Bounded distance equivalence of cut-and-project sets

Dirk Frettlöh

Technische Fakultät
Universität Bielefeld

The 13th International Conference on Discrete Mathematics:
Discrete Geometry and Convex Bodies

Bucharest 4. September 2017

joint work with Alexey Garber (UTRGV Brownsville, Texas)

- Basics
- Dimension 1
- Higher dimensions
- New result

Delone set: point set Λ in $\mathbb{R}^d$, with $R > r > 0$ such that

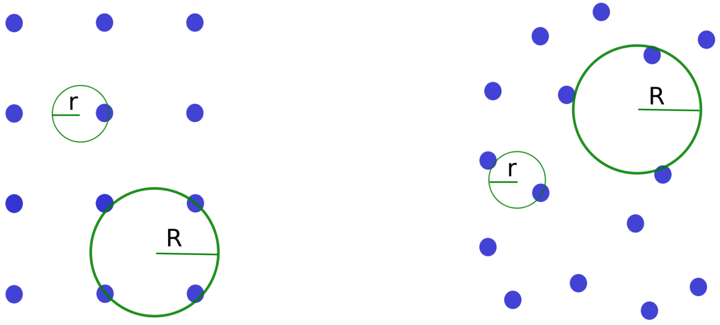
- each ball of radius r contains at most one point of Λ (*uniformly discrete*)
- each ball of radius R contains at least one point of Λ (*relatively dense*)

(Aka "separated nets". Can also live in $\mathbb{H}^d$, $(\mathbb{Q}_p)^d$...)

Delone set: point set Λ in $\mathbb{R}^d$, with $R > r > 0$ such that

- each ball of radius r contains at most one point of Λ (*uniformly discrete*)
- each ball of radius R contains at least one point of Λ (*relatively dense*)

(Aka "separated nets". Can also live in $\mathbb{H}^d$, $(\mathbb{Q}_p)^d$...)



crystallographic disordered

Relation between Delone sets:

$\Lambda \stackrel{\text{bd}}{\sim} \Lambda'$ *(bounded distance equivalent)*:

There is $g : \Lambda \to \Lambda'$ bijective with

$$\exists C > 0 \quad \forall x \in \Lambda : \quad |x - g(x)| < C$$

Relation between Delone sets:

$\Lambda \overset{\mathrm{bd}}{\sim} \Lambda'$ *(bounded distance equivalent)*:

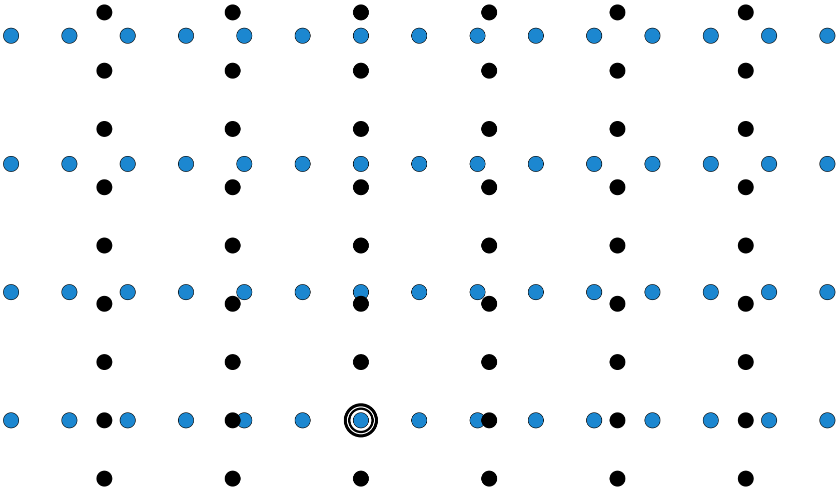
There is $g : \Lambda \to \Lambda'$ bijective with

$$\exists C > 0 \quad \forall x \in \Lambda : \quad |x - g(x)| < C$$

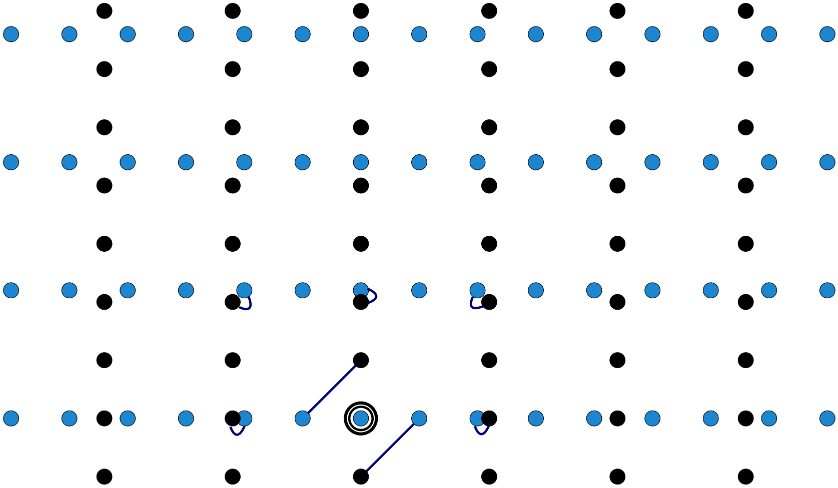
Lemma

Bounded distance equivalence is an equivalence relation.

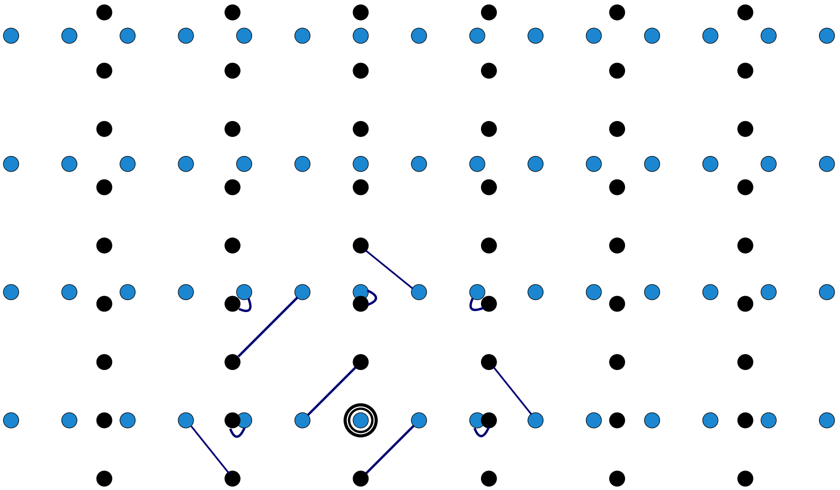
Example: Two rectangular lattices Λ, Λ'. Is $\Lambda \overset{\mathrm{bd}}{\sim} \Lambda'$?



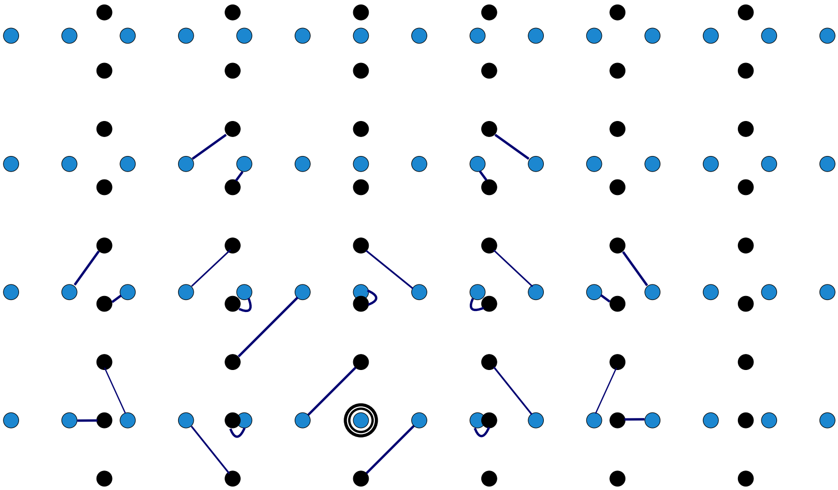
Example: Two rectangular lattices Λ, Λ'. Is $\Lambda \overset{\text{bd}}{\sim} \Lambda'$?



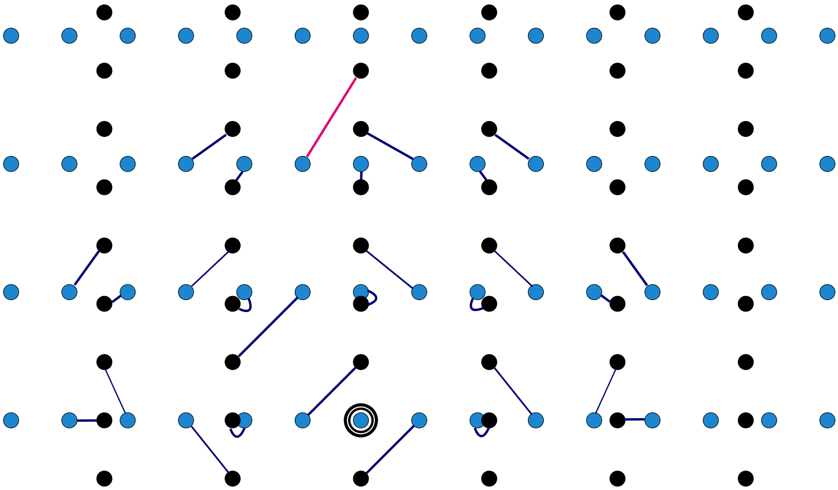
Example: Two rectangular lattices Λ, Λ'. Is $\Lambda \overset{\mathrm{bd}}{\sim} \Lambda'$?



Example: Two rectangular lattices Λ, Λ'. Is $\Lambda \overset{\text{bd}}{\sim} \Lambda'$?



Example: Two rectangular lattices Λ, Λ'. Is $\Lambda \overset{\mathrm{bd}}{\sim} \Lambda'$?



Dimension 1

Question: Let $\Lambda, \Lambda' \subset \mathbb{R}$. When is $\Lambda \stackrel{\text{bd}}{\sim} \Lambda'$? Always?

Dimension 1

Question: Let $\Lambda, \Lambda' \subset \mathbb{R}$. When is $\Lambda \overset{\text{bd}}{\sim} \Lambda'$? Always? No:

Examples:

- $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\} \overset{\text{bd}}{\not\sim} \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$
- $\{\ldots -3, -2, -1, 0, 2, 4, 6, \ldots\} \overset{\text{bd}}{\not\sim} \{\ldots -6, -4, -2, 0, 1, 2, 3, \ldots\}$

Dimension 1

Question: Let $\Lambda, \Lambda' \subset \mathbb{R}$. When is $\Lambda \stackrel{\text{bd}}{\sim} \Lambda'$? Always? No:

Examples:

- $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\} \stackrel{\text{bd}}{\not\sim} \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$
- $\{\ldots -3, -2, -1, 0, 2, 4, 6, \ldots\} \stackrel{\text{bd}}{\not\sim} \{\ldots -6, -4, -2, 0, 1, 2, 3, \ldots\}$

Density matters. Preliminary definition:

$$\text{dens}(\Lambda) := \lim_{r \to \infty} \frac{1}{2r} \#(\Lambda \cap [-r, r]),$$

if it exists.

Dimension 1

Question: Let $\Lambda, \Lambda' \subset \mathbb{R}$. When is $\Lambda \overset{\text{bd}}{\sim} \Lambda'$? Always? No:

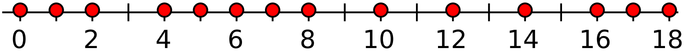
Examples:

- $\{\dots -3, -2, -1, 0, 1, 2, 3, \dots\} \overset{\text{bd}}{\not\sim} \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$
- $\{\dots -3, -2, -1, 0, 2, 4, 6, \dots\} \overset{\text{bd}}{\not\sim} \{\dots -6, -4, -2, 0, 1, 2, 3, \dots\}$

Density matters. Preliminary definition:

$$\text{dens}(\Lambda) := \lim_{r \to \infty} \frac{1}{2r} \#(\Lambda \cap [-r, r]),$$

if it exists. Does not need to exist:



Oscillates between $\frac{2}{3}$ and $\frac{5}{6}$.

Question: If dens(Λ)=dens(Λ'), is $\Lambda \overset{\mathrm{bd}}{\sim} \Lambda'$?

Question: If dens(Λ)=dens(Λ'), is $\Lambda \overset{\text{bd}}{\sim} \Lambda'$?

Theorem (Duneau-Oguey 1990)

Let Λ, Λ' be periodic. Then dens(Λ)=dens(Λ') implies $\Lambda \overset{\text{bd}}{\sim} \Lambda'$.
(True even in $\mathbb{R}^d$ for $d \geq 2$)

Interesting examples are non-periodic.

Question: If dens(Λ)=dens(Λ'), is $\Lambda \overset{\text{bd}}{\sim} \Lambda'$?

Theorem (Duneau-Oguey 1990)

Let Λ, Λ' be periodic. Then dens(Λ)=dens(Λ') implies $\Lambda \overset{\text{bd}}{\sim} \Lambda'$.
(True even in $\mathbb{R}^d$ for $d \geq 2$)

Interesting examples are non-periodic.

Theorem (Kesten 1966)

Let $\xi \in [0,1]$, $0 \leq a < b \leq 1$ and define

$$\Lambda := \{k \in \mathbb{Z} \mid a \leq \big(k\xi \bmod 1\big) < b\}.$$

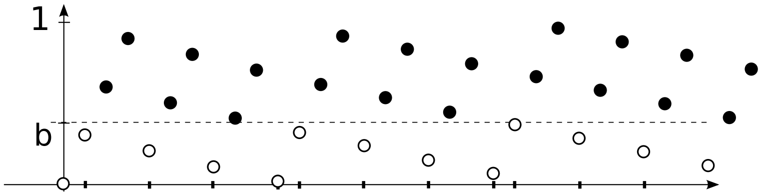
Then the deficiency $D(n) := \#(\Lambda \cap [1,n]) - n(b-a)$ is bounded, if and only if $b - a = k\xi \mod 1$ for some $k \in \mathbb{Z}$.

(if-part: Hecke 1921, Ostrowski 1927)

Choose $\xi \in [0,1]$ irrational, let $0 < b \leq 1$ and define

$$\Lambda_b := \{k \in \mathbb{Z} \mid 0 \leq \big(k\xi \mod 1\big) < b\}.$$

Then the deficiency $D(n) := \#(\Lambda \cap [1,n]) - nb$ is bounded, if and only if $b = k\xi \mod 1$ for some $k \in \mathbb{Z}$.

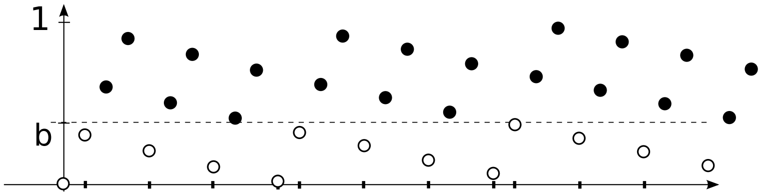


The image shows $\{(k, k\xi \text{ mod } 1) \mid k = 0, 1, 2, \ldots\}$.

Choose $\xi \in [0,1]$ irrational, let $0 < b \le 1$ and define

$$\Lambda_b := \{k \in \mathbb{Z} \mid 0 \le \big(k\xi \mod 1\big) < b\}.$$

Then the deficiency $D(n) := \#(\Lambda \cap [1,n]) - nb$ is bounded, if and only if $b = k\xi \mod 1$ for some $k \in \mathbb{Z}$.



The image shows $\{(k, k\xi \text{ mod } 1) \mid k = 0,1,2,\ldots\}$.

In particular:

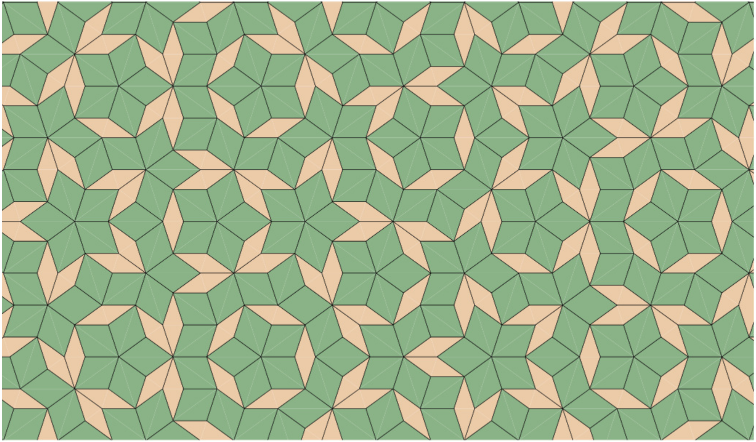
- Deficiency bounded $\Leftrightarrow \Lambda_b \overset{\text{bd}}{\sim} \frac{1}{b}\mathbb{Z}$,
- Any $b \neq k\xi$ mod 1 yields a (nonperiodic!) Delone set Λ_b such that $\Lambda_b \overset{\text{bd}}{\not\sim} c\mathbb{Z}$. Even when $\text{dens}(\Lambda_b)$ exists!

Higher Dimensions

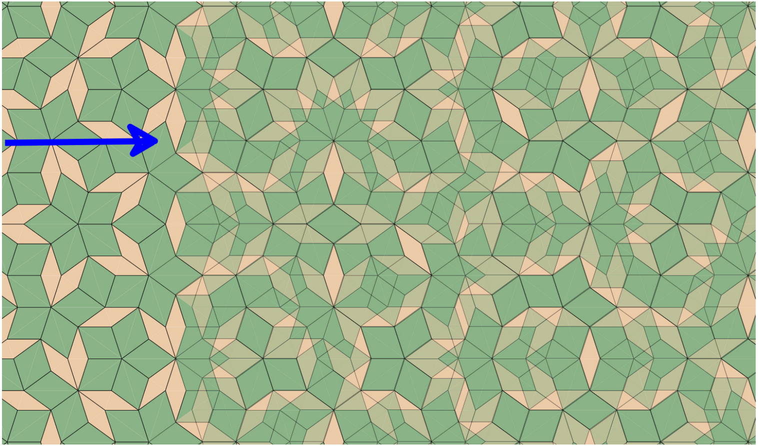
Cool! Alexey Garber and I started to study some problems in this field. E.g.

1. Are the vertices of the Penrose tiling bounded distance equivalent to some lattice?
2. Which cut-and-project sets are bounded distance equivalent to some lattice?
3. Which substitution tilings (resp. their vertex sets) are bounded distance equivalent to some lattice?

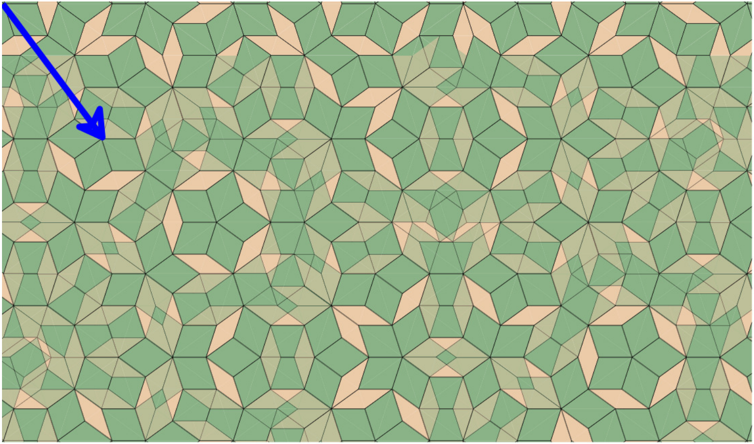
Recall: Interesting examples are non-periodic.
Like the Penrose tiling:



The Penrose tiling is indeed non-periodic:



The Penrose tiling is indeed non-periodic:



Theorem (F-Garber 2011 unpublished)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^2$, then $\Lambda \overset{\text{bil}}{\sim} \mathbb{Z}^2$.
(where $\overset{\text{bd}}{\sim}$ implies $\overset{\text{bil}}{\sim}$)

Theorem (F-Garber 2011 unpublished)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^2$, then $\Lambda \overset{\text{bil}}{\sim} \mathbb{Z}^2$.
(where $\overset{\text{bd}}{\sim}$ implies $\overset{\text{bil}}{\sim}$)

Theorem (Aliste-Prieto, Coronel, Gambaudo 2011)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^d$, then $\Lambda \overset{\text{bil}}{\sim} \mathbb{Z}^d$.

Theorem (F-Garber 2011 unpublished)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^2$, then $\Lambda \overset{\mathrm{bil}}{\sim} \mathbb{Z}^2$.
(where $\overset{\mathrm{bd}}{\sim}$ implies $\overset{\mathrm{bil}}{\sim}$)

Theorem (Aliste-Prieto, Coronel, Gambaudo 2011)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^d$, then $\Lambda \overset{\mathrm{bil}}{\sim} \mathbb{Z}^d$.

Corollary (F-Garber 2011 unpublished)

Let Λ_P be the vertices of the Penrose tiling. $\Lambda_P \overset{\mathrm{bil}}{\sim} \mathbb{Z}^2$.

Theorem (F-Garber 2011 unpublished)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^2$, then $\Lambda \overset{\text{bil}}{\sim} \mathbb{Z}^2$.
(where $\overset{\text{bd}}{\sim}$ implies $\overset{\text{bil}}{\sim}$)

Theorem (Aliste-Prieto, Coronel, Gambaudo 2011)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^d$, then $\Lambda \overset{\text{bil}}{\sim} \mathbb{Z}^d$.

Corollary (F-Garber 2011 unpublished)

Let Λ_P be the vertices of the Penrose tiling. $\Lambda_P \overset{\text{bil}}{\sim} \mathbb{Z}^2$.

Theorem (Solomon 2007)

$\Lambda_P \overset{\text{bd}}{\sim} c\mathbb{Z}^2$.

Theorem (F-Garber 2011 unpublished)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^2$, then $\Lambda \overset{\text{bil}}{\sim} \mathbb{Z}^2$.
(where $\overset{\text{bd}}{\sim}$ implies $\overset{\text{bil}}{\sim}$)

Theorem (Aliste-Prieto, Coronel, Gambaudo 2011)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^d$, then $\Lambda \overset{\text{bil}}{\sim} \mathbb{Z}^d$.

Corollary (F-Garber 2011 unpublished)

Let Λ_P be the vertices of the Penrose tiling. $\Lambda_P \overset{\text{bil}}{\sim} \mathbb{Z}^2$.

Theorem (Solomon 2007)

$\Lambda_P \overset{\text{bd}}{\sim} c\mathbb{Z}^2$.

Theorem (Deuber-Simonovits-Sós 1995)

$\Lambda_P \overset{\text{bd}}{\sim} c\mathbb{Z}^2$.

Theorem (F-Garber 2011 unpublished)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^2$, then $\Lambda \stackrel{\text{bil}}{\sim} \mathbb{Z}^2$. (where $\stackrel{\text{bd}}{\sim}$ implies $\stackrel{\text{bil}}{\sim}$)

Theorem (Aliste-Prieto, Coronel, Gambaudo 2011)

If Λ is a linearly repetitive Delone set in $\mathbb{R}^d$, then $\Lambda \stackrel{\text{bil}}{\sim} \mathbb{Z}^d$.

Corollary (F-Garber 2011 unpublished)

Let Λ_P be the vertices of the Penrose tiling. $\Lambda_P \stackrel{\text{bil}}{\sim} \mathbb{Z}^2$.

Theorem (Solomon 2007)

$\Lambda_P \stackrel{\text{bd}}{\sim} c\mathbb{Z}^2$.

Theorem (Deuber-Simonovits-Sós 1995)

$\Lambda_P \stackrel{\text{bd}}{\sim} c\mathbb{Z}^2$.

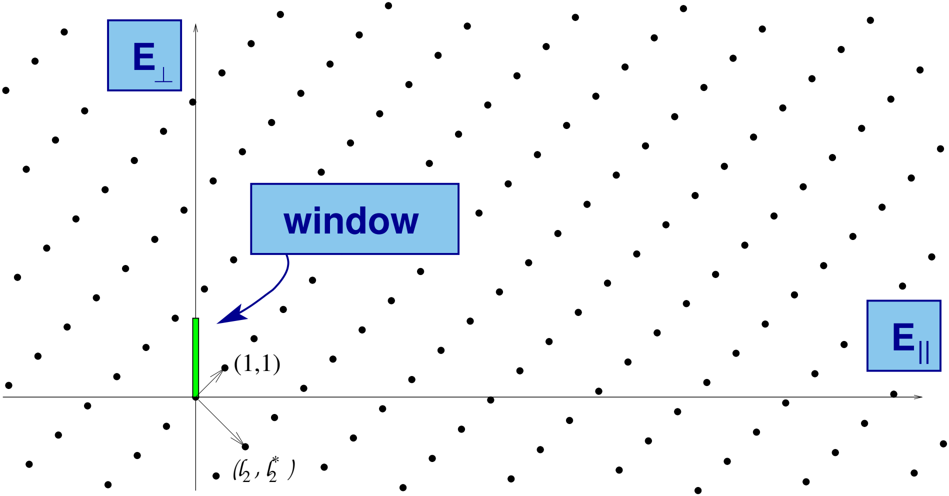
Well. Then let us generalise Kesten's Theorem to higher dimensions.

Cut-and-Project Sets

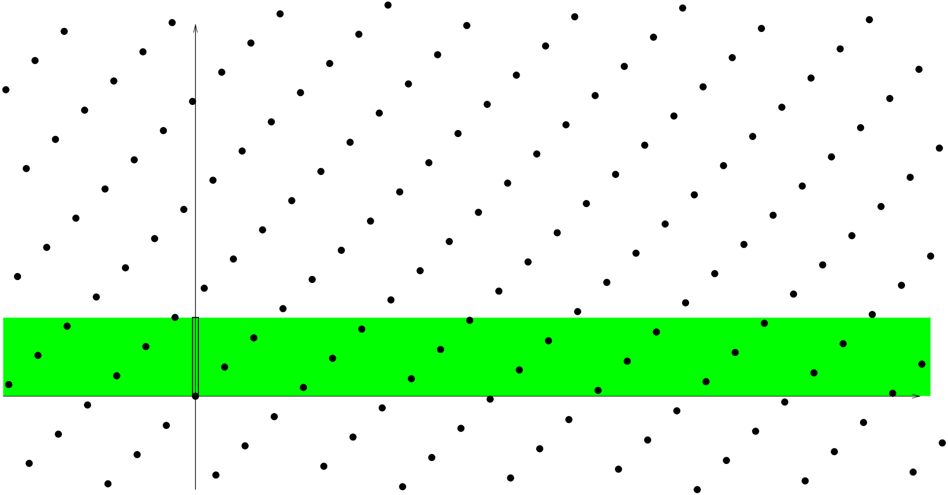
$$\begin{array}{ccccc} E_{||} = \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{R}^e & \xrightarrow{\pi_2} & \mathbb{R}^e = E_\perp \\ \cup & & \cup & & \cup \\ \Lambda & & \Gamma & & W \end{array}$$

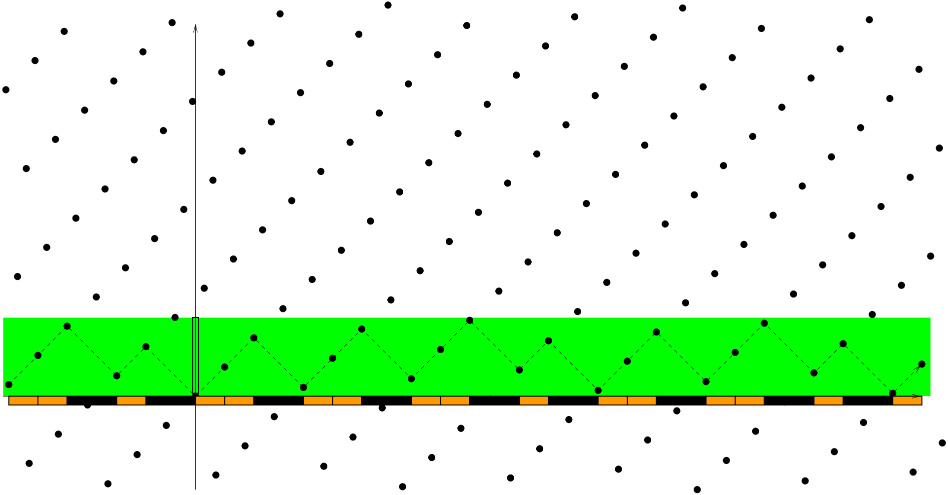
- Γ a *lattice* in $\mathbb{R}^d \times \mathbb{R}^e$
- π_1, π_2 *projections*
  - $\pi_1|_\Gamma$ injective
  - $\pi_2(\Gamma)$ dense
- W *compact* ("window", somehow nice, e.g. ∂W has zero measure)

Then $\Lambda = \{\pi_1(x) \,|\, x \in \Lambda, \pi_2(x) \in W\}$ is a (regular) *cut-and-project set* (CPS).



E⊥
window
(1,1)
E||





The last one uses $d = e = 1$ ($E_{||} = \mathbb{R}^1, E_\perp = \mathbb{R}^1$).

An example with $d = 1$, $e = 2$:

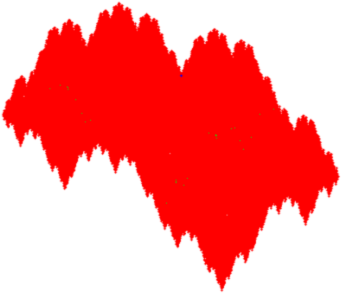
$$\sigma : \quad S \to ML, \quad M \to SML, \quad L \to LML$$

The last one uses $d = e = 1$ ($E_{||} = \mathbb{R}^1, E_\perp = \mathbb{R}^1$).

An example with $d = 1$, $e = 2$:

$$\sigma : \quad S \to ML, \quad M \to SML, \quad L \to LML$$

...uses a window W that looks like a fractal:

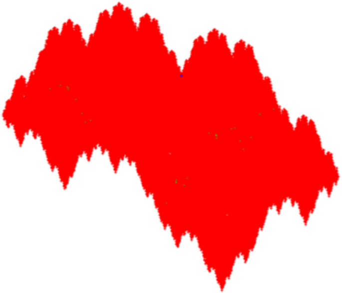


The last one uses $d = e = 1$ ($E_{||} = \mathbb{R}^1, E_\perp = \mathbb{R}^1$).

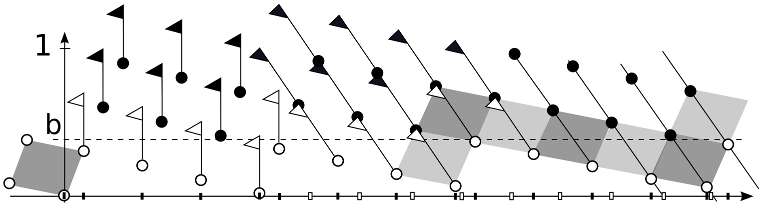
An example with $d = 1$, $e = 2$:

$$\sigma: \quad S \to ML, \quad M \to SML, \quad L \to LML$$

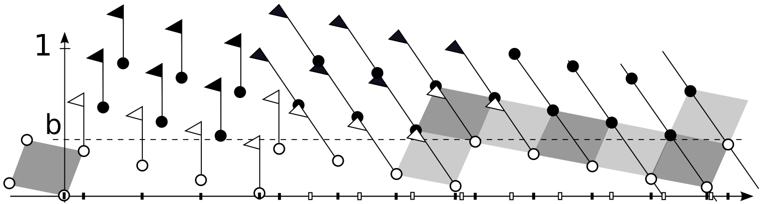
...uses a window W that looks like a fractal:



Now let us generalize Kesten to $\mathbb{R}^d$ (at least "if"-part)



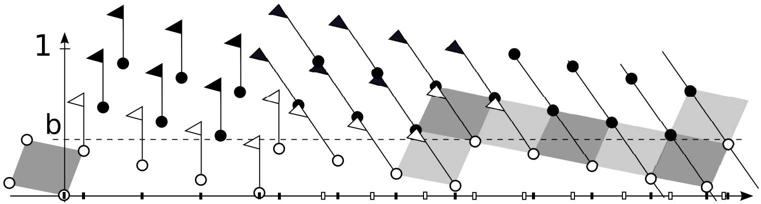
(looks almost like a cut-and-project set!)



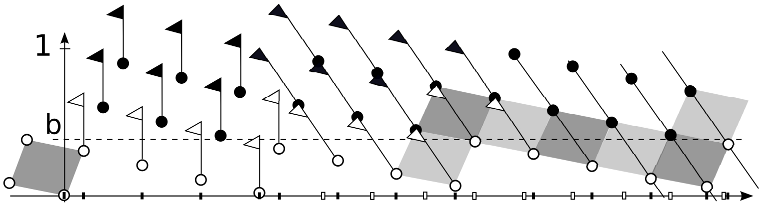
(looks almost like a cut-and-project set!)

One can state the argument in purely algebraic terms:

- $X = \mathbb{R}^{d+1}$
- $X = V_p + V_i$ (here: horizontal + vertical), $W \subset V_i$ compact set (here $W = [0, b]$),
- π_p projection to V_p (here: $\downarrow$),
- π_i projection to V_i (here: $\leftarrow$),
- Γ discrete cocompact subgroup (here: black and white points)



- $X = \mathbb{R}^{d+1}$,
- $X = V_p + V_i$ (here: horizontal + vertical), $W \subset V_i$ compact set (here $W = [0, b]$),
- π_p projection to V_p (here: $\downarrow$),
- π_i projection to V_i (here: $\leftarrow$),
- Γ discrete cocompact subgroup (here: black and white points)



- $X = \mathbb{R}^{d+1}$,
- $X = V_p + V_i$ (here: horizontal + vertical), $W \subset V_i$ compact set (here $W = [0, b]$),
- π_p projection to V_p (here: $\downarrow$),
- π_i projection to V_i (here: $\leftarrow$),
- Γ discrete cocompact subgroup (here: black and white points)
- $Y = \pi_i^{-1}(W) \cap \Gamma$ (here: white points),
- $\Lambda = \pi_p(Y)$
- Z subgroup of X with $V_p + Z = X$, $Z/(Z \cap \Gamma)$ compact (here "lattice direction" for projection)
- π_Z corresponding projection etc...

...then $\pi_p(Y) \overset{\mathrm{bd}}{\sim} \pi_Z(Y)$.

Other colleagues had the same idea: Haynes-Koivusalo 2014, Haynes-Kelly-Koivusalo 2017.

...then $\pi_p(Y) \stackrel{\text{bd}}{\sim} \pi_Z(Y)$.

Other colleagues had the same idea: Haynes-Koivusalo 2014, Haynes-Kelly-Koivusalo 2017.

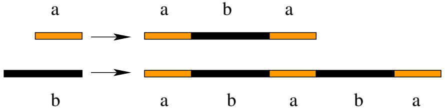
Last October I've learned from Alan Haynes that this was done already in

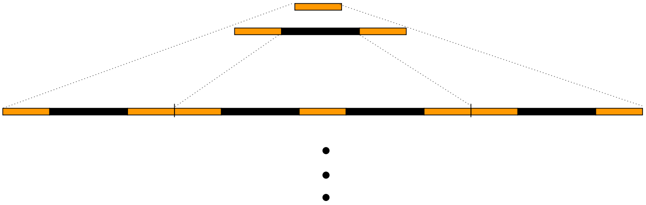
C. Godrèche and C. Oguey:
Construction of average lattices for quasiperiodic structures by the section method, *J. Phys. France* 51 (1990) 21-37

So much on Question 2.

Only briefly regarding Question 3:

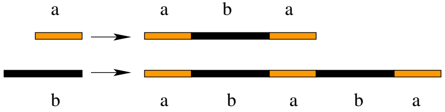
A one-dimensional *tile substitution*, producing tilings of the line by intervals. The endpoints form some Delone set.

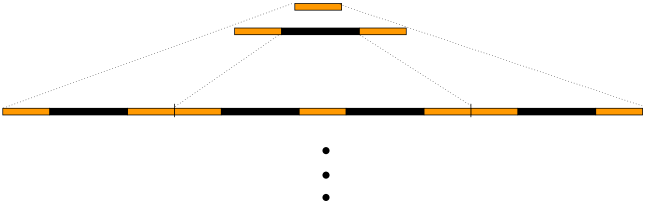




Only briefly regarding Question 3:

A one-dimensional *tile substitution*, producing tilings of the line by intervals. The endpoints form some Delone set.





- $M_\sigma = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$
- Inflation factor $2 + \sqrt{3}$
- length$(a) = 1$, length$(b) = \sqrt{3}$

A one-dimensional substitution tiling with inflation factor λ is a *Pisot substitution*, if all eigenvalues of M_σ other than λ are less than one in modulus.

A one-dimensional substitution tiling with inflation factor λ is a *Pisot substitution*, if all eigenvalues of M_σ other than λ are less than one in modulus.

Theorem (F-Garber 2017 preprint)

All one-dimensional Pisot substitution tilings are bounded distance equivalent to some lattice.

A one-dimensional substitution tiling with inflation factor λ is a *Pisot substitution*, if all eigenvalues of M_σ other than λ are less than one in modulus.

Theorem (F-Garber 2017 preprint)

All one-dimensional Pisot substitution tilings are bounded distance equivalent to some lattice.

Unfortunately:

Theorem (Holton-Zamboni 1998)

All one-dimensional Pisot substitution tilings are bounded distance equivalent to some lattice.

A one-dimensional substitution tiling with inflation factor λ is a *Pisot substitution*, if all eigenvalues of M_σ other than λ are less than one in modulus.

Theorem (F-Garber 2017 preprint)

All one-dimensional Pisot substitution tilings are bounded distance equivalent to some lattice.

Unfortunately:

Theorem (Holton-Zamboni 1998)

All one-dimensional Pisot substitution tilings are bounded distance equivalent to some lattice.

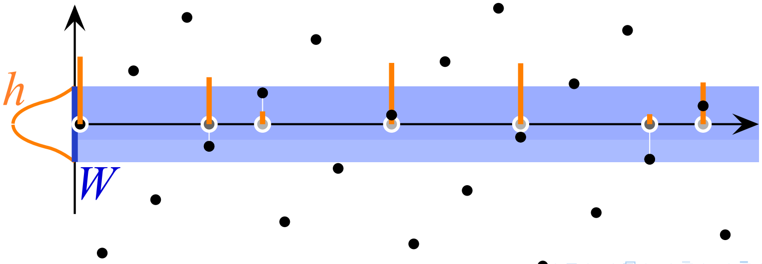
We did not give up....

New result

Take some CPS Λ and give each point a weight. One convenient way to write it: *Dirac comb*

$$\delta_{w,\Lambda} = \sum_{x \in \Lambda} w(x)\delta_x \qquad (w(x) \in \mathbb{R}, \delta_x \text{ the Dirac measure in } x)$$

If $w(x) = h(x^*)$ for $h : W \to \mathbb{R}$ continuous, then $\delta_{w,\Lambda}$ is called a *weighted CPS*.



Theorem (F-Garber 2017 preprint)

Let $\delta_{w,\Lambda}$ be a weighted CPS with $e = d = 1$. Let $W = [a, b]$, $w(x) = h(x^*)$ and $h(a) = h(b) = 0$. If h is

1. *piecewise linear, or*
2. *twice differentiable,*

then $\delta_{w,\Lambda}$ is bounded distance equivalent to $c\mu$ for some $c > 0$, where μ denotes the one-dimensional Lebesgue measure.

Finally, our first new result!

Theorem (F-Garber 2017 preprint)

Let $\delta_{w,\Lambda}$ be a weighted CPS with $e = d = 1$. Let $W = [a, b]$, $w(x) = h(x^*)$ and $h(a) = h(b) = 0$. If h is

1. *piecewise linear, or*
2. *twice differentiable,*

then $\delta_{w,\Lambda}$ is bounded distance equivalent to $c\mu$ for some $c > 0$, where μ denotes the one-dimensional Lebesgue measure.

Finally, our first new result! (At least we hope so...)

More here:

D.F., Alexey Garber:
Bounded distance and bilipschitz equivalence of Delone sets, preprint,
www.math.uni-bielefeld.de/∼frettloe/papers/bilip-draft.pdf

and references therein.

*   *
*

Thank you!