HIGHLY SYMMETRIC FUNDAMENTAL DOMAINS FOR LATTICES IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

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ABSTRACT. It is shown that most lattices Γ in $\mathbb{R}^2$ and $\mathbb{R}^3$ possess a fundamental domain F for the action of Γ on $\mathbb{R}^2$, respectively $\mathbb{R}^3$, having more symmetries than the point group $P(\Gamma)$ of Γ. In particular, $P(\Gamma)$ is a subgroup of the symmetry group $S(F)$ of F of index 2 in these cases. Exceptions are cubic lattices in the three-dimensional case, where such an F does not exist. Possible exceptions are rectangular lattices in the plane case, and orthorhombic lattices in $\mathbb{R}^3$, where we did not find a proof that the constructions presented here do work in all cases.

1. Introduction

This work is dedicated to the question "Given a lattice Γ in $\mathbb{R}^d$, how many symmetries can a fundamental domain for the action of Γ on $\mathbb{R}^d$ have?". In fact this work aims to extending the beautiful constructions of Veit Elser of exceptionally symmetric lattices [9] to all lattices in $\mathbb{R}^2$ and $\mathbb{R}^3$ and providing precise proofs. Let us first provide basic facts in order to state our main results.

A *lattice* in $\mathbb{R}^d$ is the $\mathbb{Z}$-span of d linearly independent vectors in $\mathbb{R}^d$. The point group $P(\Gamma)$ of a lattice Γ in $\mathbb{R}^d$ is the set of Euclidean isometries fixing both Γ and the origin. In other words, $P(\Gamma) \subset O(d)$ is the set of orthogonal maps fixing Γ. It is clear that each lattice Γ has a fundamental domain having $P(\Gamma)$ as its symmetry group, see Proposition 1.4. (For more detailed definitions see below.) For instance, consider the square lattice $\mathbb{Z}^2$ in the plane $\mathbb{R}^2$. Its point group $P(\mathbb{Z}^2)$ is the dihedral group D_4 of order eight, containing rotations by 0, $\pi/2$, π, $3\pi/2$, together with four reflections. One possible fundamental domain of $\mathbb{Z}^2$ is a unit square, centered at the origin. Clearly the symmetry group of this unit square is D_4 as well.

In this paper we show that most lattices in $\mathbb{R}^2$ and $\mathbb{R}^3$ possess fundamental domains with more symmetry than the point group of the lattice. In general, these fundamental domains will be neither simply connected, nor will their interiors be connected. Some of these domains are of fractal appearance. The two main results are the following.

Theorem 1.1. *Let $\Gamma \subset \mathbb{R}^2$ be a lattice with point group $P(\Gamma)$, such that Γ is not a rectangular lattice with irrational base length ratio. Then there is a compact fundamental domain F of Γ with symmetry group $S(F)$ such that $P(\Gamma)$ is a subgroup of $S(F)$ of index $[S(F) : P(\Gamma)] = 2$.*

Remark 1. Section 2 covers also compact fundamental domains F of rectangular lattices Γ such that $[S(F) : P(\Gamma)] = 2$. But if the ratio of the basis vectors is irrational we were not able to find a complete proof that this construction indeed yields the desired fundamental domain. Hence we must leave this point open. Since we consider the other results of decent interest we decided finally to submit this work nevertheless.

Theorem 1.2. *Let $\Gamma \subset \mathbb{R}^3$ be a lattice with point group $P(\Gamma)$, such that Γ is neither a cubic lattice nor a primitive orthorhombic lattice. Then there is a compact fundamental domain F of Γ with symmetry group $S(F)$ such that $P(\Gamma)$ is a subgroup of $S(F)$ of index $[S(F) : P(\Gamma)] = 2$.*

Remark 2. The exception of the cubic lattices in Theorem 1.2 has group theoretical reasons: there is no appropriate symmetry group in $\mathbb{R}^3$ with the required number of elements, hence a fundamental domain with the required properties does not exist.

The exception of the primitive orthorhombic lattice is due to the fact that Theorem 1.1 excludes rectangular lattices with irrational base length ratio. We will construct a fundamental domain with higher symmetries for the primitive orthorhombic lattice as well, and it will work fine if at least two lengths of the base vectors have rational ratio. But lacking a proof for the general rectangular case in $\mathbb{R}^2$ we cannot prove that the construction works for all possible orthorhombic lattices.

In the remainder of this section the necessary definitions and notations are introduced. Section 2 is dedicated to the proof of Theorem 1.1. Section 3 contains the proof of Theorem 1.2, which makes heavy use of Theorem 1.1. Section 4 contains some remarks and further questions.

Notation: We denote the cyclic group of order n by C_n, and the dihedral group of order $2n$ by D_n. The orthogonal group over $\mathbb{R}^d$ is denoted by $O(d)$. This group can be identified with the group of Euclidean isometries fixing the origin (Euclidean isometries are isometries of $\mathbb{R}^d$ including reflections). The closure of a set $A \subset \mathbb{R}^d$ is denoted by $\mathrm{cl}(A)$. For any set $X \subset \mathbb{R}^d$, let $S(X)$ denote the symmetry group of X, that is, the set of all Euclidean isometries (including reflections and translations) $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ with $\varphi(X) = X$. The *cone* centered at x spanned by m vectors $v_1, \dots v_m \in \mathbb{R}^d$ is defined by

$$\mathrm{cone}(x; v_1, \dots, v_m) = \{x + \sum_{i=1}^{m} \lambda_i v_i : \lambda_i \geq 0\}.$$

A sum $A + B$, where $A, B \subset \mathbb{R}^d$, always means the *Minkowski sum*

$$A + B = \{a + b \,|\, a \in A, b \in B\}.$$

The line segment with endpoints $x, y \in \mathbb{R}^d$ is denoted by $[x, y]$, while $[x, y] \setminus \{x, y\}$ is denoted (x, y). A *lattice* in $\mathbb{R}^d$ is a discrete cocompact subgroup of $\mathbb{R}^d$. Any lattice in $\mathbb{R}^d$ can be written as $\langle b_1, \dots, b_d \rangle_{\mathbb{Z}}$, where $b_1, \dots, b_d$ span $\mathbb{R}^d$. Such a set $\{b_1, \dots, b_d\}$ is called a *basis* of the lattice. A basis of a given lattice is not unique.

A fundamental domain for the action of a lattice Γ on $\mathbb{R}^d$ is a set F such that F contains exactly one representative for each element of $\mathbb{R}^d/\Gamma$. Here we want to consider nice geometric representations of fundamental domains. In particular we want to consider compact sets. Hence we allow a fundamental domain F to contain more than one representative for some element in $\mathbb{R}^d/\Gamma$ if these representatives are all contained in the boundary of F. For instance, a proper fundamental domain of the action of $\mathbb{Z}^d$ on $\mathbb{R}^d$ is the d-torus, and a fundamental domain in the sense of this paper of the action of the lattice $\mathbb{Z}^d$ on $\mathbb{R}^d$ is the d-dimensional unit cube $[0, 1]^d$. A particular fundamental domain of a lattice $\Gamma = \langle b_1, \dots, b_d \rangle_{\mathbb{Z}}$ is the *fundamental parallelepiped* $[0, b_1] + \ldots + [0, b_d]$. Note that for any fundamental domain F of a lattice Γ, $\{F + g \,|\, g \in \Gamma\}$ is a tiling of $\mathbb{R}^d$. A *tiling* of $\mathbb{R}^d$ is a packing of $\mathbb{R}^d$ which is also a covering of $\mathbb{R}^d$. In other words, a tiling is a covering of $\mathbb{R}^d$ by pairwise non-overlapping compact sets T_i. Two compact sets are *non-overlapping* if their interiors are disjoint.

Trivially, the symmetry group $S(\Gamma)$ of any lattice contains a subgroup isomorphic to Γ, namely, the group of all translations by elements of Γ. The subgroup $P(\Gamma) = S(\Gamma)/\Gamma$ is called *point group* of Γ. For lattices in $\mathbb{R}^d$, one has:

$$S(\Gamma) = P(\Gamma) \ltimes \Gamma.$$

The following fact is usually called the crystallographic restriction (see for instance [7], Section 4.5).

Proposition 1.3. *Rotations fixing a lattice in $\mathbb{R}^2$ or $\mathbb{R}^3$ are either 2-fold, 3-fold, 4-fold or 6-fold.*

In order to speak about "high symmetry" of a fundamental domain let us first make precise what its "usual symmetry" means. The following result is well-known, but we did not find a reference in the literature. Hence we provide a proof here.

Proposition 1.4. *If Γ is a lattice in $\mathbb{R}^d$ then Γ has a fundamental domain F such that $S(F) = P(\Gamma)$.*

Proof. Recall that the *Voronoi cell* of a lattice point x in $\mathbb{R}^d$ is the set of points in $\mathbb{R}^d$ whose distance to x is not greater than their distance to any other lattice point. It is easy to see that for any lattice $\Gamma \subset \mathbb{R}^d$ the closed Voronoi cell $V = V(0)$ of 0 is a fundamental domain of Γ with $P(\Gamma) \subseteq S(V)$: for each $\sigma \in S(\Gamma)$ holds by definition that $\sigma(\Gamma) = \Gamma$. Hence the Voronoi cell $V(0)$ in Γ equals the Voronoi cell $V'(0)$ in $\sigma(\Gamma)$, hence $\sigma\big(V(0)\big) = V(0)$.

Now, let $\sigma \in S(V)$. Let $v_1, \ldots, v_n$ be the vectors in Γ defining the $d-1$-faces $F_1, \ldots, F_n$ of V. That is: For $v \in \Gamma$, let π_v be the perpendicular bisector of $[0, v]$, and H_v be the half-space delineated by π_v containing 0. Then, $V = \bigcap_{v \in \Gamma \setminus \{0\}} H_v$. In particular, there exists a minimal set $v_1, \ldots, v_n \in \Gamma$ such that $V = \bigcap_{i=1}^{n} H_{v_i}$. Then the $d-1$-faces of V are $F_i = V \cap H_{v_i}$. Each H_{v_i} is called a *supporting hyperplane* of V.

We claim that $\langle v_1, \ldots, v_n \rangle_{\mathbb{Z}} = \Gamma$. First, we note that $\Gamma' = \langle v_1, \ldots, v_n \rangle_{\mathbb{Z}}$ is a lattice of full rank d: otherwise $v_1, \ldots, v_n$ are all contained in one hyperplane in $\mathbb{R}^d$ and $\bigcap_{i=1}^{n} H_{v_i}$ cannot be a fundamental domain for Γ. Moreover, Γ' cannot be a proper sublattice of Γ, because the Voronoi cell of 0 in Γ' must also be $\bigcap_{i=1}^{n} H_{v_i}$.

Since $\sigma(V) = V$, σ is a permutation the $d-1$-faces of V. Hence σ permutes the vectors v_i as well. Thus by linearity of σ, we get

$$\sigma(\Gamma) = \sigma(\langle v_1, \ldots, v_n \rangle_{\mathbb{Z}}) = \langle \sigma(v_1), \ldots, \sigma(v_n) \rangle_{\mathbb{Z}}) = \Gamma.$$

Hence $\sigma \in S(\Gamma)$. □

We will use orbifold notation to denote planar symmetry groups in the sequel, compare [3]. For instance, $*442$ denotes the symmetry group $S(\mathbb{Z}^2)$ of the square lattice $\mathbb{Z}^2$, and $*432$ denotes the symmetry group of the cube. For a translation of orbifold notation into your favorite notation, see [3] or [20]. In principle we can denote cyclic groups C_n and dihedral groups D_n in orbifold notation, too. Since the symbol for C_n—regarded as the symmetry group of some object in the plane—is just n in orbifold notation, we will rather use the former abbreviation for the sake of clarity.

2. **Dimension 2**

It is well known that each finite group of Euclidean isometries in the plane is either C_n or D_n. By the crystallographic restriction (Proposition 1.3) there are just 10 candidates for such groups being point groups of a planar lattice, namely

$$C_1, C_2, C_3, C_4, C_6, D_1, D_2, D_3, D_4, D_6.$$

Note that C_2 and D_1 are equal as abstract groups, since there is only one group of order two up to isomorphisms. But since we are dealing with groups of Euclidean isometries, we will use the convention that a cyclic group C_n contains rotations only, and a dihedral group D_n contains n rotations (including the identity) and n reflections. The fact that each planar lattice is fixed under a rotation through π about the origin implies that C_1, C_3, D_1 and D_3 cannot be point groups of any planar lattice. Some further thought yields the following result.

Proposition 2.1. *If Γ is a lattice in $\mathbb{R}^2$, then $P(\Gamma) \in \{C_2, D_2, D_4, D_6\}$, and $S(\Gamma) \in \{*632$, $*442$, $*2222$, $2*22$, $2222\}$.*

This result is well known. Nevertheless, since we are not aware of a decent reference, we will sketch the proof here.

Proof. We consider the distinct possibilities of properties of basis vectors of Γ. First, if Γ has a basis of two orthogonal vectors of equal length, this yields (up to similarity) the square lattice $\mathbb{Z}^2$, with point group D_4 and symmetry group $*442$. Second, if Γ has a basis of two vectors of equal length with angle $\pi/3$, this yields (up to similarity) the hexagonal lattice $A_2 = \langle (1,0)^T, (\frac{1}{2}, \frac{\sqrt{3}}{2})^T \rangle_{\mathbb{Z}}$, with point group D_6 and symmetry group $*632$. Third, if Γ has a basis of two vectors of equal length, but neither with angle $\pi/3$ nor $\pi/2$ nor $2\pi/3$, then Γ is called *rhombic lattice* and has point group D_2 and symmetry group $2*22$. A planar lattice which has orthogonal basis vectors of different length (but not of equal length) is called *rectangular lattice*. It has also point group D_2, but its symmetry group is $*2222$. In particular, the entire symmetry group of a rhombic lattice is not isomorphic to the entire symmetry group of a rectangular lattice, even though their point groups agree. (Compare [15], p 210.) All other lattices are called *oblique lattices* and have point group C_2, and symmetry group 2222. □

We will prove Theorem 1.1 by considering the five different types of lattices above. The case of rhombic lattices has been proven in [8] already:

Proposition 2.2 ([8])**.** *Let Γ be a rhombic lattice. Then there exists a fundamental domain F for Γ such that $S(F) \cong D_4$, that is,* $[\mathrm{S}(F) : P(\Gamma)] = 2$.

The remaining four cases are dealed with in this paper. The first two cases—the square lattice and the hexagonal lattice—are due to Veit Elser [9]. To the knowledge of the authors his proof has not been published anywhere, so we give a detailed proof here.

Proposition 2.3 (Elser)**.** *The square lattice $\mathbb{Z}^2$ has a fundamental domain $F_\square$ such that $S(F_\square) = D_8$.*

Proof. The point group of the square lattice $\mathbb{Z}^2$ is D_4. The claim is proved by constructing a fundamental domain $F_\square$ of $\mathbb{Z}^2$ with symmetry group D_8.

Let $\Gamma = \mathbb{Z}^2$. Consider a regular octagon of edge length $\ell = \sqrt{2} - 1$ oriented such that it has edges parallel to the coordinate axes. Let P_1 be the packing of the plane by copies of the octagon, with every point in Γ having one copy centered at it. See Figure 1 left. The packing looks like the Archimedean tiling 4.8^2 by octagons and squares, where the squares are the holes of the packing. By the choice of ℓ and the orientation of the octagons, a pair of octagons intersect if and only if their centers are one unit apart, and the two intersect only at a common edge. That is, the octagons are pairwise non-overlapping. For $n \geq 2$, let P_n be the packing by octagons having the same orientation as those in P_1 and of edge length ℓ^n, centered at each vertex of every octagon in P_{n-1}. Similarly, a pair of octagons in the nth step intersect if and only if their centers are consecutive vertices in some octagon in P_{n-1}, and the two intersect only at a common edge. In Figure 1, we have $\bigcup\limits_{k=1}^{n} P_k$ for $n \leq 4$.

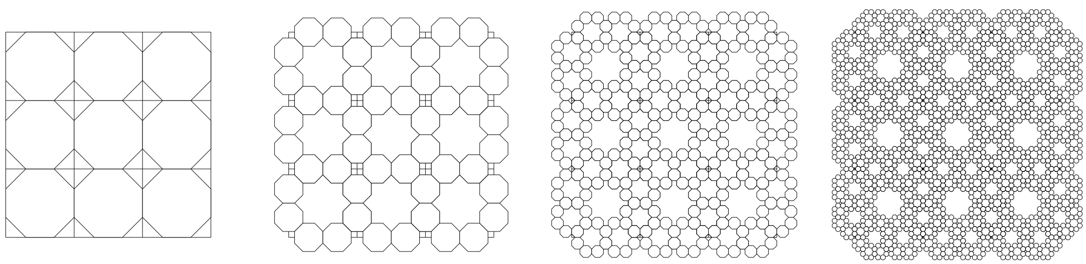

FIGURE 1. Filling the plane with octagons.

Define P_1^* as the octagon in P_1 that is centered at the origin. For $n \geq 2$, let P_n^* consist of the octagons of P_n centered at the vertices of all octagons in P_{n-1}^*. See Figure 2 for illustrations of $\bigcup\limits_{k=1}^{n} P_k^*$ for $n \leq 4$.

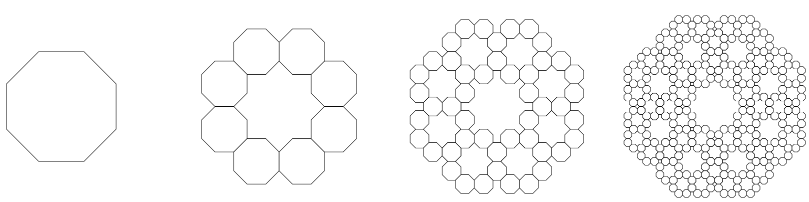

FIGURE 2. Octagons generated by the "central" octagon in P_1.

Clearly, $P_n^* + \Gamma = P_n$, and $\bigcup\limits_{k=1}^{n} P_k^* + \Gamma = \bigcup\limits_{k=1}^{n} P_k$. Observe as well that $S(P_n^*) = D_8$ for all n. We will construct for each n a compact subset F_n of P_n^* satisfying the following properties:

S1. For every n, $\mathrm{S}(F_n) = D_8$.

S2. For every n, $F_n + \Gamma = \bigcup\limits_{k=1}^{n} P_k$.

S3. For every n and any nontrivial $\mathbf{v} \in \Gamma$, $\mathrm{int}(F_n) \cap \mathrm{int}(\mathbf{v} + F_n) = \varnothing$.
S4. The sequence $\{F_n\}_{n=1}^{\infty}$ is Cauchy in $H(\mathbb{R}^2)$, the space of non-empty compact subsets of $\mathbb{R}^2$ equipped with the induced Hausdorff metric.

If there exist such F_n, let $F_\square = \lim\limits_{n\to\infty} F_n$, which is well-defined by completeness of $H(\mathbb{R}^2)$. By conditions S2 and S3, and the fact that $\mathrm{cl}\left(\bigcup\limits_{k=1}^{\infty} P_k\right) = \mathbb{R}^2$, $F_\square$ is a fundamental domain for Γ. Condition S1 and continuity of isometries in $H(X)$ imply that $\mathrm{S}(F_\square) = D_8$.

To this end, color P_1^* red and let $R_1 = P_1^*$ and $F_1 = R_1$. Then, F_1 satisfies S1, S2, and S3. At any succeeding step, an octagon O with center x will be colored purely red, purely white, or in the following manner: First, divide O into eight congruent slices, namely $S_i = O \cap \mathrm{cone}(x; e_i, e_{i+1})$ for $0 \le i \le 7$. Color S_k, where k is even, all red or all white, and color the rest of the slices oppositely. Thus, the slices are colored in alternating fashion. See Figure 3.

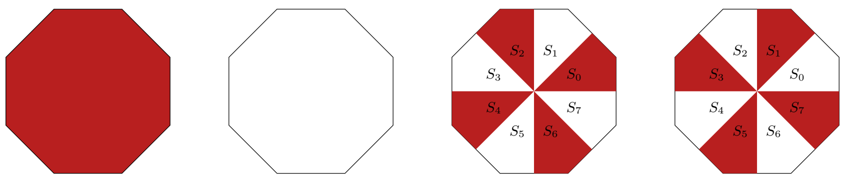

FIGURE 3. All possible colorings of any octagon at any step.

After coloring the octagons in step n according to the rule to be described below, call the union of the red pieces R_n, and define $F_n = \mathrm{cl}\left((F_{n-1} \setminus P_n^*) \cup R_n\right)$.

We now describe the coloring procedure for step n, where $n \ge 2$, with the assumption that all octagons in previous steps were colored in one of the four ways in Figure 3. If O is an octagon in P_n with center x, then x is the vertex of either one or two octagons in P_{n-1}^*, as in Figure 4.

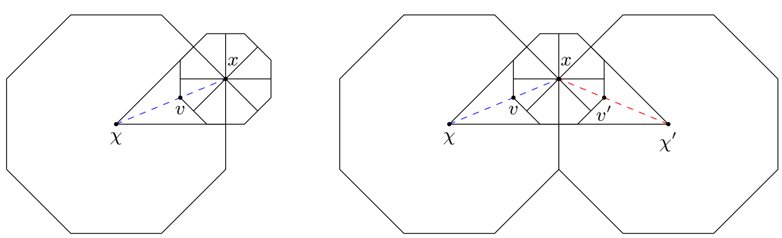

FIGURE 4. An octagon may be centered at an unshared or a shared vertex.

Suppose there is exactly one such octagon $\mathcal{O}$, with center χ. Now, there is a unique vertex v of O such that v lies on (x, χ). Let S be the slice of O containing v, that is, $v \in S := O \cap \mathrm{cone}(x; e_i, e_{i+1})$ for some unique $i \in \{0, 1, 2, \ldots, 7\}$. Then, because (x, χ) is contained in a slice of $\mathcal{O}$, either $(x, \chi) \subseteq F_{n-1}$ or (x, χ) is disjoint with F_{n-1}. Furthermore, $\mathrm{int}(O \setminus \mathcal{O})$ is contained in either $\mathbb{R}^2 \setminus \bigcup\limits_{k=1}^{n-1} P_k^*$, or in a portion of a slice of an octagon in some previous P_k^*, $k < n - 1$, that has not been re-colored after the kth step. In either case, either $\mathrm{int}(O \setminus \mathcal{O}) \subseteq F_{n-1}$ or $\mathrm{int}(O \setminus \mathcal{O})$ is disjoint with F_{n-1}.

We divide O into two subsets $\bigcup_{i=0}^{3} O \cap \mathrm{cone}(x; e_{2i}, e_{2i+1})$ and $\bigcup_{i=0}^{3} O \cap \mathrm{cone}(x; e_{2i+1}, e_{2i+2})$ with non-overlapping interiors. As mentioned previously, each of these will be colored purely red or purely white. If the subset containing (x, v) is denoted by $\mathcal{C}_v$, then the other subset is $\mathrm{cl}\,(O \setminus \mathcal{C}_v)$.

1. Color $\mathcal{C}_v$ red if and only if $(x, \chi) \subseteq F_{n-1}$.
2. Color $\mathrm{cl}\,(O \setminus \mathcal{C}_v)$ red if and only if $\mathrm{int}(O \setminus \mathcal{O}) \subseteq F_{n-1}$.

In other words, if (x, χ) and $\mathrm{int}(O \setminus \mathcal{O})$ have the same color after step $n-1$, then in step n we color O the same way. On the other hand, if (x, χ) and $\mathrm{int}(O \setminus \mathcal{O})$ are oppositely colored after step $n-1$, we color the slices of O alternately red and white such that S inherits the color of (x, χ).

Suppose there are two octagons in P^*_{n-1} having x as a vertex, say $\mathcal{O}$ and $\mathcal{O}'$ with centers χ and χ', respectively. Let v and v' be the vertices of O lying on (x, χ) and (x, χ'), respectively, and $\mathcal{C}_v$ and $\mathrm{cl}\,(O \setminus \mathcal{C}_v)$ be as before.

1. Color $\mathcal{C}_v$ red if and only if $(x, \chi) \subseteq F_{n-1}$.
2. Color $\mathrm{cl}\,(O \setminus \mathcal{C}_v)$ red if and only if $(x, \chi') \subseteq F_{n-1}$.

This is well-defined because v' is either in $O \cap \mathrm{cone}(x; e_{i+3}, e_{i+4})$ or $O \cap \mathrm{cone}(x; e_{i-3}, e_{i-2})$.

In order to demonstrate step 2, let O be any octagon in P^*_2. Refer to Figure 5. Then, its center is an unshared vertex, and $\mathcal{O} = F_1 = P^*_1$, and $\mathrm{int}(O \setminus \mathcal{O}) \subseteq \mathbb{R}^2 \setminus P^*_1$. Thus, the associated $(x, \chi) \subseteq F_1$ and $\mathrm{int}(O \setminus \mathcal{O})$ is disjoint with F_1. Therefore, we color $\mathcal{C}_v$ red and $\mathrm{cl}\,(O \setminus \mathcal{C}_v)$ white. Note that F_2 satisfies S1, S2, and S3.

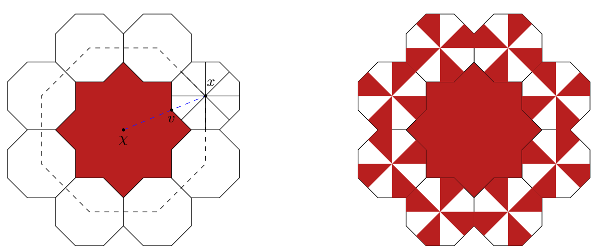

Figure 5. Second step of construction of a highly symmetric fundamental domain for $\mathbb{Z}^2$.

For $n \geq 3$, assume that F_{n-1} satisfies S1, S2, and S3. We prove that so does F_n. First, we show that it satisfies S1. Because $\mathrm{S}(F_{n-1}) = D_8$, then for any $\sigma \in D_8$ and octagon O in P^*_n, (x, χ) and $\sigma(x, \chi)$ have the same color in the preceding step. The same is true of $\mathrm{int}(O \setminus \mathcal{O})$ and $\sigma(\mathrm{int}(O \setminus \mathcal{O}))$ for unshared octagons, and of (x, χ') and $\sigma(x, \chi')$ for shared octagons. Thus, for any pair O and $\sigma(O)$, the octagons will be colored such that the red pieces are invariant under σ. From this, $\mathrm{S}(R_n) = D_8$ and it follows that $\mathrm{S}(F_n) = D_8$.

We now prove that F_n satisfies S2 and S3. Let O in P_n^* with center x and consider all $\mathbf{v}+O$, $\mathbf{v}\in\Gamma$, such that $\mathbf{v}+O$ is also in P_n^*. Then, exactly one of the following is true:

(1.) There exists a unique octagon $\mathcal{O}$ in P_{n-1}^* such that for all such $\mathbf{v}$, $\mathbf{v}+\mathcal{O}$ is the unique octagon in P_{n-1}^* having $\mathbf{v}+x$ as a vertex. For example, in Figure 6, for each $\mathbf{v}$ such that $\mathbf{v}+A$ is in P_3^*, the center of $\mathbf{v}+A$ is a vertex of $\mathbf{v}+\mathcal{O}$ and of no other octagon in P_2^*.

(2.) There exist exactly two octagons $\mathcal{O}$ and $\mathcal{O}'$ in P_{n-1}^* such that for all such $\mathbf{v}$, $\mathbf{v}+x$ is a vertex shared by $\mathbf{v}+\mathcal{O}$ and $\mathbf{v}+\mathcal{O}^*$ in P_{n-1}^*. In Figure 6, for each $\mathbf{v}$ such that $\mathbf{v}+B$ is in P_3^*, the center of $\mathbf{v}+B$ is a shared vertex of $\mathbf{v}+\mathcal{O}$ and $\mathbf{v}+\mathcal{O}_B$.

(3.) There exist octagons $\mathcal{O}$ and $\mathcal{O}'$ such that for all such $\mathbf{v}$, $\mathbf{v}+x$ is a vertex of $\mathbf{v}+\mathcal{O}$ or $\mathbf{v}+\mathcal{O}^*$ in P_{n-1}^*, but there exists $\mathbf{v}^*$ such that exactly one of $\mathbf{v}^*+\mathcal{O}$ and $\mathbf{v}^*+\mathcal{O}'$ is in P_{n-1}^*. In Figure 6, note that the center of C is shared by $\mathcal{O}$ and $\mathcal{O}_C$ but the center of $\binom{-1}{0}+C$ is an unshared vertex of $\binom{-1}{0}+\mathcal{O}$.

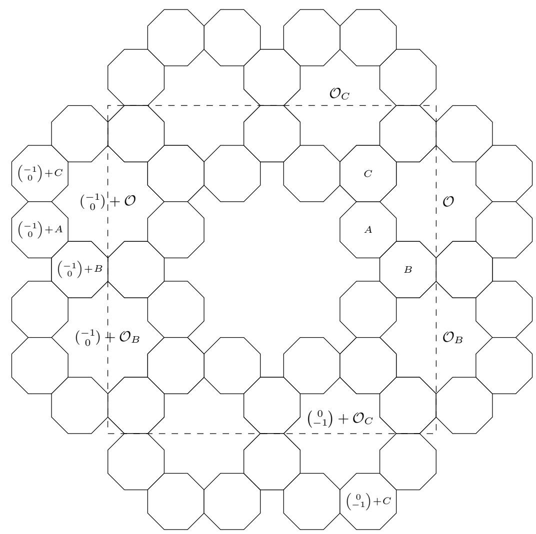

FIGURE 6. Selected octagons with their translates in P_3^*.

We consider each case.

(1.) By S2 and S3, each of the associated (x,χ) and $\operatorname{int}(O\setminus\mathcal{O})$ has exactly one translate that is red in the previous step. Thus, in the current step, each of $\mathcal{C}_v$ and $\operatorname{cl}(O\setminus\mathcal{C}_v)$ will have exactly one translate that will be colored red.

(2.) Similarly, each of the associated (x,χ) and (x,χ') has exactly one translate that is red in the previous step.

(3.) Consider $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $\mathbf{v}_1+\mathcal{O}$ and $\mathbf{v}_2+\mathcal{O}'$ are in P_{n-1}^*, as in Figure 7. Here, $\mathbf{v}_1+\mathcal{O}'$ and $\mathbf{v}_2+\mathcal{O}$ are not necessarily in P_{n-1}^*. Let χ and χ' be the centers of $\mathcal{O}$ and $\mathcal{O}'$, respectively. Let v and v' be the vertices of O such that $\mathbf{v}_1+v$ lies on $\mathbf{v}_1+(x,\chi)$ and $\mathbf{v}_2+v'$ lies on $\mathbf{v}_2+(x,\chi')$. We note that $\mathbf{v}_1+(x,v')\subseteq\mathbf{v_1}+\operatorname{int}(O\setminus\mathcal{O})$. Thus, either there is exactly one translate of $\mathcal{O}'$ such that the slice containing the corresponding translate of (x,v') is red, or there is exactly one translate of $\operatorname{int}(O\setminus\mathcal{O})$ that is contained in a red slice of some octagon in an earlier step. This implies that $\mathrm{cl}\,(O \setminus \mathcal{C}_v)$ will have exactly one translate that will be colored red. Similarly, exactly one translate of $\mathcal{C}_v$ will be colored red.

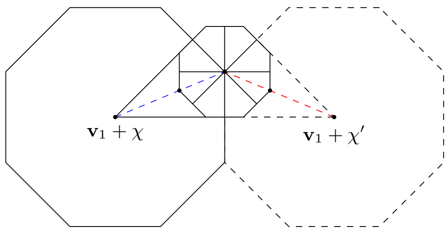

FIGURE 7. The case when a translate of an octagon in P^*_{n-1} is not necessarily in P^*_{n-1}.

This shows that for any octagon O in P^*_n with center x, exactly one translate of each of $\bigcup_{i=0}^{3} O \cap \mathrm{cone}(x; e_{2i}, e_{2i+1})$ and $\bigcup_{i=0}^{3} O \cap \mathrm{cone}(x; e_{2i+1}, e_{2i+2})$ will be colored red. Then, $R_n + \Gamma = P_n$ and $\mathrm{int}(R_n) \cap \mathrm{int}(\mathbf{v} + R_n) = \varnothing$ for any nontrivial $\mathbf{v}$. It follows that F_n satisfies S2 and S3. In Figure 8, we illustrate the third and fourth construction steps.

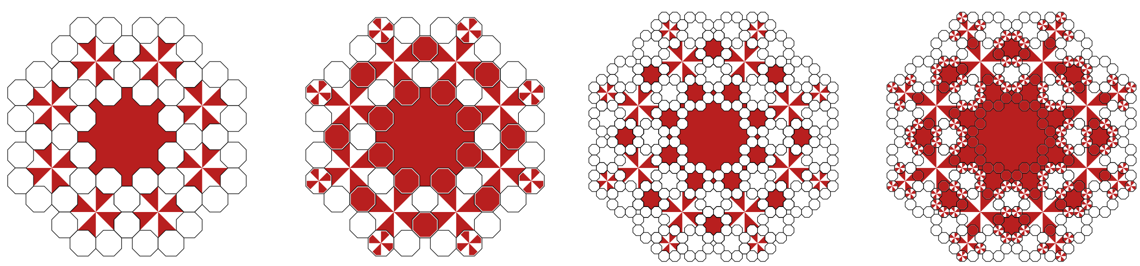

FIGURE 8. Third and fourth steps of construction of a highly symmetric fundamental domain for $\mathbb{Z}^2$.

We note that F_n is compact, as it is closed by definition, and $\mathrm{diam}(F_n) = \sum_{k=1}^{n} h\ell^{k-1}$, where $h = 2\sqrt{1 - \frac{\sqrt{2}}{2}}$. Furthermore, in the Hausdorff metric-equipped space $H(\mathbb{R}^2)$ of non-empty compact subsets of $\mathbb{R}^2$, the sequence $\{F_n\}_{n=1}^{\infty}$ is Cauchy, seeing that the distance between F_n and F_m for $n > m$ is $\sum_{k=m+1}^{n} h\ell^{k-1}$. Here, it is also important to note that for an octagon O with center x in P^*_n, by construction, $\bigcup_{i=n+1}^{\infty} P^*_i$ will not cover O, with $O \setminus \bigcup_{i=n+1}^{\infty} P^*_i$ being a connected set containing the ball centered at x with radius $1 - \frac{\sqrt{2}}{2}$ times that of O. This means that $O \setminus \bigcup_{i=n+1}^{\infty} P^*_i$ will not be affected by any succeeding step. Because this is true for any octagon at any stage, we find that in the limit, the union of the portion of each $O \setminus \bigcup_{i=n+1}^{\infty} P^*_i$

that overlaps with the unit square fundamental domain covers this unit square up to a set of measure zero. From this, the boundary of the limit $F_\square$ of $\{F_n\}_{n=1}^{\infty}$ in $H(\mathbb{R}^2)$ has measure zero.

We conclude that $F_\square$ is a fundamental domain for Γ with the desired symmetry group D_8. See Figure 9 for the tile F together with some of its translates. $\square$

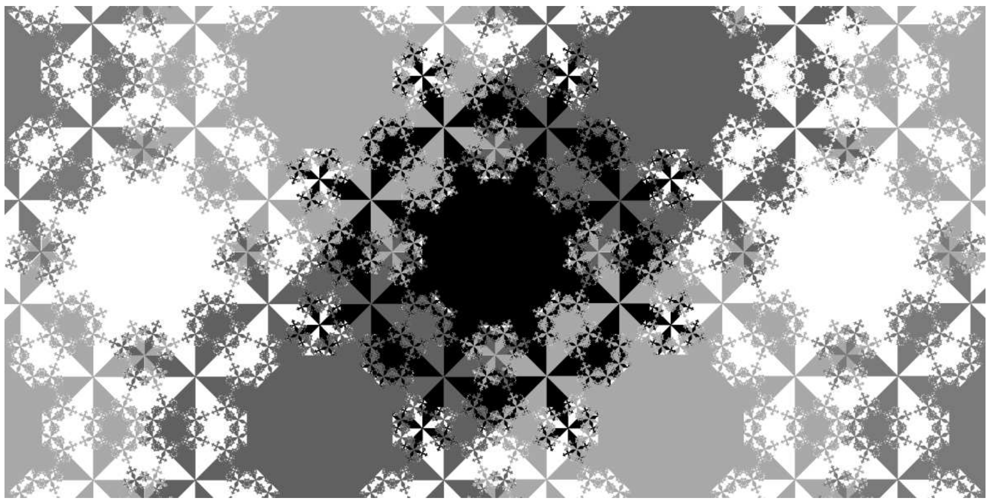

FIGURE 9. The fundamental domain $F_\square$ (black) of $\mathbb{Z}^2$ and some of its copies, illustrating how they form a tiling .

Proposition 2.4 (Elser)**.** *The hexagonal lattice A_2 has a compact fundamental domain $F_\triangle$ with $S(F_\triangle) = D_{12}$.*

Proof (sketch). The hexagonal lattice case is analogously treated. The plane is first packed by dodecagons inscribed in hexagonal fundamental domains for the hexagonal lattice A_2 defined by $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}\cos\frac{\pi}{3}\\ \sin\frac{\pi}{3}\end{pmatrix}$ (corresponding to the Archimedean tiling 3.12^2 by triangles and dodecagons). Further generations are obtained by placing dodecagons centered at the vertices of the dodecagons of the preceding generation. Here, the "holes" of the union of the first n packings are always equilateral triangles, and these also vanish eventually in the progression. A central dodecagon and the dodecagons arising from it are considered, and subsets are taken at each step analogously as in the coloring procedure described for the square lattice case. See Figure 10. $\square$

Interestingly, the set $F_\triangle$ appears in an entirely different context in [1] and [4], where it serves as a *window* respectively *atomic surface* for mathematical quasicrystals. The tiling property of $F_\triangle$ is not mentioned in these texts, and not obvious from the constructions used there. Let us now prove Theorem 1.1.

Proof (of Theorem 1.1). We consider the five cases where Γ is an oblique lattice, a rhombic lattice, a square lattice, a hexagonal lattice, or a rectangular lattice. The first four cases will be settled completely. In case of rectangular lattices we provide a construction that works

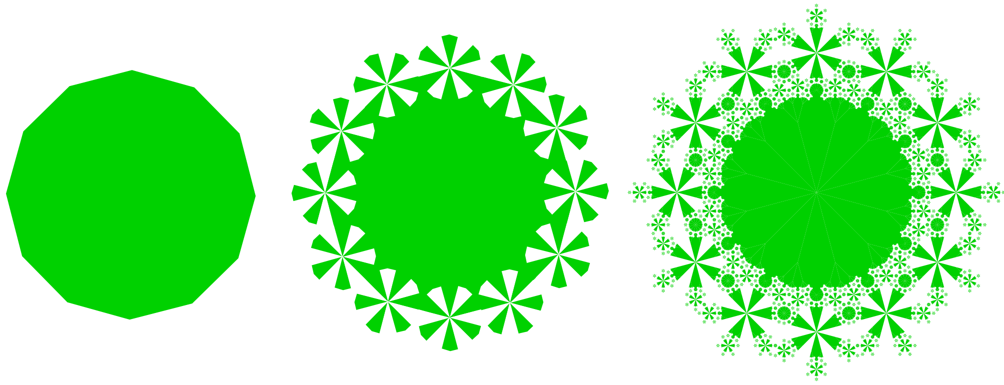

FIGURE 10. The first two iterates of the construction of the 12-fold fundamental domain $F_\triangle$ of the hexagonal lattice A_2 (left and middle), and a higher iterate (right).

for sure if the ratio of the lengths of the basis vectors of the rectangular lattice is a rational number. The construction might work in general, but there is a gap in the proof. This is explained in more detail in Remark 3 below.

Case 1: Let Γ be an oblique lattice. Without loss of generality one basis of Γ is $b_1 = \binom{x}{0}$, $b_2 = \binom{y}{z}$, $z \neq 0$. Then, let F be the rectangle with vertices $\binom{x/2}{z/2}$, $\binom{-x/2}{z/2}$, $\binom{-x/2}{-z/2}$, $\binom{x/2}{-z/2}$, see Figure 11.

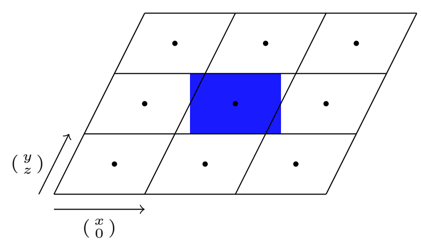

FIGURE 11. A fundamental domain for an oblique lattice, with D_2-symmetry.

It is easy to see that $\Gamma + F = \mathbb{R}^2$, and the copies of F do not overlap. Thus F is a fundamental domain of Γ. We have $P(\Gamma) = C_2$ and $S(F) = D_2$, unless $x = z$, in which case $S(F) = D_4$. In the last case, one can use a rectangle with one edge having the same length as and parallel to $\binom{y}{z}$ and suitable perpendicular edge length to obtain $S(F) = D_2$.

Case 2: Rhombic lattices. This is Proposition 2.2. We have $P(\Gamma) = D_2$ and $S(F) = D_4$.

Case 3: $\Gamma = \mathbb{Z}^2$: This is Proposition 2.3. We have $P(\Gamma) = D_4$ and $S(F_\square) = D_8$.

Case 4: $\Gamma = A_2$: This is Proposition 2.4. We have $P(\Gamma) = D_6$ and $S(F_\triangle) = D_{12}$.

Case 5: The algorithm to be presented for the rectangular lattice case is for the most part analogous to those of the square and hexagonal cases. The idea is to cover more and more of the plane such that squares of a current generation are centered at the vertices of those of the previous generation. The sequences of sizes of the squares remain nondecreasing, but this time the squares are allowed to be of the same size as those in the previous generation. Thus we distinguish between "iterations" and "steps". One step consists of one or more iterations where squares of the same size are used, and squares in a current step are smaller than those in the previous step.

Let Γ be a rectangular lattice with basis vectors $\binom{b}{0}$, $\binom{0}{a}$, with $b < a$. Let $r_0 = a$, $r_1 = b$, $v_1 = \left\lfloor \frac{r_0}{r_1} \right\rfloor$, $r_2 = r_0 - v_1 r_1$. The first step shall consist of v_1 iterations, and squares of edge length r_1 will be used. Let $P_{1,1}$ be the packing by squares of edge length b centered at each lattice point. If $v_1 = 1$, then the next packing belongs to the second step. Otherwise, for each k from 2 to v_1, let $P_{1,k}$ be the collection of squares of edge length r_1 centered at the vertices of the squares of $P_{1,k-1}$.

If $r_2 = 0$, then there are no further steps to consider. Otherwise, the portion of the plane that remains uncovered by the squares may be viewed as a union of $r_1/2 \times r_2/2$ rectangles. If $r_1 \times r_1$ squares are placed at the vertices of the current packing, then there would be nontrivial overlapping regions. Thus, smaller squares will be used.

In general, suppose $r_j \neq 0$, where $j \geq 2$, and the uncovered portions are $r_{j-1}/2 \times r_j/2$ rectangles. Let $v_j = \left\lfloor \frac{r_{j-1}}{r_j} \right\rfloor$, $r_{j+1} = r_{j-1} - v_j r_j$. For each k from 1 to v_j, let $P_{j,k}$ be the collection of squares of edge length r_j centered at the vertices of the squares in $P_{j,k-1}$, where we define $P_{j,0}$ naturally as $P_{j-1,v_{j-1}}$. After the jth step, the uncovered portion of the plane, if any, consists of $r_j/2 \times r_{j+1}/2$ rectangles. Note that the sequence of r_j's is the output of the Euclidean algorithm. Thus, $r_{n+1} = 0$ for some n if and only if $\frac{a}{b} \in \mathbb{Q}$, and the process terminates after step n in this case. In the case that $\frac{a}{b} \notin \mathbb{Q}$, the sequence of r_j's converges to 0, so that the process also produces squares that cover the plane in the limit.

As before, let $P^*_{1,1}$ be the square of edge length r_1 centered at the origin, and for each j, k, let $P^*_{j,k}$ consist of the squares of $P_{j,k}$ centered at the vertices of $P^*_{j,k-1}$. Again, $P^*_{j,k} + \Gamma = P_{j,k}$, $\bigcup\limits_j \bigcup\limits_{k=1}^{v_j} P^*_{j,k} + \Gamma = \bigcup\limits_j \bigcup\limits_{k=1}^{v_j} P_{j,k}$, and $\mathrm{S}(P^*_{j,k}) = D_4$ for all j, k. We will construct for each j, k a subset $F_{j,k}$ of $P^*_{j,k}$ satisfying the following properties:

- R1. For every j, k, $\mathrm{S}(F_{j,k}) = D_4$.
- R2. For every j, k, $F_{j,k} + \Gamma = \bigcup\limits_j \bigcup\limits_{k=1}^{v_j} P_{j,k}$.
- R3. For every j, k and any nontrivial $\mathbf{v} \in \Gamma$, $\mathrm{int}(F_{j,k}) \cap \mathrm{int}(\mathbf{v} + F_{j,k}) = \varnothing$.
- R4. The sequence $\{F_{j,k}\}$ is Cauchy in $H(\mathbb{R}^2)$.

Color $P^*_{1,1}$ red and let $R_{1,1} = P^*_{1,1}$ and $F_{1,1} = R_{1,1}$. This region satisfies R1, R2, and R3. Similarly, any square O with center x will be colored purely red, colored purely white, or divided into four congruent slices $S_i = O \cap \mathrm{cone}(x; e_i, e_{i+1})$ for $i = 0, 1, 2, 3$ and have two non-adjacent slices colored red. Here, $e_i = \begin{pmatrix} \cos \frac{\pi i}{2} \\ \sin \frac{\pi i}{2} \end{pmatrix}$. The union of the red pieces at the kth iteration of the jth step will be denoted by $R_{j,k}$, and $F_{j,k}$ is taken to be $\mathrm{cl}\left((F_{j,k-1} \setminus P^*_{j,k}) \cup R_{j,k}\right)$.

This time, if a square O in $P^*_{j,k}$ has center x, then x is the vertex of either one, two, or four squares in $P^*_{j,k-1}$. For the first two cases, we apply the natural analogues of the procedures for the square and hexagonal lattices. In the last case, for every vertex v of O and every square $\mathcal{O}$ having x as vertex, color the slice of O containing (x, v) according to the color of

(x, χ) in the previous iteration, where χ is the center of $\mathcal{O}$. In simpler terms, the slices of O retain the way they are colored previously.

The fact that $\mathrm{S}(F_{j,k}) = D_4$ for all j, k is proved similarly as in the previous cases. We now prove that $F_{j,k}$ satisfies R2 and R3. Let O be in $P^*_{j,k}$ with center x and consider all $\mathbf{v} \in \Gamma$ such that $\mathbf{v} + O$ is also in $P^*_{j,k}$. Then, exactly one of the following is true:

(1.) There exist either one, two, or four squares in $P_{j,k-1}$ such that for all such $\mathbf{v}$, $\mathbf{v} + x$ is a vertex of the translates by $\mathbf{v}$ of the one, two, or four squares, respectively, and only of these squares.
(2.) There exist squares $\mathcal{O}$ and $\mathcal{O}'$ such that for all such $\mathbf{v}$, $\mathbf{v}+x$ is a vertex of $\mathbf{v}+\mathcal{O}$ or $\mathbf{v}+\mathcal{O}'$ in $P^*_{j,k-1}$, but there exists $\mathbf{v}^*$ such that exactly one of $\mathbf{v}^* + \mathcal{O}$ and $\mathbf{v}^* + \mathcal{O}'$ is in $P^*_{j,k-1}$.
(3.) There exist four squares $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$ such that for all such $\mathbf{v}$, $\mathbf{v} + x$ is a vertex in $P^*_{j,k-1}$ of either the pair $\mathbf{v} + \mathcal{O}_1$ and $\mathbf{v} + \mathcal{O}_2$, or $\mathbf{v} + \mathcal{O}_3$ and $\mathbf{v} + \mathcal{O}_4$, or both pairs, but there exists $\mathbf{v}^*$ such that exactly one pair has a translate by $\mathbf{v}^*$ in $P^*_{j,k-1}$.

The first two cases are dealt with analogously as in the square and hexagonal cases. As for the third case, if $\mathbf{v} + x$ is a vertex of $\mathbf{v} + \mathcal{O}_3$ and $\mathbf{v} + \mathcal{O}_4$ but not of $\mathbf{v} + \mathcal{O}_1$ and $\mathbf{v} + \mathcal{O}_2$, then $\mathbf{v} + \mathcal{O}_1$ and $\mathbf{v} + \mathcal{O}_2$ must be in an "extreme" portion of $P^*_{j,k-1}$, that is, in an extreme vertical or extreme horizontal position. Moreover, in $P^*_{j,k-1}$, either $k - 1 \geq 2$ or $k - 1 = 0$ and $P^*_{j,k-1} = P^*_{j-1,v_{j-1}}$ where $v_{j-1} \geq 2$. That is, $P^*_{j,k}$ comes after an iteration that is at least the second in its step. And because $\mathbf{v} + x$ is a shared vertex, it is not a "corner" of $P^*_{j,k-1}$. Thus, $\mathbf{v} + O$ will be colored purely white because from the coloring procedure, among the squares in the extreme of $P^*_{j,k-1}$, only the squares centered at the corners of $P^*_{j,k-1}$ will have red portions. This means that the non-corner squares in the extreme may be ignored, and the case is reduced to the first case where each translate of O is shared by the corresponding translates of four squares. See for example Figure 12.

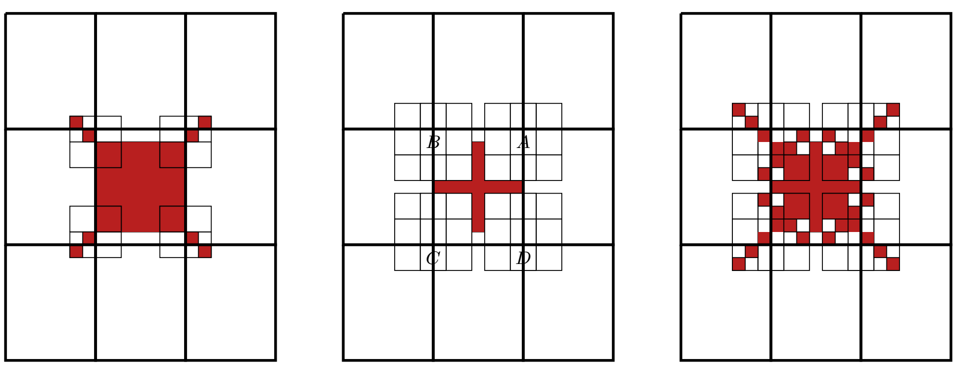

Figure 12. Squares A, B, C, and D are equivalent under the action of Γ, with A and B shared by four squares and C and D shared by two squares.

We note that the limit F is compact even if $\frac{a}{b}$ is irrational, as the horizontal length of F is $\sum_{j=1}^{\infty} v_j r_j = \sum_{j=1}^{\infty} r_{j-1} - r_{j+1} = \lim_{j\to\infty} (r_0 + r_1 - r_j - r_{j+1}) = (a + b)$. If $\frac{a}{b}$ is rational, there exists a minimal m such that $r_{m+1} = 0$, and the horizontal length of F is $\sum_{j=1}^{m} v_j r_j = a + b - r_m$.

We thus conclude as in the square and hexagonal cases. Shown in Figure 13 are the steps in constructing a fundamental domain with the desired symmetry group for a particular rectangular lattice. Figure 14 illustrates the tiling induced by the fundamental domain. □

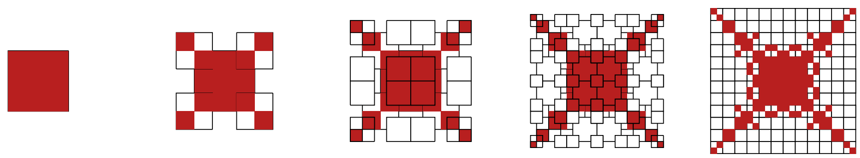

FIGURE 13. Construction of a D_4-symmetric fundamental domain for a rectangular lattice with $a = 8$, $b = 5$.

Remark 3. The gap in the proof above is that we cannot guarantee R3 to be fulfilled if the construction requires infinitely many steps. It is known that the limit of a sequence of compact sets F_n all of area 1 can have area larger than 1. See Figure 15 for an example: Each iterate has area 1, but the limit in the above sense has area 3. We are able to exclude this phenomenon for the square case and the hexagonal case since we can quantify the amount of area added in each step of the construction (see the last part of the proof of Proposition 2.3). We do not see how we can achieve this in the rectangular case in general.

3. **Dimension 3**

Similar to the proof of Theorem 1.1, the proof of Theorem 1.2 consists of considering all possible cases. Fortunately, we can utilize Theorem 1.1 to cover most cases: Create a right prism having one of the two-dimensional fundamental domains as base and with an appropriately chosen height. Then stack copies of this prism such that there is one such solid centered at each three-dimensional lattice point.

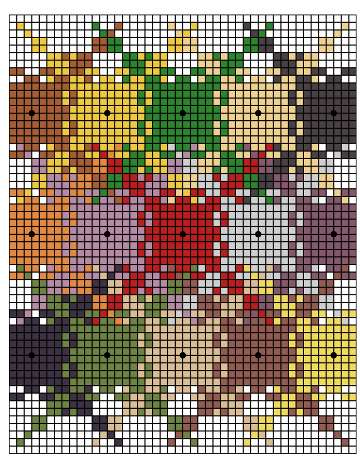

FIGURE 14. Tiling by a D_4-symmetric fundamental domain for a rectangular lattice.

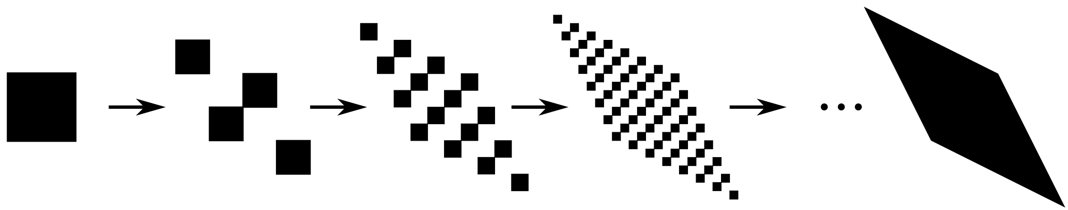

FIGURE 15. A sequence of compact sets of area 1 each, where the (compact) limit is of area 3.

In $\mathbb{R}^3$ there are 32 finite groups of Euclidean isometries obeying the crystallographic restriction in Proposition 1.3, see [7], Section 15.6. Only seven of them occur as point groups of lattices. Table 1 summarizes the situation. The second column contains the name of the lattice, more precisely: the name of the family of lattices with a common symmetry group (the names as being used in crystallography). The third column contains the point group of the lattice in orbifold notation, the fourth column contains the order of the point group. The last column indicates the two-dimensional fundamental domain of Theorem 1.1 which yields a three-dimensional fundamental domain F for the current three-dimensional lattice, and the order $|S(F)|$ in parentheses.

Since the list of finite groups of Euclidean isometries in $\mathbb{R}^3$ is known, we know that there is no such group containing the group $*432$ as a subgroup of index 2. (The only candidates—the ones of order 96—are the groups C_{96}, D_{48} and $C_2 \times D_{24}$, regarded as symmetry groups of solids in $\mathbb{R}^3$.) The corresponding lattices are the so-called *cubic lattices*: the *primitive cubic lattice*

Nr	Name	Point group	Order	2-dim fundamental domain (number of symmetries $\lvert S(F)\rvert$)
1	$\mathbb{Z}^3$	$*432$	48	—
2	body centered cubic	$*432$	48	—
3	face centered cubic	$*432$	48	—
4	Hexagonal	$*622$	24	12fold (48)
5	Tetragonal primitive	$*422$	16	8fold (32)
6	Tetragonal body-centered	$*422$	16	8fold (32)
7	Rhombohedral	$2*3$	12	6fold (24) / 12fold (48)
8	Orthorhombic primitive	$*222$	8	- / 4fold rectangular* (16)
9	Orthorhombic base-centered	$*222$	8	4fold rhombic (16)
10	Orthorhombic body-centered	$*222$	8	4fold rhombic (16)
11	Orthorhombic face-centered	$*222$	8	4fold rhombic (16)
12	Monoclinic primitive	$2*$	4	2fold (8)
13	Monoclinic base-centered	$2*$	4	2fold (8) / 4fold rhombic (16)
14	Triclinic primitive	2	2	mon.(4) / 2fold (8)

TABLE 1. The 14 Bravais types of lattices, their point groups, the order of the point groups, and the two-dimensionsal fundamental domain used to achieve a higher order (the order given in brackets). The $*$ indicates that this works only if one of $\frac{a}{b}, \frac{a}{c}, \frac{b}{c}$ is rational.

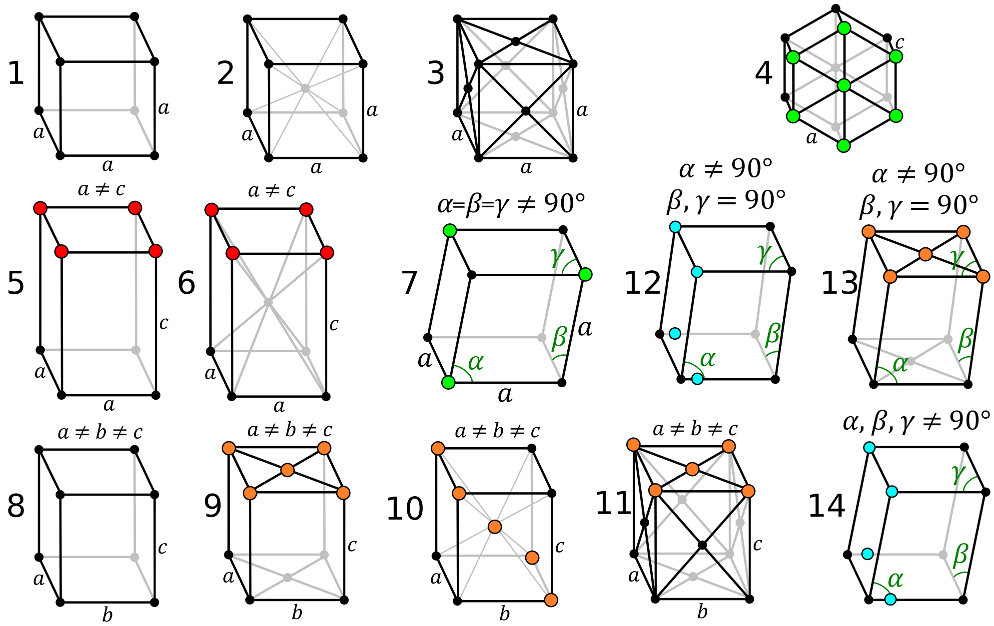

FIGURE 16. Illustrations of the 14 Bravais types of lattices in $\mathbb{R}^3$. The shaded nodes indicate how the lattices consist of layers of two-dimensional lattices. Angles omitted in the figure are assumed to be $\pi/2$ (or $\pi/3$ in 4). Edges labelled with equal letters are of equal length. The image is taken from [19] and only slightly modified.

$\mathbb{Z}^3$, the *body centered cubic lattice* $\mathbb{Z}^3 \cup \left(\mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T\right)$ (bcc) and the *face centered cubic lattice* (fcc). So we cannot expect to find fundamental domains for these three cubic lattices possessing a symmetry group that contains their point group $*432$ as a proper subgroup of finite index.

Proof (of Theorem 1.2). We consider 6 cases (numbers 4-14 in Table 1 and Figure 16, identified if they have equal point groups). This will yield the entries of the last column of Table 1, which shows the name of the two-dimensional fundamental domain used, and the order of the symmetry group of the corresponding three-dimensional fundamental domain (in parentheses).

Case 1: Hexagonal (4). The lattice consists of equidistant layers of hexagonal lattices. Attaching a thickened version of the fundamental domain $F_\triangle$—say, $F := F_\triangle \times [-\ell/2, \ell/2]$, where ℓ is the distance between two adjacent layers—to each lattice point yields a tiling of $\mathbb{R}^3$. The symmetry group $S(F)$ of F is $*12\ 2\ 2$, the new symmetries coming from rotating F along an axis which is parallel to the layers of hexagonal lattices about π ("turning F upside down").

Case 2: Tetragonal (5-6). These lattices consist of equidistant layers of square lattices. Thus we can use the thickened version of the fundamental domain $F_\square$ of the square lattice, its symmetry being $*822$, having order 32.

Case 3: Rhombohedral (7). This lattice consists of equidistant layers of the hexagonal lattice A_2. So we can either use a thickened fundamental domain of A_2 with D_6-symmetry, yielding a three-dimensional fundamental domain F with $S(F) = *622$, $|S(F)| = 24$ and index $[S(F) : P(\Gamma)] = 2$. Or we can use a thickened version of $F_\triangle$ as in case 1, yielding a fundamental domain with $S(F) = *12\ 2\ 2$, $|S(F)| = 48$ and index $[S(F) : P(\Gamma)] = 4$.

Case 4: Orthorhombic (8-11). Cases 9-11 consist of equidistant layers of rhombic lattices. Thus we can use the thickened version of the fundamental domain of the rhombic lattice, its symmetry group being $*422$ of order 16. The rhombic lattices are indicated in Figure 16 by shaded points.

For Case 8 we can use thickened versions of the fundamental domain of the rectangular lattice, if one of a/b, a/c, or b/c is rational.

Case 5: Monoclinic (12-13). These lattices consist of equidistant layers of oblique lattices. We can use rectangular cuboids as fundamental domains, having symmetry group $*222$ of order 8.

Case 13 consists of equidistant layers of rhombic lattices. Thus here we can reason as in the preceding case, obtaining a fundamental domain F with $S(F) = 16$.

Case 6: Triclinic (14). This lattice—or rather: these lattices—consist of layers of oblique lattices. We may use a right prism over a parallelogram as a fundamental domain. It has symmetry group $2*$ of order 4. Or we may even use a cuboid (erected on the rectangles of Figure 11). This yields a fundamental domain with symmetry group $*222$ of order 8. $\square$

4. **Conclusions and Outlook**

The above results motivate several further questions. We list below a few that, to our knowledge, are completely open.

4.1. **Even more symmetry.** Are there fundamental domains F with $[S(F) : P(G)] > 2$? We have found a few: In the case of oblique plane lattices the rectangular fundamental F might be a square. (This happens if $x = z$ in Figure 11.) In this case we obtain $[S(F) : P(G)] = 4$. In the case of the triclinic primitive lattice there is always a cuboidal fundamental domain with $[S(F) : P(G)] = 4$. In analogy to the oblique lattices in the plane, this cuboid might be a square prism or even a cube in some particular cases. This would yield $[S(F) : P(G)] = 8$ or $[S(F) : P(G)] = 24$, respectively. What is the maximal value of the index $[S(F) : P(G)]$ in $\mathbb{R}^d$ ($d \geq 2$)? Is the maximal index always obtained by the lattices with the smallest point group?

4.2. **Higher Dimensions.** The results in the present paper have been obtained by considering all different classes of lattices with respect to their symmetry group. There are 5 such classes in $\mathbb{R}^2$, 14 such classes in $\mathbb{R}^3$, 64 such classes in $\mathbb{R}^4$, 189 such classes in $\mathbb{R}^5$ and 826 such classes in $\mathbb{R}^6$ [2, 10, 14, 17, 18]. At some point it seems desirable to find more general arguments than case-by-case considerations. However, it is very likely that in higher dimensions there are several lattices Γ with fundamental domains F such that $P(\Gamma)$ is a proper subgroup of $S(F)$.

4.3. **Fractal Dimension.** The fundamental domains $F_\square$ of the square lattice, $F_\triangle$ of the hexagonal lattice and those of the rectangular lattices with incommensurate basis lengths are of fractal appearance. It might be possible to compute the Hausdorff dimensions of the boundaries of these fundamental domains, as well as other fractal dimensions, like the box-counting dimension or the affinity dimension [11], see also [16] and references therein. The two latter dimensions are particularly easy to compute if one finds an iterated function system (IFS) generating the fractal under consideration, see [16]. Up to the knowledge of the authors, no IFS for $F_\square$ or $F_\triangle$ or the fundamental domains of rectangular lattices are known yet.

4.4. **Alternative Constructions.** The constructions used in this paper can be altered in many ways. For instance, there are other ways to partition the octagons, dodecagons and squares into two regions of different colours than the one used in the proof of Theorem 1.1. All that is required is to keep the mirror symmetry of the partition, and take care that no overlaps occur. One possibility is just to interchange the colours in the polygons of mixed colour.

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