Quasicrystals and symmetry

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1. Aperiodic tilings
2. Substitution tilings with tiles in infinitely many orientations
3. Dense tile orientations (DTO) in Dim. 2
4. Tilings with rotational symmetry and DTO in Dim. 2
5. Dimension 3

1. Aperiodic tilings

A tiling is a covering of $\mathbb{R}^2$ that is also a packing.

I.e., a tiling is a collection of (usually compact) sets that cover $\mathbb{R}^2$ without overlap (except at the boundary).

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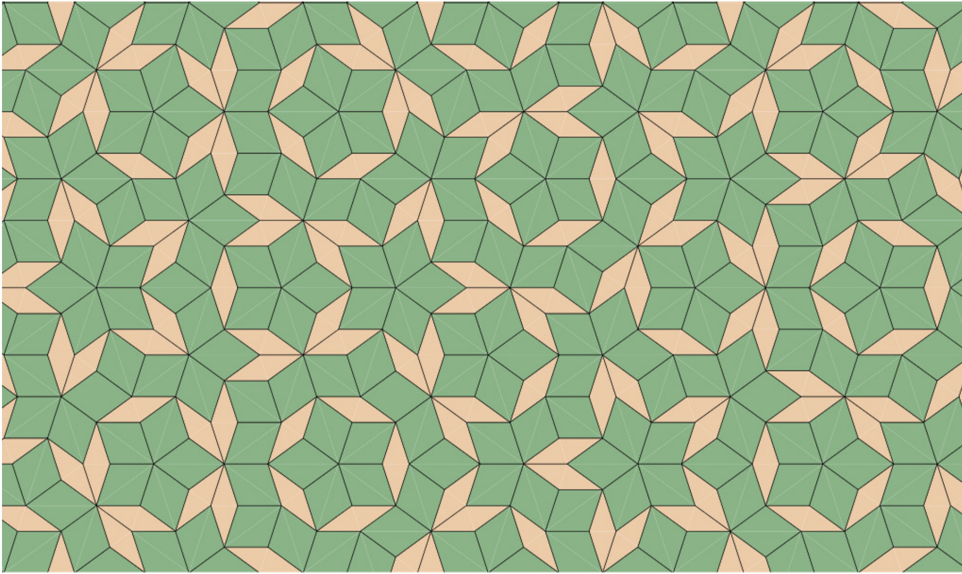
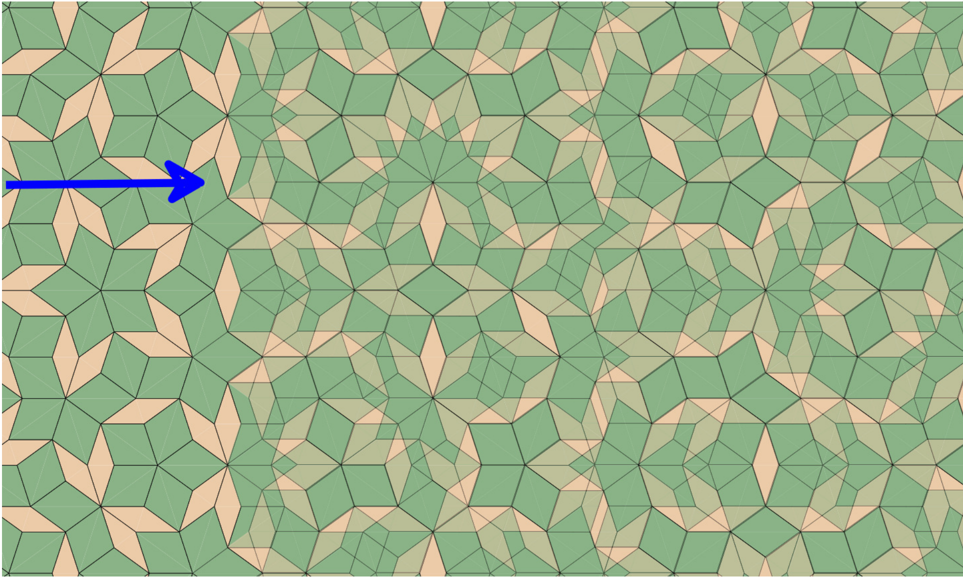
A tiling is *aperiodic* if no translation maps the tiling to itself.

An aperiodic tiling: the Penrose tiling



Aperiodic



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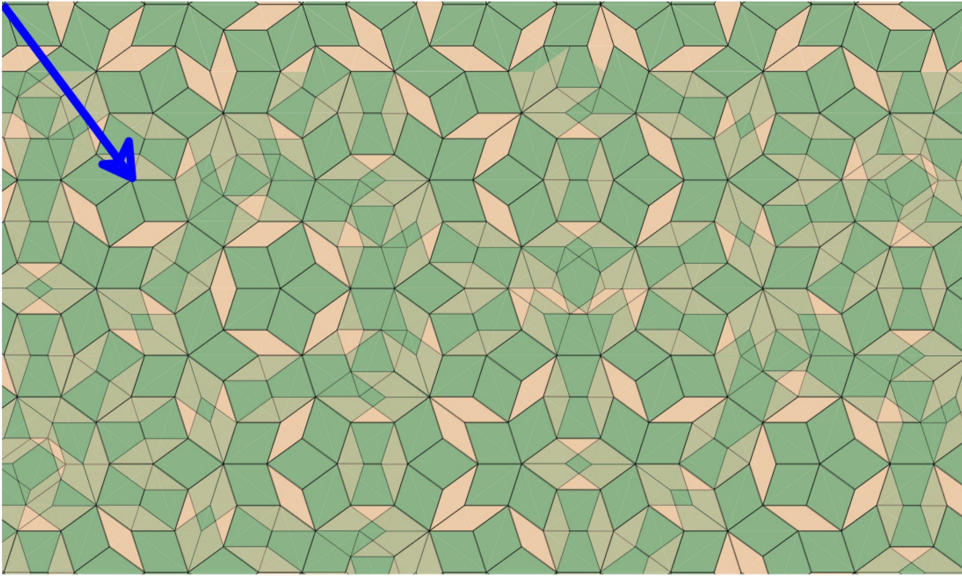
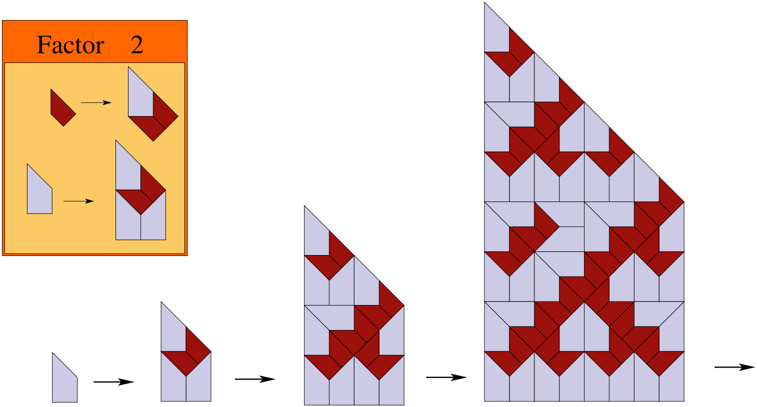


A simple way to generate aperiodic tilings: substitution tilings.

Substitution tiling with *substitution factor* 2, and two *prototiles*:



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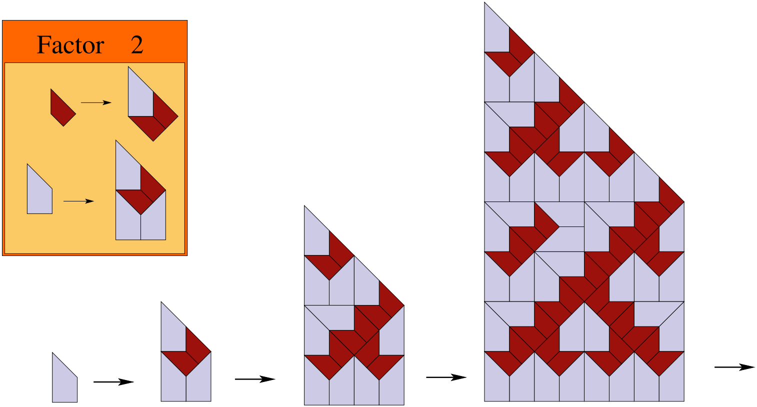
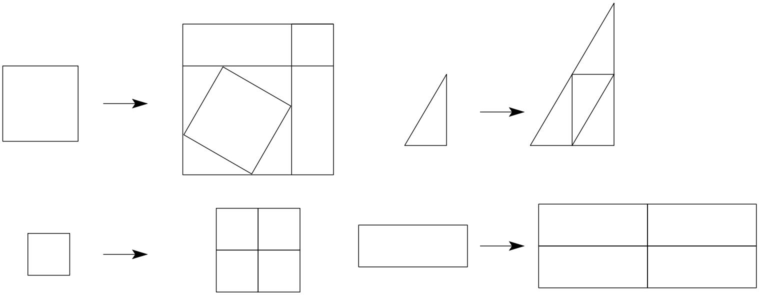


Substitution matrix here $M = \left(\begin{smallmatrix} 2 & 2 \\ 1 & 3 \end{smallmatrix}\right)$.

Fact: if λ is the substitution factor, then λ^2 is the largest eigenvalue of the substitution matrix.

2. Substitution tilings with tiles in infinitely many orientations

Usually, tiles occur in finitely many different orientations only.
Not always. Cesi's example (1990):



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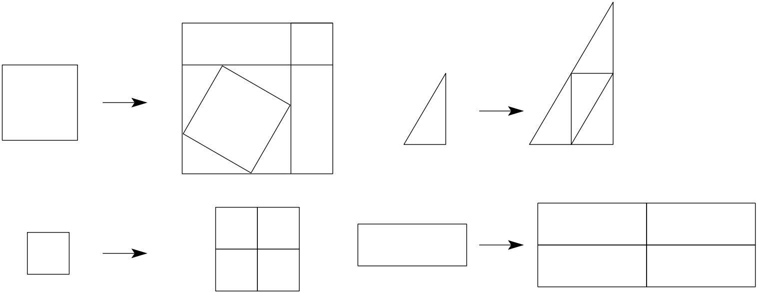
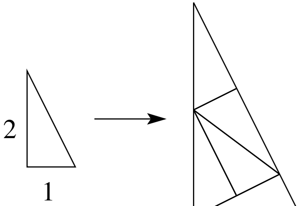
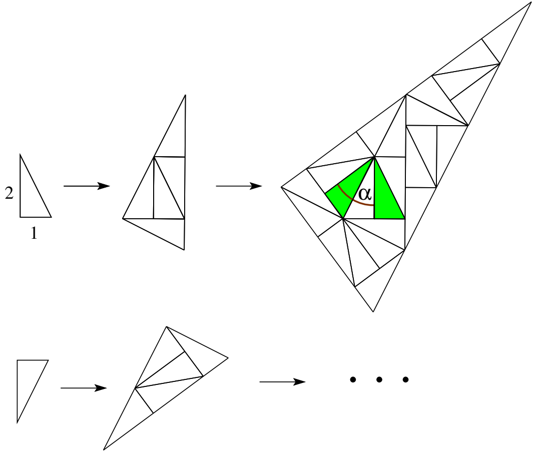


A substitution σ is *primitive*, if for any tile T there is $k \geq 1$ such that $\sigma^k(T)$ contains all tile types.

So this substitution is not primitive.

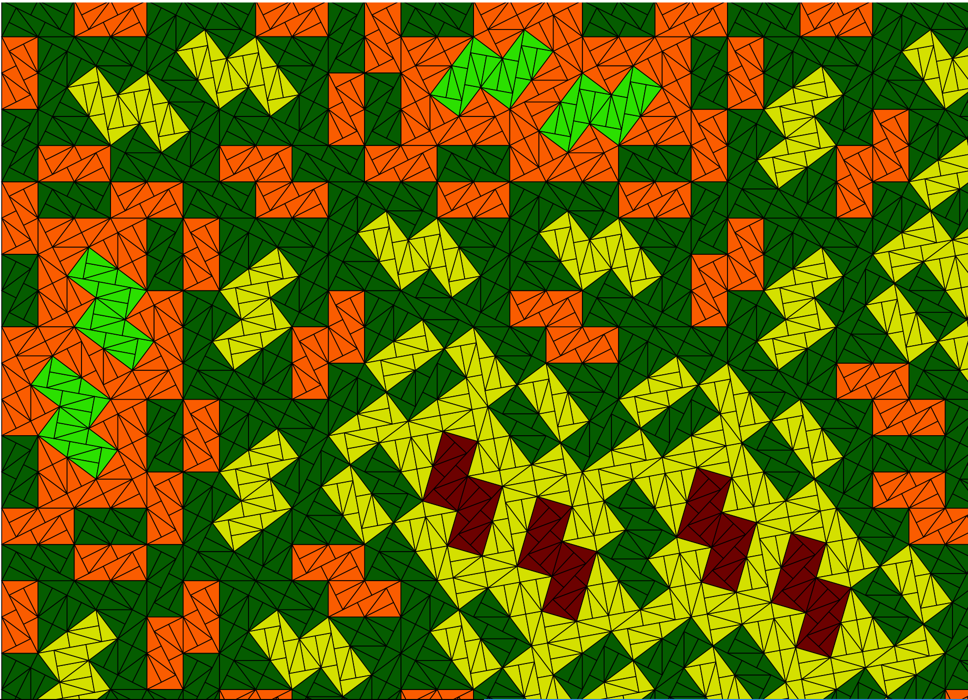
Conway's Pinwheel substitution (1991):



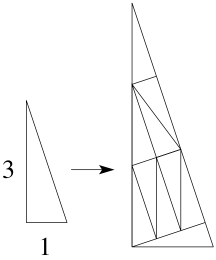


The angle α is *irrational*; that is, $\alpha \notin \pi\mathbb{Q}$.

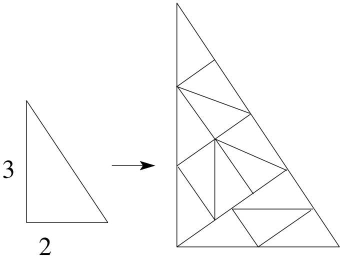
Hence all multiples of α are different.



Obvious generalizations: Pinwheel (n, k)



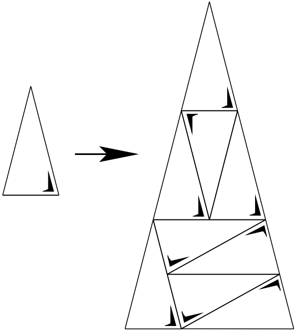
$n = 3, k = 1$

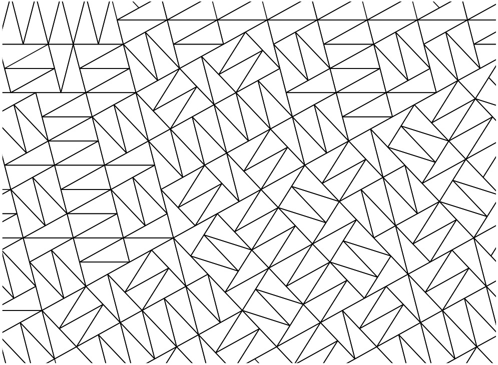


$n = 3, k = 2$

etc.

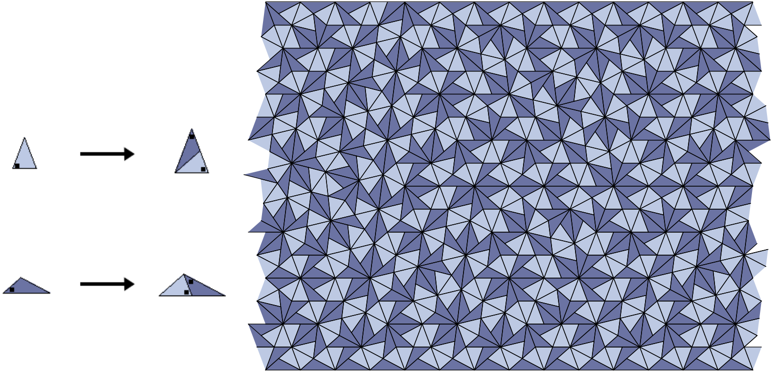
Another example:



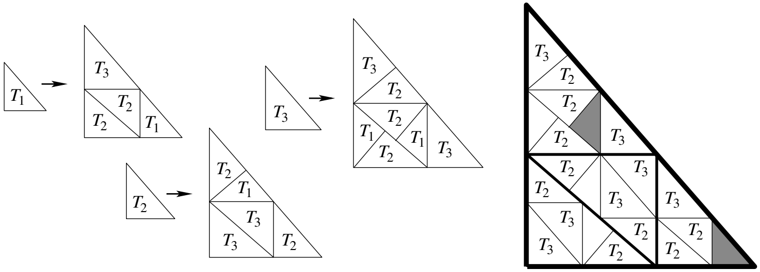


(+ obvious generalizations)

C. Goodman-Strauss, L. Danzer (ca. 1996):



Pythia (m, j), here: $m = 3$, $j = 1$.



3. Dense Tile Orientations (DTO)

For all examples: the orientations are dense in $[0, 2\pi[$.

Even more: The orientations are equidistributed in $[0, 2\pi[$.

Theorem (F. '08)

In each primitive substitution tiling with tiles in infinitely many orientations, the orientations are equidistributed in $[0, 2\pi[$.

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Recall: $(\alpha_j)_j$ is *equidistributed* in $[0, 1[$, if for all $0 \le a < b < 1$ holds:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{[a,b]}(\alpha_j) = b - a$$

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Here: in a tiling $\mathcal{T} = T_1, T_2, \ldots$ the orientations of the tiles are equidistributed, if for all $0 \le a < b < 2\pi$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} 1_{[a,b]}(\alpha(T_j)) = \frac{b-a}{2\pi}$$

where $\alpha(T_j)$ is the angle of tile T_j (wrt some fixed copy of T_j).

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Because the sum is not absolutely convergent, the order matters!

Here it is OK to order the tiles wrt distance from 0.

Proof needs:

Weyl's criterion: (a_n) equidistributed mod 1 iff

$$\forall \, \ell \in \mathbb{Z} \setminus \{0\} : \qquad \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i \ell a_j} = 0.$$

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Perron's Theorem: $M \in \mathbb{R}^{n\times n} \geq 0$ (i.e., non-negative entries only) and $M^k > 0$ for some k, then

- There is a biggest eigenvalue $\mu \in \mathbb{R}$ with $\mu > 0$
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- There is a biggest eigenvalue $\mu \in \mathbb{R}$ with $\mu > 0$
- μ has a positive eigenvector v
- $\lim\limits_{n\to\infty} \frac{1}{\mu^n} M^n$ exists, the columns are multiples of v
- If $0 \leq A \leq M$, $A \neq M$, then the biggest eigenvalue of A is less than μ.

Sketch of proof: Let M be the substitution matrix, with biggest eigenvalue μ.

$$\text{Let } A(\ell) = \Big(\sum_{j=1}^{M_{km}} e^{i\alpha(T_j)\ell} \Big)_{km} \qquad (\ell \in \mathbb{Z})$$

be the matrix containing the orientations $\alpha(T_j)$ times ℓ.
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By irrationality of the angles

$$|A(\ell)|^n \leq M^n \text{ and } |A(\ell)|^n \neq M^n \qquad \text{(from some } n \text{ on)}$$

We need to show:

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Corollary

In each primitive substitution tiling with tiles in infinitely many orientations, the orientations are dense in $[0, 2\pi[$.

Which tile shapes do forbid DTO?

So far: tiles are always triangles. No surprise:

Theorem (F.-Harriss, 2013)

Let $\mathcal{T}$ be a tiling in $\mathbb{R}^2$ with finitely many prototiles (i.e., finitely many different tile shapes). Let all prototiles be centrally symmetric convex polygons. Then each prototile occurs in a finite number of orientations in $\mathcal{T}$.

So in particular: In a tiling consisting of parallelograms only, the tiles occur in finitely many orientations only.

4. Tilings with rotational symmetry and DTO

Several people (Franz Gähler, Lorenzo Sadun, Johannes Kellendonk...) compute cohomologies of tiling spaces.

(...which means: consider the set of all tilings to a given substitution. Define when two tilings are "close". This yields a topological object whose cohomologies can be computed. This is standard now for tiling with tiles in finitely many orientations, but still challenging for tilings with DTO.)

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Since rotational symmetry causes additional problems, they (J. Hunton, J. Savinien) asked:

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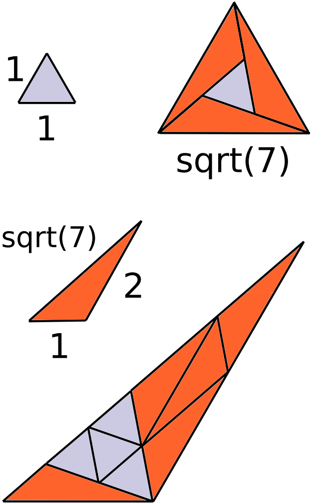
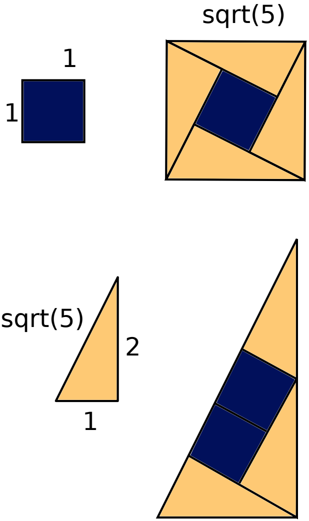
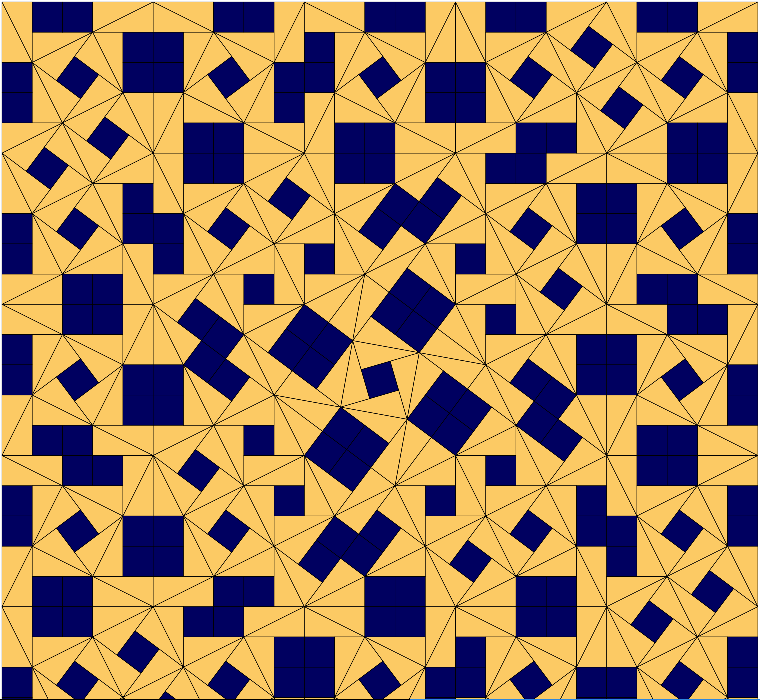
Answer: Yes. At least for $n \in \{3, 4, 5, 6, 7, 8\}$.

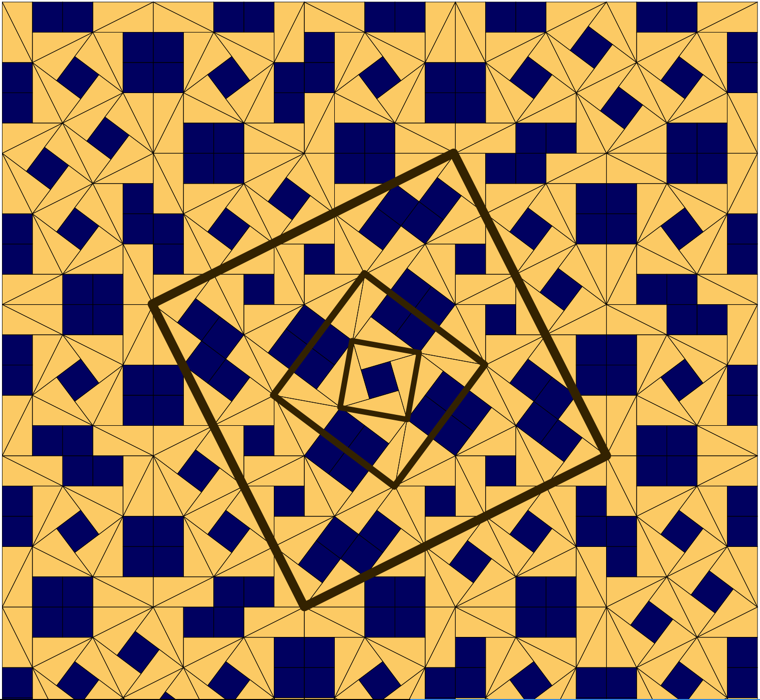
Example for $n = 3$



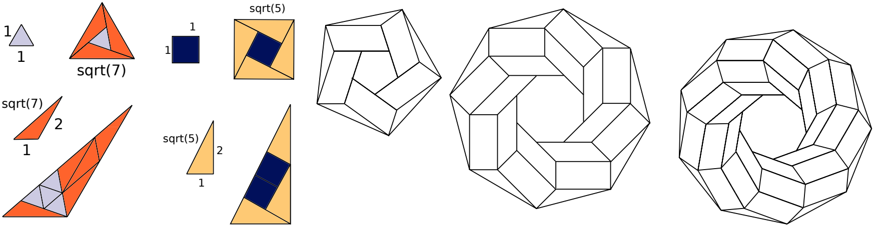
Example for $n = 4$



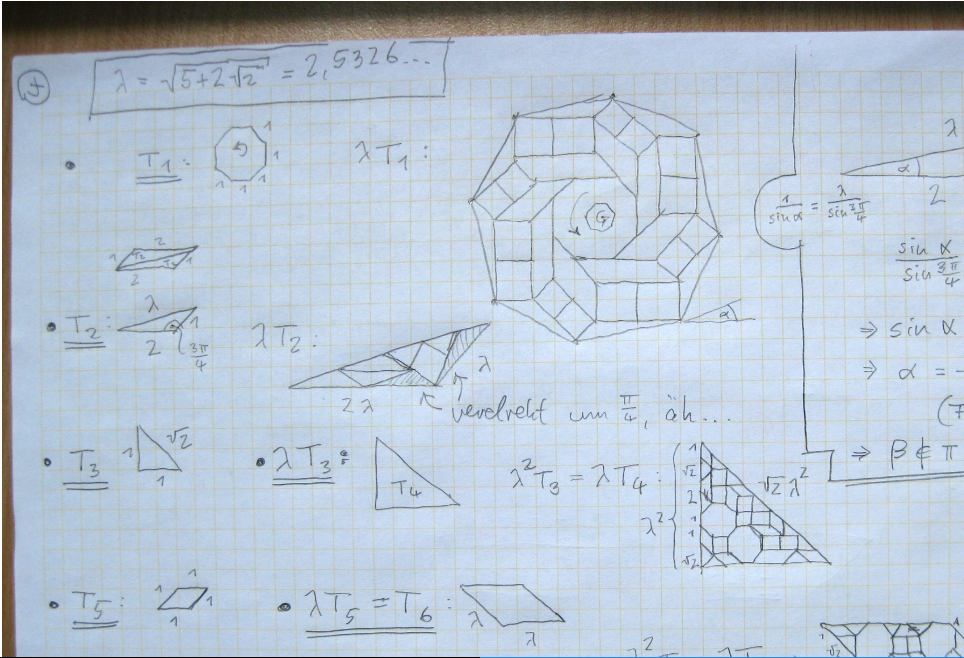




Consider the analogues for larger n



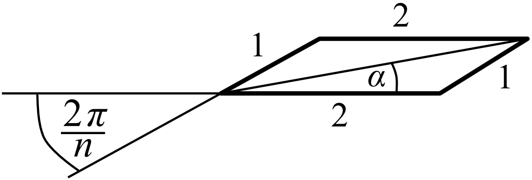
A tile substitution for $n = 8$:



In each case we need a certain angle to be irrational. Starting from this we need found (rediscovered?):

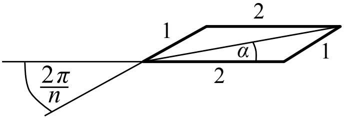
Theorem (F.-Say-Awen-de las Peñas 2017)

In a parallelogram with edge lengths 1 and 2, and interior angle β: If $\beta = \frac{2\pi}{n}$ ($n \geq 4$) then $\alpha \notin \pi\mathbb{Q}$.



Proof: Embed the parallelogram in the complex plane:

- lower left vertex: 0
- upper left vertex: $\xi_n := e^{2\pi i/n}$
- upper right vertex: $z := 2 + \xi_n$



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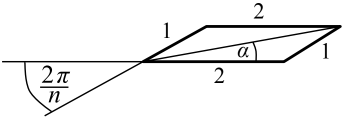

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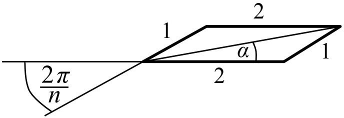

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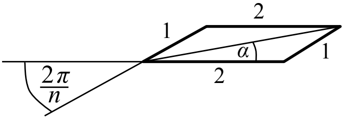

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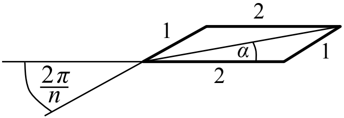

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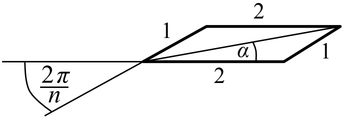

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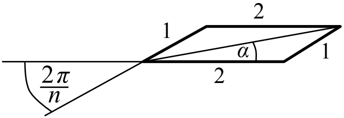

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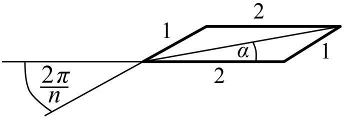

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Clearly, $\frac{z}{\overline{z}} \in \mathbb{Q}(\xi_n)$.

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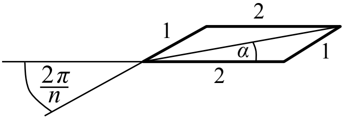

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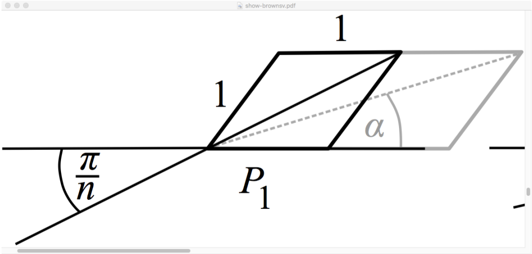
Theorem: All roots of unity in $\mathbb{Q}(\xi_n)$ are of the form $\pm\xi_n^k$.

Hence $m = n$ or $m = 2n$ (if m is even and n is odd)

Since $\arg(\frac{z}{|z|}) = \alpha$, we have $\arg(\frac{z}{\bar{z}}) = 2\alpha$,

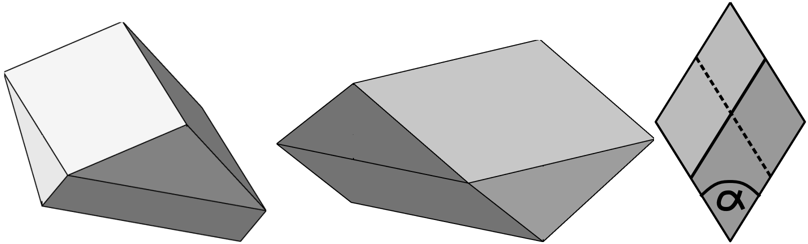
Altogether: $2\alpha = \frac{2k\pi}{n}$, hence $\alpha = \frac{k\pi}{n}$



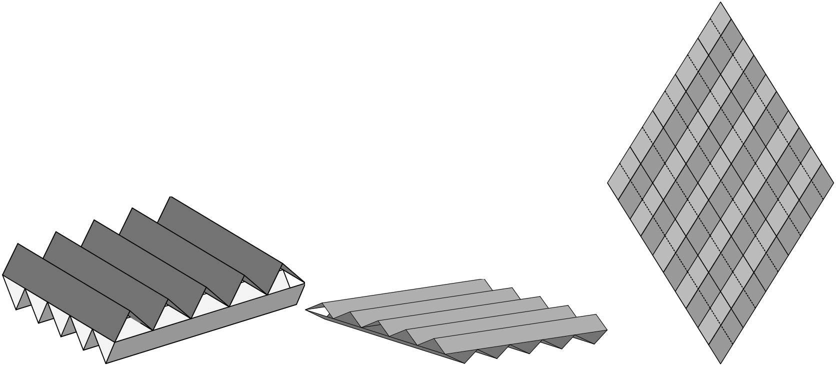
But (see image): $\alpha < \frac{\pi}{n}$ (too small!),

Contradiction. Hence $\alpha \notin \pi\mathbb{Q}$.

SCD tile (Schmitt-Conway-Danzer)



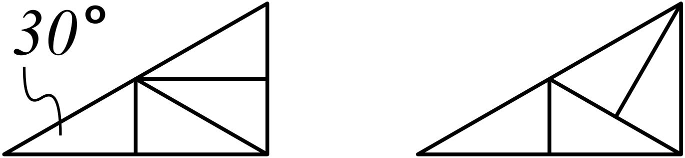
Tiles can be assembled to layers, layers can be stacked.



Infinitely many orientations, but dense only in a 2-dimensional plane, not in the sphere.

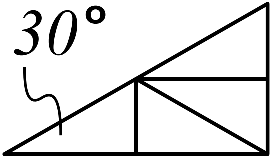
Real DTO: the Pinwheel sandwich.
(Conway-Radin: "Quaquaversal tilings and rotations" 1995)

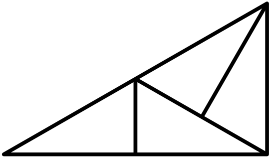
Start with the following dissections:



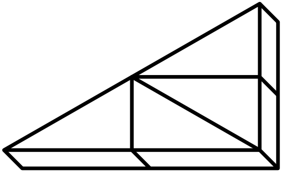
Real DTO: the Pinwheel sandwich.
(Conway-Radin: "Quaquaversal tilings and rotations" 1995)

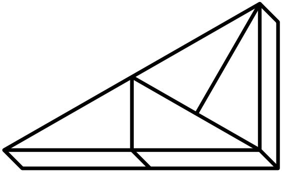
Start with the following dissections:



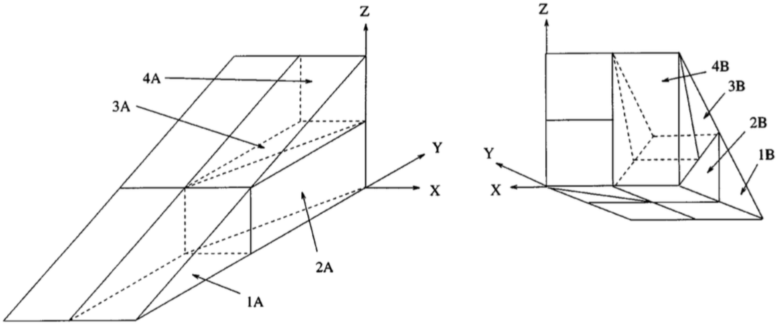


Consider thickened 3-dimensional versions:





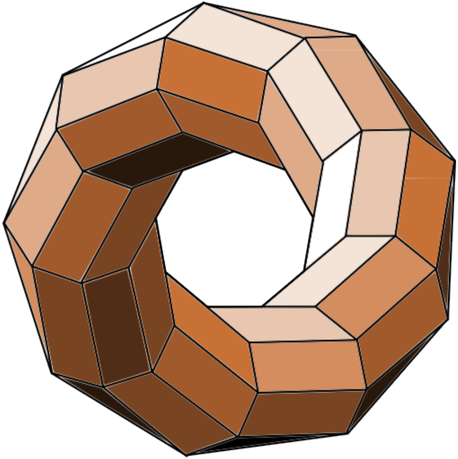
Stacking them (and adding one tweak) yields a tile substitution in $\mathbb{R}^3$:



Based on rational angles: $\frac{\pi}{2}, \frac{\pi}{6}$.

But all combinations in $\mathbb{R}^3$ are dense on the sphere!

One of the few DTO tilings in $\mathbb{R}^3$ (but see Exercises)



Thank you!