8) of S and the boundedness of $\{Du_{\Delta t}\}$ in $L^{p'}(Q')^{d\times d}$ (even in

$L^{p'}(\Omega \times (0,T))^{d\times d}$) we can, if necessary, pass to a weakly convergent subsequence. Let us denote the limit of this sequence with $\tilde{S} \in L^{p'}(Q')^{d\times d}$. Then for $w \in L^p(0,T;W_0^{1,p}(G')^d)$ there holds

$$\langle a, w\rangle = \int_{Q'} \tilde{S} : Dw \,\mathrm{d}(x,t)\,.$$

Studying the convergence of the (symmetric parts of the) gradients $\{Du_{\Delta t}\}$ leads to the question of convergence of the term

$$\int_{Q'} \big(S(Du) - S(Du_{\Delta t})\big) : D(u-u_{\Delta t})\zeta \,\mathrm{d}(x,t)\,,$$

see e.g. [41, Chapter 2, Lemma 2.2, p.184] for a similar problem without the function ζ.

For that, we test (8.18) with the truncation $\zeta \mathcal{T} w_{\Delta t} \in L^\infty(0,T;W_0^{1,\infty}(G')^d)$ and receive

$$\begin{aligned}\langle Au - Au_{\Delta t}, \zeta\mathcal{T}w_{\Delta t}\rangle = &-\big\langle a - Au, \zeta\mathcal{T}w_{\Delta t}\big\rangle\\ &-\big\langle (u - v_{\Delta t} + \nabla\hat{\pi}_{h,\Delta t})', \zeta\mathcal{T}w_{\Delta t}\big\rangle\\ &+\big\langle f - f_{\Delta t} + \nabla\pi_{f,\Delta t}, \zeta\mathcal{T}w_{\Delta t}\big\rangle\\ &+\big\langle Bu_{\Delta t} - Bu - \nabla\pi_{B,\Delta t}, \zeta\mathcal{T}w_{\Delta t}\big\rangle\\ &-\big\langle \nabla\pi_{A,\Delta t}, \zeta\mathcal{T}w_{\Delta t}\big\rangle.\end{aligned}$$

It is easy to verify the identity

$$\nabla(\zeta\mathcal{T}w_{\Delta t}) = \nabla(\mathcal{T}w_{\Delta t})\zeta + \mathcal{T}w_{\Delta t} \otimes \nabla\zeta.$$

For the symmetric part of the gradient it then follows by simple calculations

$$D(\zeta\mathcal{T}w_{\Delta t}) = D(\mathcal{T}w_{\Delta t})\zeta + \frac{1}{2}\left(\mathcal{T}w_{\Delta t} \otimes \nabla\zeta + \nabla\zeta \otimes \mathcal{T}w_{\Delta t}\right).$$

Having this in mind, we can write

$$\begin{aligned}&\langle Au - Au_{\Delta t}, \zeta\mathcal{T}w_{\Delta t}\rangle\\ &\quad= \int_{Q'} \big(S(Du) - S(Du_{\Delta t})\big) : D(\mathcal{T}w_{\Delta t})\zeta \,\mathrm{d}(x,t)\\ &\quad\quad+ \frac{1}{2}\int_{Q'} \big(S(Du) - S(Du_{\Delta t})\big) : \left(\mathcal{T}w_{\Delta t} \otimes \nabla\zeta + \nabla\zeta \otimes \mathcal{T}w_{\Delta t}\right) \mathrm{d}(x,t)\,.\end{aligned}$$

Altogether, we end up with the equation

$$\begin{aligned}&\int_{Q'} \big(S(Du) - S(Du_{\Delta t})\big) : D(\mathcal{T}w_{\Delta t})\zeta \,\mathrm{d}(x,t)\\ &\quad= -\frac{1}{2}\int_{Q'} \big(S(Du) - S(Du_{\Delta t})\big) : \left(\mathcal{T}w_{\Delta t} \otimes \nabla\zeta + \nabla\zeta \otimes \mathcal{T}w_{\Delta t}\right) \mathrm{d}(x,t)\\ &\quad\quad- \int_{Q'} \big(\tilde{S} - S(Du)\big) : D(\zeta\mathcal{T}w_{\Delta t}) \,\mathrm{d}(x,t)\\ &\quad\quad- \big\langle (u - v_{\Delta t} + \nabla\hat{\pi}_{h,\Delta t})', \zeta\mathcal{T}w_{\Delta t}\big\rangle\\ &\quad\quad+ \big\langle f - f_{\Delta t} + \nabla\pi_{f,\Delta t}, \zeta\mathcal{T}w_{\Delta t}\big\rangle\\ &\quad\quad+ \big\langle Bu_{\Delta t} - Bu - \nabla\pi_{B,\Delta t}, \zeta\mathcal{T}w_{\Delta t}\big\rangle\\ &\quad\quad- \big\langle \nabla\pi_{A,\Delta t}, \zeta\mathcal{T}w_{\Delta t}\big\rangle.\end{aligned} \tag{8.23}$$

Lemma 8.15. *There holds*

$$\limsup_{\Delta t\to 0}\int_{Q'}\big(S(Du)-S(Du_{\Delta t})\big):D(\mathcal{T}w_{\Delta t})\zeta\,\mathrm{d}(x,t)\\ \leq c\,2^{-k/p}-\liminf_{\Delta t\to 0}\big\langle(u-v_{\Delta t}+\nabla\hat{\pi}_{h,\Delta t})',\mathcal{T}w_{\Delta t}\zeta\big\rangle$$

Proof. We consider each term on the right-hand side of (8.23). For the first term, we estimate

$$\begin{aligned}\frac{1}{2}&\left|\int_{Q'}\big(S(Du)-S(Du_{\Delta t})\big):(\mathcal{T}w_{\Delta t}\otimes\nabla\zeta+\nabla\zeta\otimes\mathcal{T}w_{\Delta t})\,\mathrm{d}(x,t)\right|\\ &\leq\frac{1}{2}\|\tilde{S}-S(Du_{\Delta t})\|_{L^{p'}(Q')^{d\times d}}\|\mathcal{T}w_{\Delta t}\otimes\nabla\zeta+\nabla\zeta\otimes\mathcal{T}w_{\Delta t}\|_{L^p(K)^{d\times d}}\\ &\leq c\,\|\tilde{S}-S(Du_{\Delta t})\|_{L^{p'}(Q')^{d\times d}}\|\nabla\zeta\|_{L^\infty(K)^d}\|\mathcal{T}w_{\Delta t}\|_{L^p(K)^d}\\ &\leq c\,\|\tilde{S}-S(Du_{\Delta t})\|_{L^{p'}(Q')^{d\times d}}\|\nabla\zeta\|_{L^\infty(K)^d}\|\mathcal{T}w_{\Delta t}\|_{L^\infty(K)^d}.\end{aligned}$$

This term vanishes with $\Delta t\to 0$, since $\{S(Du_{\Delta t})\}$ is bounded in $L^{p'}(Q')^{d\times d}$. Moreover, ζ is fixed and with (8.20) there holds $\mathcal{T}w_{\Delta t}\to 0$ in $L^\infty(K)^d$.

For the second term we remind that Theorem 6.4 states $\zeta\mathcal{T}w_{\Delta t}\rightharpoonup 0$ in $L^s(0,T;W_0^{1,s}(G')^d)$ for any $s\in[1,\infty)$. Hence,

$$\int_{Q'}\big(\tilde{S}-S(Du)\big):D(\zeta\mathcal{T}w_{\Delta t})\,\mathrm{d}(x,t)\;\to 0.$$

We know from Lemma 8.12, that $f_{\Delta t}-f$ converges strongly towards zero. We estimate

$$\begin{aligned}&|\langle f_{\Delta t}-f,\zeta\mathcal{T}w_{\Delta t}\rangle|\\ &\quad\leq\|f_{\Delta t}-f\|_{L^{p'}(0,T;(W_0^{1,p}(G')^d)')}\|\zeta\mathcal{T}w_{\Delta t}\|_{L^p(0,T;W_0^{1,p}(G')^d)}\\ &\quad=\|f_{\Delta t}-f\|_{L^{p'}(0,T;(W_0^{1,p}(G')^d)')}\|\nabla(\zeta\mathcal{T}w_{\Delta t})\|_{L^p(K)^{d\times d}}\\ &\quad\leq c\,\|f_{\Delta t}-f\|_{L^{p'}(0,T;(W_0^{1,p}(G')^d)')}\|\nabla(\zeta\mathcal{T}w_{\Delta t})\|_{L^\infty(K)^{d\times d}}.\end{aligned}$$

Due to (8.21) this converges to zero with $\Delta t\to 0$. To show the convergence of the term involving $\pi_{f,\Delta t}$, one proceeds analogously.

The term $\big\langle Bu_{\Delta t}-Bu-\nabla p_{B,\Delta t},\zeta\mathcal{T}w_{\Delta t}\big\rangle$ can be handled in the same way, replacing p' by r', with $r'<p(d+2)/(2d)$ chosen in Section 3.2.4.

The pressure term pertaining to the term $a-Au_{\Delta t}$ is more difficult to handle, because this pressure does not converge strongly. Instead we will be exploiting the fact that it vanishes where $\mathcal{T}w_{\Delta t}=w_{\Delta t}$ thanks to $\operatorname{div}w_{\Delta t}=0$. The measure of the remaining set is small enough. First we split the divergence of the product with the product rule:

$$\begin{aligned}|\langle\nabla p_{A,\Delta t},\mathcal{T}w_{\Delta t}\rangle|&=\left|\int_{Q'}p_{A,\Delta t}\operatorname{div}(\zeta\mathcal{T}w_{\Delta t})\,\mathrm{d}(x,t)\right|\\ &\leq\left|\int_{Q'}p_{A,\Delta t}\,\operatorname{div}(\mathcal{T}w_{\Delta t})\zeta\,\mathrm{d}(x,t)\right|\\ &\quad+\left|\int_{Q'}p_{A,\Delta t}\,(\mathcal{T}w_{\Delta t}\cdot\nabla\zeta)\,\mathrm{d}(x,t)\right|.\end{aligned}$$

Employing the convergence of $\{\mathcal{T}w_{\Delta t}\}$ in $L^\infty(K)^d$ (see (8.20)) and the bound on the pressure (see Lemma 8.14), we get for the second term

$$
\begin{aligned}
&\lim_{\Delta t \to 0} \left| \int_{Q'} p_{A,\Delta t} \left(\mathcal{T}w_{\Delta t} \cdot \nabla\zeta\right) \,\mathrm{d}(x,t) \right| \\
&\leq \lim_{\Delta t \to 0} \|p_{A,\Delta t}\|_{L^{p'}(Q')} \|\mathcal{T}w_{\Delta t}\|_{L^p(K)^d} \|\nabla\zeta\|_{L^\infty(K)^d} \\
&\leq \lim_{\Delta t \to 0} \|p_{A,\Delta t}\|_{L^{p'}(Q')} \|\mathcal{T}w_{\Delta t}\|_{L^\infty(K)^d} \|\nabla\zeta\|_{L^\infty(K)^d} \\
&= 0.
\end{aligned}
$$

For the first term, we note $\mathcal{T}w_{\Delta t} = w_{\Delta t}$ on $Q' \setminus E_{k,\Delta t}$ and thus we conclude $\operatorname{div}\mathcal{T}w_{\Delta t} = \operatorname{div} w_{\Delta t} = \operatorname{div} u - \operatorname{div} u_{\Delta t} = 0$. Hence, the problem reduces to the small set $E_{k,\Delta t}$ and we observe

$$
\begin{aligned}
&\left| \int_{Q'} p_{A,\Delta t} \operatorname{div}(\mathcal{T}w_{\Delta t})\zeta \,\mathrm{d}(x,t) \right| \\
&= \left| \int_{E_{k,\Delta t}} p_{A,\Delta t} \operatorname{div}(\mathcal{T}w_{\Delta t})\zeta \,\mathrm{d}(x,t) \right| \\
&\leq \|p_{A,\Delta t}\|_{L^{p'}(Q')} \|\operatorname{div}\mathcal{T}w_{\Delta t}\|_{L^p(E_{k,\Delta t}\cap K)} \|\zeta\|_{L^\infty(K)} \\
&\leq \|p_{A,\Delta t}\|_{L^{p'}(Q')} \|\nabla\mathcal{T}w_{\Delta t}\|_{L^p(E_{k,\Delta t}\cap K)^{d\times d}} \|\zeta\|_{L^\infty(K)}.
\end{aligned}
$$

Since $\{\|p_{A,\Delta t}\|_{L^{p'}(Q')}\}$ is bounded by a constant, (8.22) then implies together with the results on the second term

$$
\limsup_{\Delta t \to 0} |\langle \nabla p_{A,\Delta t}, \mathcal{T}w_{\Delta t}\rangle| \leq c\, 2^{-k/p}.
$$

□

The rest of the proof of convergence will be given after a short discussion of the remaining term on the right-hand side in Lemma 8.15.

8.8.1 Open Problem

Unfortunately, we have not yet been able to show

$$
-\liminf_{\Delta t \to 0} \left\langle (u - v_{\Delta t} + \nabla\hat{\pi}_{h,\Delta t})', \mathcal{T}w_{\Delta t}\zeta \right\rangle \leq c\, 2^{-k/p}.
$$

The problem here lies in a suitable rule of integration by parts similar to the one in [22, Theorem 3.21]. Only truncating the function $w_{\Delta t}$, which implicitly means $u_{\Delta t}$, will not admit such a result, since $u_{\Delta t}$ does not have a full time derivative in some dual space.

One possible method would be to truncate $u_{\Delta t}$ and $v_{\Delta t}$ simultaneously, i.e. sharing one cutoff-set $E_{k,\Delta t}$, and use the rule of integration by parts developed in [22, Theorem 3.21]. The problem in this method lies in the order of differentiability $\bar{\sigma}$ of $u_{\Delta t}$. With $\bar{\sigma} \leq \frac{1}{2}$ we are not able to bound the term $\partial_t \mathcal{T}w_{\Delta t}$ properly since we scale the metric ϱ_σ with the exponent $\sigma < \bar{\sigma} \leq \frac{1}{2}$. Scaling with $\frac{1}{2}$ would be necessary. This however leads to problems when employing Poincaré's inequality from Lemma 2.11.

Another method one could think of is truncating $u_{\Delta t}$ and $v_{\Delta t}$ independently, say with truncation operators $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Then we could use $\mathcal{T}_1 w_{\Delta t}$ as a test function as before and employ a rule of integration by parts for $\mathcal{T}_2(u - v_{\Delta t} + \nabla\hat{\pi}_{h,\Delta t})$, which would then be possible, since we could scale the metric with $\frac{1}{2}$ for $\mathcal{T}_2$, see e.g. [22]. In this case, there would be a term

$$\langle (u - v_{\Delta t} + \nabla\hat{\pi}_{h,\Delta t})', \mathcal{T}_1 w_{\Delta t} - \mathcal{T}_2 v_{\Delta t}\rangle$$

remaining to be estimated.

Unfortunately, in the course of this thesis, we were not able to study these or other approaches to fill this gap.

Postulation 8.16. In what follows, we will postulate

$$-\liminf_{\Delta t\to 0} \big\langle (u - v_{\Delta t} + \nabla\hat{\pi}_{h,\Delta t})', \mathcal{T} w_{\Delta t}\zeta\big\rangle \le c\, 2^{-k/p}.$$

Under the foregoing postulate, the Lemma 8.15 implies

$$\limsup_{\Delta t\to 0} \int_{Q'} \big(S(Du) - S(Du_{\Delta t})\big) : D(\mathcal{T} w_{\Delta t})\zeta \,\mathrm{d}(x,t) \le c\, 2^{-k/p}. \tag{8.24}$$

We will now use $\mathcal{T} w_{\Delta t} = w_{\Delta t} = u - u_{\Delta t}$ on $Q'\setminus E_{k,\Delta t}$. Then we can apply the pointwise monotonicity of S. The integral over the rest $E_{k,\Delta t}$ is small enough:

Lemma 8.17. *There exists a subsequence $\{(\Delta t)_k\}$ of $\{\Delta t\}$ such that for every $k \in \mathbb{N}$ holds*

$$\left| \int_{Q'\setminus E_{k,(\Delta t)_k}} \big(S(Du) - S(Du_{(\Delta t)_k})\big) : D(u - u_{(\Delta t)_k})\zeta \,\mathrm{d}(x,t) \right| \le c\, 2^{-k/p}$$

and

$$\lambda^p_{k,(\Delta t)_k} \mu_{d+1}(E_{k,(\Delta t)_k}) \le c\, 2^{-k}.$$

Proof. First, let us consider the small set $E_{k,\Delta t}$. For this, we obtain from (8.22)

$$\begin{aligned}
&\limsup_{\Delta t\to 0} \left| \int_{E_{k,\Delta t}} \big(S(Du) - S(Du_{\Delta t})\big) : D(\mathcal{T} w_{\Delta t})\zeta \,\mathrm{d}(x,t) \right| \\
&\quad\le \limsup_{\Delta t\to 0} \|S(Du) - S(Du_{\Delta t})\|_{L^{p'}(Q')^{d\times d}} \|\zeta\|_{L^\infty(Q')} \|\nabla \mathcal{T} w_{\Delta t}\|_{L^p(K\cap E_{k,\Delta t})^{d\times d}} \\
&\quad\le c\, 2^{-k/p}.
\end{aligned}$$

This together with (8.24) implies

$$\begin{aligned}
&\limsup_{\Delta t\to 0} \left| \int_{Q'\setminus E_{k,\Delta t}} \big(S(Du) - S(Du_{\Delta t})\big) : D(w_{\Delta t})\zeta \,\mathrm{d}(x,t) \right| \\
&\quad= \limsup_{\Delta t\to 0} \left| \int_{Q'\setminus E_{k,\Delta t}} \big(S(Du) - S(Du_{\Delta t})\big) : D(\mathcal{T} w_{\Delta t})\zeta \,\mathrm{d}(x,t) \right| \\
&\quad\le \limsup_{\Delta t\to 0} \left| \int_{Q'} \big(S(Du) - S(Du_{\Delta t})\big) : D(\mathcal{T} w_{\Delta t})\zeta \,\mathrm{d}(x,t) \right| \\
&\qquad + \limsup_{\Delta t\to 0} \left| \int_{E_{k,\Delta t}} \big(S(Du) - S(Du_{\Delta t})\big) : D(\mathcal{T} w_{\Delta t})\zeta \,\mathrm{d}(x,t) \right| \\
&\quad\le c\, 2^{-k/p}.
\end{aligned}$$

We remind that $w_{\Delta t} = u - u_{\Delta t}$. Finally, if the limes superior is smaller than $c\,2^{-k/p}$, we find for each $k \in \mathbb{N}$ a number $(\Delta t)_k \in \mathbb{N}$ such that

$$\left| \int_{Q' \setminus E_{k,(\Delta t)_k}} \big(S(Du) - S(Du_{(\Delta t)_k})\big) : D(u - u_{(\Delta t)_k})\zeta \, \mathrm{d}(x,t) \right| \le \tilde{c}\, 2^{-k/p}.$$

With (8.19) the same holds for $\lambda^p_{k,(\Delta t)_k} \mu_{d+1}(E_{k,(\Delta t)_k})$. □

The next step is to show the convergence of the sets $E_{k,(\Delta t)_k}$. We accomplish this by introducing the sequence of functions

$$\zeta_k = \zeta \, \chi_{Q' \setminus E_{k,(\Delta t)_k}}.$$

With this definition, the first statement of Lemma 8.17 implies

$$\lim_{k \to \infty} \left| \int_{Q'} \big(S(Du) - S(Du_{(\Delta t)_k})\big) : D(u - u_{(\Delta t)_k})\zeta_k \, \mathrm{d}(x,t) \right| = 0. \qquad (8.25)$$

Lemma 8.18. *There holds*

$$\zeta_k \to \zeta \quad \textit{almost everywhere in} \quad Q'.$$

Proof. First of all, it is clear that $\zeta_k(x,t) \to \zeta(x,t)$ for every point (x,t) that belongs to almost every set $Q' \setminus E_{k,(\Delta t)_k}$. This means, that $\zeta_k(x,t) \to \zeta(x,t)$ for every

$$x \in \bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} Q' \setminus E_{k,(\Delta t)_k}.$$

But this set already has full measure. Indeed,

$$\bigcup_{l=1}^{\infty} \bigcap_{k=l}^{\infty} Q' \setminus E_{k,(\Delta t)_k} = Q' \setminus \left(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} E_{k,(\Delta t)_k} \right)$$

and with the second statement of Lemma 8.17

$$\begin{aligned} \mu_{d+1}\left(\bigcup_{k=l}^{\infty} E_{k,(\Delta t)_k} \right) &\le \sum_{k=l}^{\infty} \mu_{d+1}\left(E_{k,(\Delta t)_k}\right) \\ &\le \sum_{k=l}^{\infty} c\, 2^{-k} \lambda^{-p}_{k,(\Delta t)_k} \\ &\le c\, 2^{-l}. \end{aligned}$$

This gives

$$\mu_{d+1}\left(\bigcap_{l=1}^{\infty} \bigcup_{k=l}^{\infty} E_{k,(\Delta t)_k} \right) = \lim_{l \to \infty} c\, 2^{-l} = 0$$

with the continuity of measures. □

Lemma 8.19. *Let S be as in Section 1.2 fulfilling the conditions* (1.8), (1.9) *and* (1.10). *For $k \in \mathbb{N}$, let $0 \le \zeta, \zeta_k \le 1$ be smooth functions with $\zeta_k \to \zeta$ almost everywhere in Q' and let $\{u_{(\Delta t)_k}\}_{k\in\mathbb{N}} \subset L^p(0,T;W^{1,p}(G')^d)$, $u \in L^p(0,T;W^{1,p}(G')^d)$ with $u_{(\Delta t)_k} \rightharpoonup u$ in $L^p(0,T;W^{1,p}(G')^d)$.*

If (8.25) *holds, i.e.*

$$\lim_{k\to\infty}\left|\int_{Q'}\big(S(Du)-S(Du_{(\Delta t)_k})\big):D(u-u_{(\Delta t)_k})\zeta_k\,\mathrm{d}(x,t)\right|=0,$$

then

$$Du_{(\Delta t)_k}(x,t)\to Du(x,t)$$

for almost every $(x,t)\in Q'$ *with* $\zeta(x,t)>0$.

Proof. We follow the proof of [41, Chapter 2, Lemma 2.2, p. 184], where one can find a similar result except for the cut-off function ζ. Let us call

$$F_{(\Delta t)_k}=\big(S(Du)-S(Du_{(\Delta t)_k})\big):D(u-u_{(\Delta t)_k})\zeta_k.$$

Due to the strict monotonicity (1.9) of S and $\zeta\geq 0$, we have $F_{(\Delta t)_k}\geq 0$ for all $k\in\mathbb{N}$. From the weak convergence of $\{u_{(\Delta t)_k}\}_{k\in\mathbb{N}}$ in $L^p(0,T;W^{1,p}(G')^d)$ follows the strong convergence in $L^p(Q')^d$. Hence, we can extract a (not relabeled) subsequence, such that for almost every $(x,t)\in Q'$ there holds

$$u_{(\Delta t)_k}(x,t)\to u(x,t),\quad F_{(\Delta t)_k}(x,t)\to 0.$$

Let $(x,t)\in Q'$ be one of those points with additionally $\zeta_k(x,t)\to\zeta(x,t)\neq 0$. We now will show, that the sequence $\{(Du_{(\Delta t)_k})(x,t)\}_{k\in\mathbb{N}}$ is bounded and thus possesses a limit point. Suppose for contradiction that there exists a subsequence (which we do not relabel) such that

$$|(Du_{(\Delta t)_k})(x,t)|\to\infty.$$

With the coercivity (1.10) and the growth condition (1.8) of S, we estimate for every $k\in\mathbb{N}$ omitting the argument (x,t)

$$\begin{aligned}
F_{(\Delta t)_k}&=\zeta_k\Big(\big(S(Du_{(\Delta t)_k}):Du_{(\Delta t)_k}\big)-\big(S(Du):Du_{(\Delta t)_k}\big)\\
&\quad-\big(S(Du_{(\Delta t)_k}):Du\big)+\big(S(Du):Du\big)\Big)\\
&\geq\zeta_k\Big(c_0|Du_{(\Delta t)_k}|^p-c\,(1+|Du|)^{p-1}|Du_{(\Delta t)_k}|\\
&\quad-c(1+|Du_{(\Delta t)_k}|)^{p-1}|Du|\big)+0\Big)\\
&\geq\zeta_k\Big(c_0|Du_{(\Delta t)_k}|^p-\tilde{c}\big(1+|Du_{(\Delta t)_k}|+|Du_{(\Delta t)_k}|^{p-1}\big)\Big),
\end{aligned}$$

where $\tilde{c}$ depends on $|Du(x,t)|$. This term tends to infinity, because $p>1$ and $\zeta(x,t)\neq 0$. This is a contradiction to $F_{(\Delta t)_k}(x,t)\to 0$ and hence there exists some limit point $\tilde{D}(x,t)<\infty$ and a corresponding (again not relabeled) subsequence converging to it.

Now the continuity of S and $F_{(\Delta t)_k}(x,t)\to 0$ give

$$\big(S(Du(x,t))-S(\tilde{D}(x,t))\big):(Du(x,t)-\tilde{D}(x,t))\zeta(x,t)=0.$$

The strict monotonicity (1.9) of S then implies $\tilde{D}(x,t)=(Du)(x,t)$ and thus $(Du_{(\Delta t)_k})(x,t)\to(Du)(x,t)$ for the chosen subsequence. Since the limit is unique, the usual argumentation by contradiction shows that this result already holds for the whole sequence. □

Lemma 8.20. *For the weak limit* $a \in L^{p'}(0,T;V_p')$ *of* $\{Au_{\Delta t}\}$ *from* (8.10)*, we have*

$$a = Au \quad \textit{in } L^{p'}(0,T;V_p').$$

Proof. Since ζ was chosen arbitrarily with compact support K in Q', we can choose for every $j \in \mathbb{N}$ a set $K_j \subset\subset Q'$ and a function $\zeta_j \in C_0^\infty(Q')$ with compact support in Q', which does not vanish on any point of K_j, such that

$$Q' = \bigcup_{l=1}^{\infty} \bigcap_{j=l}^{\infty} K_j,$$

i.e. every point of Q' lies in almost every K_j. For every of those functions ζ_j, we can proceed as for ζ in Lemmas 8.15, 8.17, 8.18 and 8.19 to receive a subsequence $\{(\Delta t)_{k,j}\}_{k\in\mathbb{N}}$ for which

$$Du_{(\Delta t)_{k,j}} \to Du, \quad \text{almost everywhere in } K_j,$$

as $k \to \infty$.

Choosing the diagonal sequence $\{(\Delta t)_{k,k}\}_{k\in\mathbb{N}}$ then yields

$$Du_{(\Delta t)_{k,k}} \to Du \quad \text{almost everywhere in } Q'.$$

Since the limit Du is uniquely determined (almost everywhere), the usual argumentation by contradiction shows the convergence of the original sequence itself.[1]

Since $G' \subset\subset \Omega$ was an arbitrary subdomain of Ω (strictly speaking, of an arbitrary subdomain G with C^2-boundary), this gives the convergence of $\{Du_{\Delta t}\}$ almost everywhere in $\Omega \times (0,T)$.

Finally, with the aid of [41, Lemma 1.3, pp. 12f.] follows

$$S(Du_{\Delta t}) \rightharpoonup S(Du) \quad \text{in } L^{p'}(\Omega \times (0,T))^{d\times d}$$

and hence

$$Au_{\Delta t} \rightharpoonup Au \quad \text{in } L^{p'}(0,T;V_p').$$

From (8.10) follows $a = Au$.

□

[1] Note that this "original sequence" itself was a subsequence, namely the one extracted in Lemma 8.11.

A

Auxiliary results

Throughout this section let $(X, \|\cdot\|_X)$ denote a reflexive Banach Space.

Theorem A.1 (Separation of variables in the weak formulation)**.** *The space of functions φ of the form*

$$\varphi(t) = \sum_{i=1}^{n} \varphi_i(t)\, x_i$$

for some $x_i \in X$, $n \in \mathbb{N}$ and $\varphi_i \in C_0^\infty(0,T)$ lies dense in $L^p(0,T;X)$.

Proof. Let $v \in L^p(0,T;X)$. By [25, Theorem 7.1.23 (ii)] the simple functions lie dense in $L^p(0,T;X)$. This means there exist some $x_i \in X$ and some measurable sets $A_i \subset (0,T)$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, such that

$$\left\| v - \sum_{i=1}^{n} \chi_{A_i} x_i \right\|_{L^p(0,T;X)} < \frac{\varepsilon}{2}.$$

Since $C_0^\infty(0,T)$ lies dense in $L^p(0,T)$, we can approximate the indicator functions χ_{A_i} by $\varphi_i \in C_0^\infty(0,T)$ such that

$$\|\chi_{A_i} - \varphi_i\|_{L^p(0,T)} \le \frac{\varepsilon}{2n\|x_i\|_X}.$$

Hence, we estimate

$$\begin{aligned}
\left\| v - \sum_{i=1}^{n} \varphi_i x_i \right\|_{L^p(0,T;X)} &\le \left\| v - \sum_{i=1}^{n} \chi_{A_i} x_i \right\|_{L^p(0,T;X)} \\
&\quad + \left\| \sum_{i=1}^{n} \chi_{A_i} x_i - \sum_{i=1}^{n} \varphi_i x_i \right\|_{L^p(0,T;X)} \\
&\le \frac{\varepsilon}{2} + \sum_{i=1}^{n} \|\chi_{A_i} x_i - \varphi_i x_i\|_{L^p(0,T;X)} \\
&\le \frac{\varepsilon}{2} + \sum_{i=1}^{n} \|x_i\|_X \, \|\chi_{A_i} - \varphi_i\|_{L^p(0,T)} \\
&\le \frac{\varepsilon}{2} + \sum_{i=1}^{n} \|x_i\|_X \frac{\varepsilon}{2n\|x_i\|_X} = \varepsilon.
\end{aligned}$$

□

Corollary A.2. *Let $A, B \in L^{p'}(0,T;X')$. Then*

$$\langle A, \Phi\rangle = \langle B, \Phi\rangle$$

for all $\Phi \in L^p(0,T;X)$ holds if and only if

$$\int_0^T \langle A(t), x\rangle \varphi(t)\,\mathrm{d}t \;=\int_0^T \langle B(t), x\rangle \varphi(t)\,\mathrm{d}t \tag{A.1}$$

for all $x \in X$ and $\varphi \in C_0^\infty(0,T)$.

In other words: $A \in L^{p'}(0,T;X')$ is uniquely defined by testing with functions φx, for $x \in X$ and $\varphi \in C_0^\infty(0,T)$.

Proof. Of course, since $\varphi(t)\,x \in L^p(0,T;X)$ for $x \in X$ and $\varphi \in C_0^\infty(0,T)$, one direction of the proof is obvious.

On the other hand let (A.1) hold. By continuity and Theorem A.1 it is sufficient to test with functions of the form $\varphi(t) = \sum_{i=1}^n \varphi_i(t)\,x_i$ for $x \in X$ and $\varphi \in C_0^\infty(0,T)$. The proof is finished by calculating

$$\begin{aligned}
\left\langle A, \sum_{i=1}^n \varphi_i\,x_i \right\rangle &= \sum_{i=1}^n \langle A, \varphi_i\,x_i\rangle \\
&= \sum_{i=1}^n \int_0^T \langle A(t), x_i\rangle \varphi_i(t)\,\mathrm{d}t \\
&= \sum_{i=1}^n \int_0^T \langle B(t), x_i\rangle \varphi_i(t)\,\mathrm{d}t \\
&= \left\langle B, \sum_{i=1}^n \varphi_i\,x_i \right\rangle.
\end{aligned}$$

□

Lemma A.3 (Product rule)**.** *Let $u, u' \in L^1(0,T;X)$ and $\varphi \in C^1([0,T])$. then $(u\varphi) \in L^1(0,T;X)$ and the product rule*

$$(u\varphi)' = u'\varphi + u\varphi'$$

holds in the weak sense.

Proof. Clearly, $u\varphi, u'\varphi, u\varphi' \in L^1(0,T;X)$. For arbitrary $\psi \in C_0^\infty(0,T)$ we have $\varphi\psi \in C_0^\infty(0,T)$. The definition of the weak derivative of u implies

$$\begin{aligned}
&\int_0^T \big(u'(t)\varphi(t) + u(t)\varphi'(t)\big)\psi(t)\,\mathrm{d}t \\
&\quad= \int_0^T u'(t)(\varphi\psi)(t)\,\mathrm{d}t \;+ \int_0^T u(t)\varphi'(t)\psi(t)\,\mathrm{d}t \\
&\quad= -\int_0^T u(t)(\varphi\psi)'(t)\,\mathrm{d}t \;+ \int_0^T u(t)\varphi'(t)\psi(t)\,\mathrm{d}t \\
&\quad= -\int_0^T u(t)\big((\varphi\psi)'(t) - \varphi'(t)\psi(t)\big)\,\mathrm{d}t \\
&\quad= -\int_0^T u(t)\varphi(t)\psi'(t)\,\mathrm{d}t\,.
\end{aligned}$$

Hence $(u\varphi)' = u'\varphi + u\varphi'$. □

Lemma A.4 (Integration by parts)**.** *Let* $u, u' \in L^1(0,T;X')$ *and* $\varphi \in C^1([0,T])$. *Then the rule of integration by parts*

$$\langle u', v\varphi\rangle + \langle u, v\varphi'\rangle = \langle u(T), v\rangle\varphi(T) - \langle u(0), v\rangle\varphi(0)$$

holds for any $v \in X$.

Proof. The fundamental theorem of calculus (cf. [24, Theorem 8.1.5]) gives

$$u(T)\varphi(T) - u(0)\varphi(0) = \int_0^T (u\varphi)' \,\mathrm{d}t\,.$$

Lemma A.3 then implies

$$u(T)\varphi(T) - u(0)\varphi(0) = \int_0^T u'(t)\varphi(t)\,\mathrm{d}t\, + \int_0^T u(t)\varphi'(t)\,\mathrm{d}t\,.$$

Interchanging the integral and the dual pairing (cf. [24, Theorem 7.1.5]) yields

$$\begin{aligned}
\langle u(T), v\rangle\varphi(T) - \langle u(0), v\rangle\varphi(0) &= \langle u(T)\varphi(T), v\rangle - \langle u(0)\varphi(0), v\rangle \\
&= \left\langle \int_0^T u'(t)\varphi(t)\,\mathrm{d}t\,, v\right\rangle + \left\langle \int_0^T u(t)\varphi'(t)\,\mathrm{d}t\,, v\right\rangle \\
&= \int_0^T \langle u'(t), v\varphi(t)\rangle\,\mathrm{d}t\, + \int_0^T \langle u(t), v\varphi'(t)\rangle\,\mathrm{d}t \\
&= \langle u', v\varphi\rangle + \langle u, v\varphi'\rangle.
\end{aligned}$$

□

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