Average lattices for quasicrystals

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joint work with Alexey Garber, Brownsville, Texas

Plan:

- 1. Lattices, Delone sets, basic notions
- 2. Dimension one
- 3. Aperiodic patterns
- 4. Average lattices for quasicrystals
- 5. Weighted cut-and-project sets

1. Lattices, Delone sets, basics

Delone set: point set Λ in $\mathbb{R}^d$, with $R > r > 0$ such that

- each ball of radius r contains at most one point of Λ (*uniformly discrete*)
- each ball of radius R contains at least one point of Λ (*relatively dense*)

(Aka "separated nets". Can also live in $\mathbb{H}^d$, $(\mathbb{Q}_p)^d$...)

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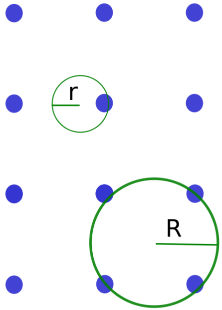
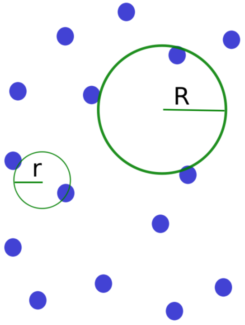


lattice
(crystallographic)



disordered

Lattice in $\mathbb{R}^d$: integer span of d linearly independent vectors $v_1, \ldots, v_d$.

Example: $\mathbb{Z}^2$, integer span of $\binom{1}{0}$, $\binom{0}{1}$, *Lattice* in $\mathbb{R}^d$: integer span of d linearly independent vectors $v_1, \ldots, v_d$.

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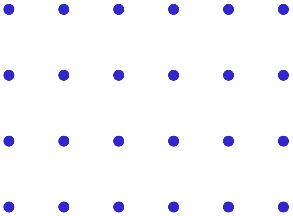
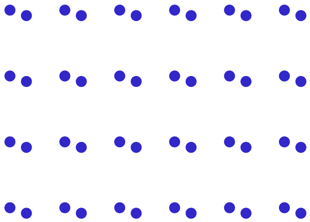
A Delone set Λ in $\mathbb{R}^d$ is *crystallographic*, if there are $v_1, \ldots, v_d$ (linearly independent) such that

$$\Lambda = \Lambda + v_i \quad (1 \leq i \leq d)$$



lattice



crystallographic, but not a lattice

Two relations between Delone sets:

$\Lambda \overset{\text{bil}}{\sim} \Lambda'$ *(bilipschitz equivalent)*:

There is $f : \Lambda \to \Lambda'$ bijective with

$$\exists c > 0 \quad \forall x, y \in \Lambda \quad \frac{1}{c}|x - y| \leq |f(x) - f(y)| \leq c|x - y|$$

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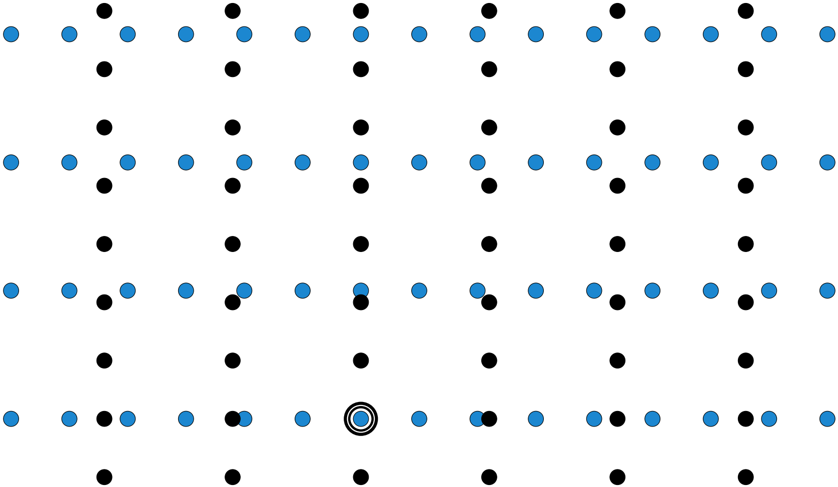
$\Lambda \stackrel{\text{bd}}{\sim} \Lambda'$ *(bounded distance equivalent)*:

There is $g : \Lambda \to \Lambda'$ bijective with

$$\exists C > 0 \quad \forall x \in \Lambda : \quad |x - g(x)| < C$$

Example: Two rectangular lattices Λ, Λ'. Is $\Lambda \overset{\text{bd}}{\sim} \Lambda'$?



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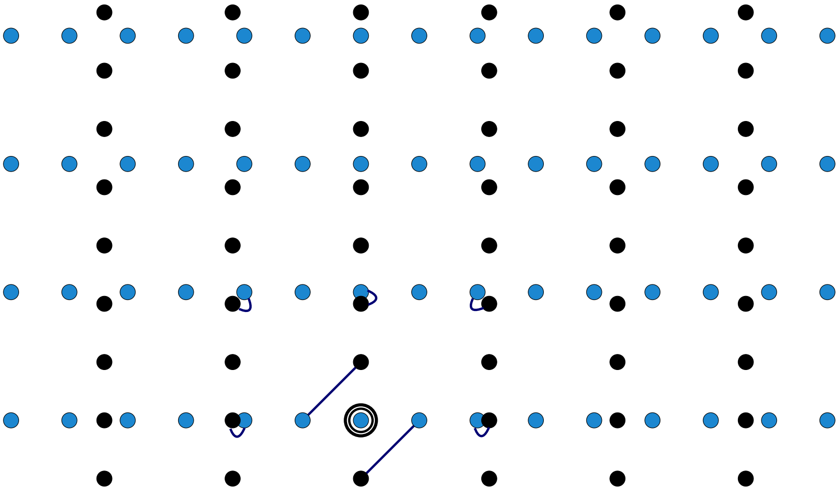

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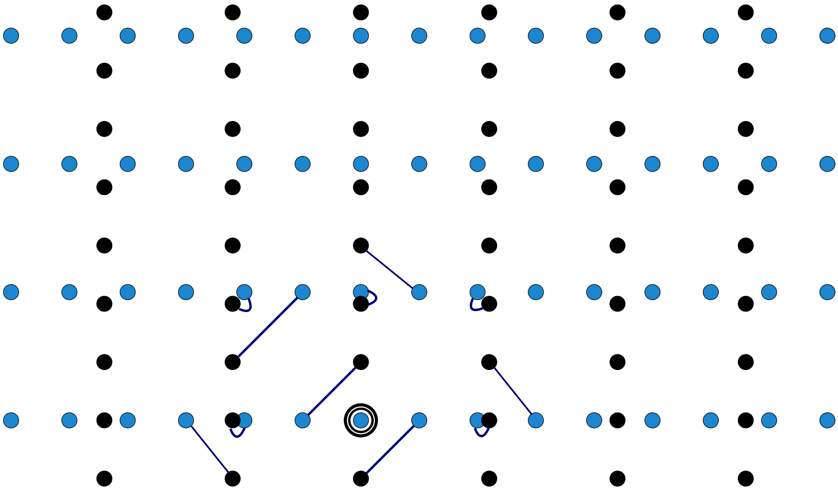

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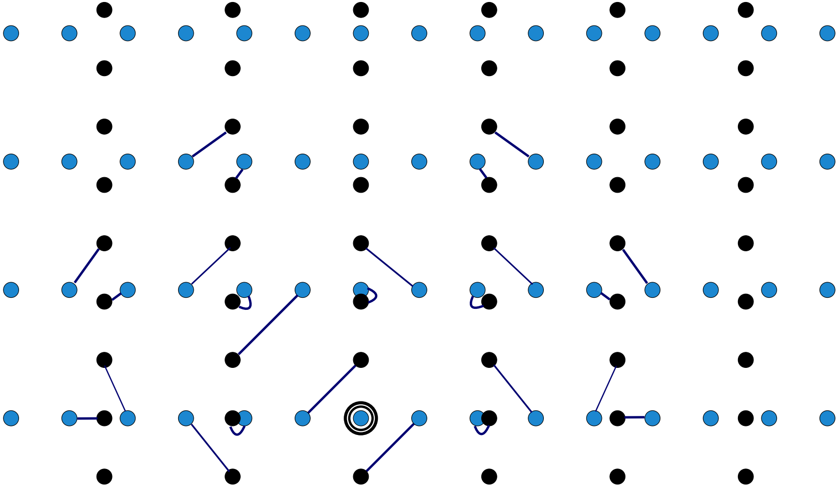

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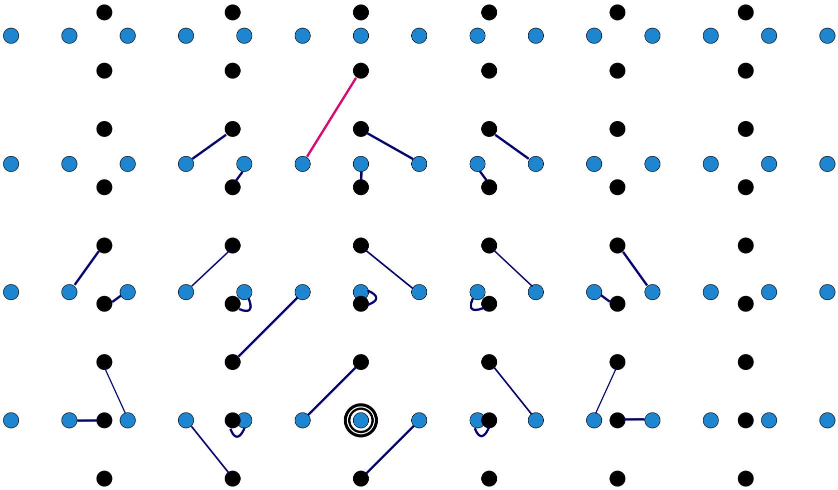

Some basic results:

Lemma (1)

Bilipschitz equivalence and bounded distance equivalence are equivalence relations.

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Lemma (2)

Let Λ, Λ' *be Delone sets in* $\mathbb{R}^d$*. If* $\Lambda \overset{\text{bd}}{\sim} \Lambda'$*, then* $\Lambda \overset{\text{bil}}{\sim} \Lambda'$*.*

2. Dimension one

Theorem (Duneau-Oguey 1990)

Let Λ, Λ' *be Delone sets in* $\mathbb{R}$ *(with Euclidean metric).*
Then $\Lambda \overset{\text{bil}}{\sim} \Lambda'$.

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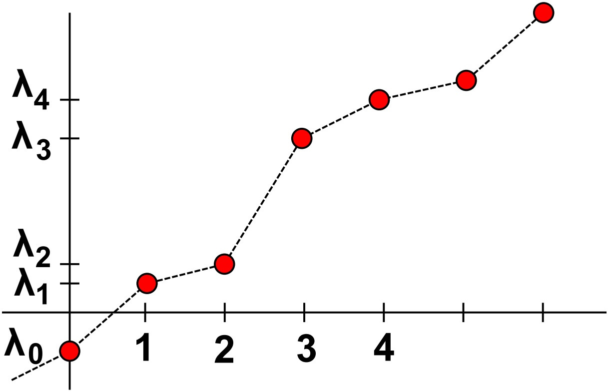
Let Λ, Λ' be Delone sets in $\mathbb{R}$ (with Euclidean metric). Then $\Lambda \overset{\text{bil}}{\sim} \Lambda'$.

Proof (by image): Show $\Lambda \overset{\text{bil}}{\sim} \mathbb{Z}$.

Let $\Lambda = \{\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \ldots\}, \quad \lambda_i < \lambda_{i+1}$. Plot (i, λ_i):



Let $\Lambda, \Lambda' \subset \mathbb{R}$. When is $\Lambda \overset{\text{bd}}{\sim} \Lambda'$? Always? No:

Examples:

- $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\} \overset{\text{bd}}{\not\sim} \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$
- $\{\ldots -3, -2, -1, 0, 2, 4, 6, \ldots\} \overset{\text{bd}}{\not\sim} \{\ldots -6, -4, -2, 0, 1, 2, 3, \ldots\}$

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Density matters. Preliminary definition ("central density"):

$$\mathrm{dens}(\Lambda) := \lim_{r\to\infty} \frac{1}{2r} \#(\Lambda \cap [-r, r]),$$

if it exists.

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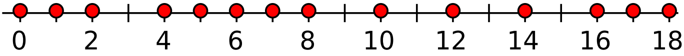
$$\text{dens}(\Lambda) := \lim_{r \to \infty} \frac{1}{2r} \#(\Lambda \cap [-r, r]),$$

if it exists. Does not need to exist:



Oscillates between $\frac{2}{3}$ and $\frac{5}{6}$.

Question: If dens(Λ)=dens(Λ'), is $\Lambda \stackrel{\text{bd}}{\sim} \Lambda'$?

Theorem (Duneau-Oguey 1990)

Let Λ, Λ' be crystallographic. Then dens(Λ)=dens(Λ') implies $\Lambda \stackrel{\text{bd}}{\sim} \Lambda'$. (True even in $\mathbb{R}^d$ for $d \geq 2$)

Interesting examples are non-periodic.

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Interesting examples are non-periodic.

Theorem (Kesten 1966)

Let $\xi \in [0,1]$, $0 \leq a < b \leq 1$ and define

$$\Lambda := \{k \in \mathbb{Z} \mid a \leq \big(k\xi \bmod 1\big) < b\}.$$

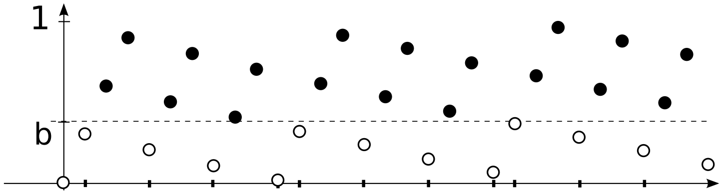
Then the deficiency $D(n) := \#(\Lambda \cap [1,n]) - n(b-a)$ is bounded, if and only if $b - a = k\xi \mod 1$ for some $k \in \mathbb{Z}$.

(if-part: Hecke 1921, Ostrowski 1927)

Choose $\xi \in [0,1]$ irrational, let $0 < b \leq 1$ and define

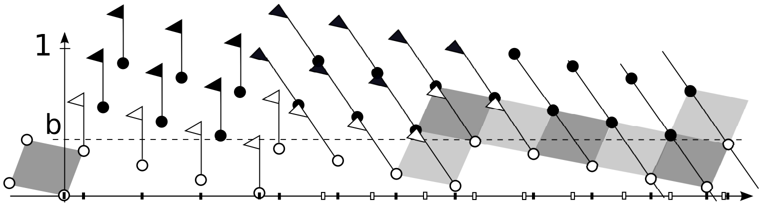
$$\Lambda_b := \{k \in \mathbb{Z} \mid 0 \leq \big(k\xi \mod 1\big) < b\}.$$

Then the deficiency $D(n) := \#(\Lambda \cap [1,n]) - nb$ is bounded, if and only if $b = k\xi \mod 1$ for some $k \in \mathbb{Z}$.



The image shows $\{(k, k\xi \text{ mod } 1) \mid k = 0,1,2,\dots\}$.

Proof (by image) of if-part: (F-Gähler 2011, Duneau-Oguey 1990):



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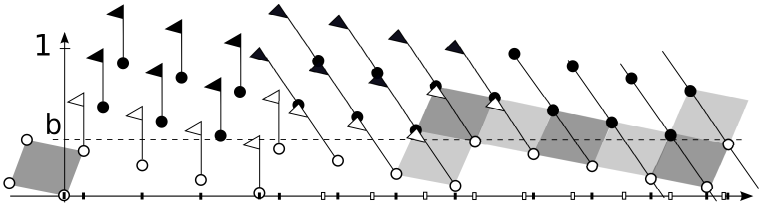

The "only if"-part of Kesten yields Delone sets Λ_b that are not bounded distance equivalent to any $c\mathbb{Z}$. Even when $\text{dens}(\Lambda_b)$ exists!

Higher dimensions:

Theorem (Bogopolski 1997)

Any two Delone sets in $\mathbb{H}^d$ ($d \geq 2$) are bounded distance equivalent, hence bilipschitz equivalent.

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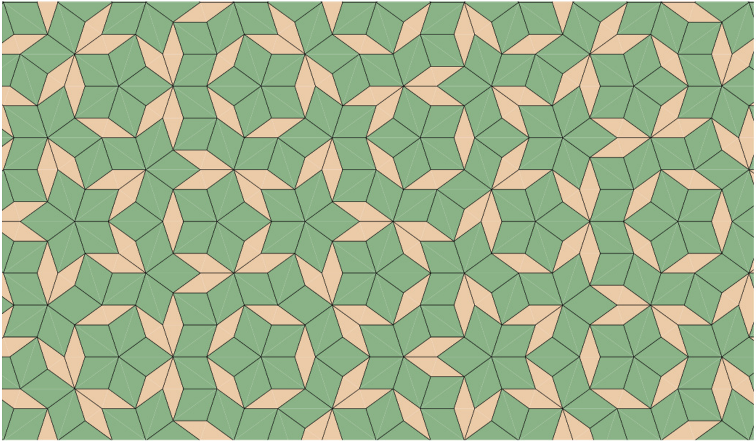
Cool! Alexey and I decided to study some problems in this field. E.g.

1. Are the vertices of the Penrose tiling bounded distance equivalent to some lattice?
2. Which substitution tilings are $\overset{\text{bd}}{\sim}$ to some lattice?
3. Which cut-and-project sets are $\overset{\text{bd}}{\sim}$ to some lattice?

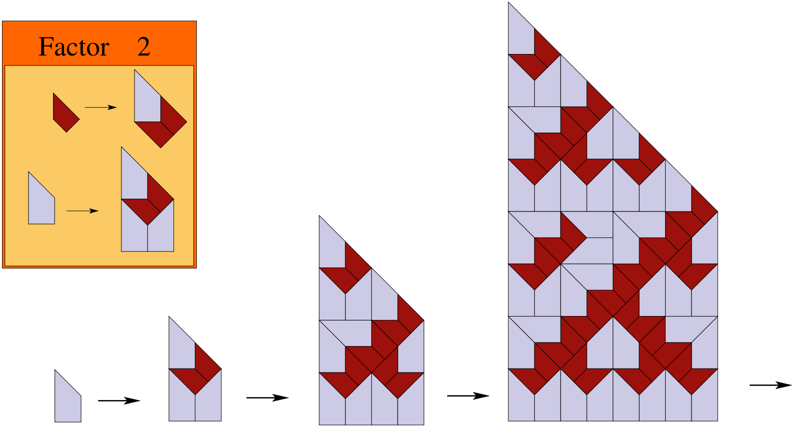
3. Aperiodic patterns

Recall: Interesting examples are non-periodic.
Like the Penrose tiling:

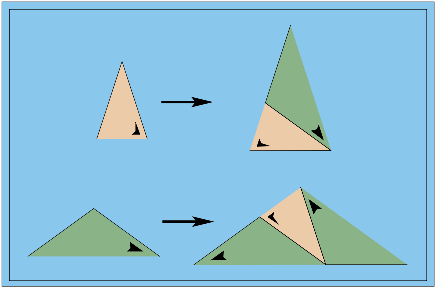


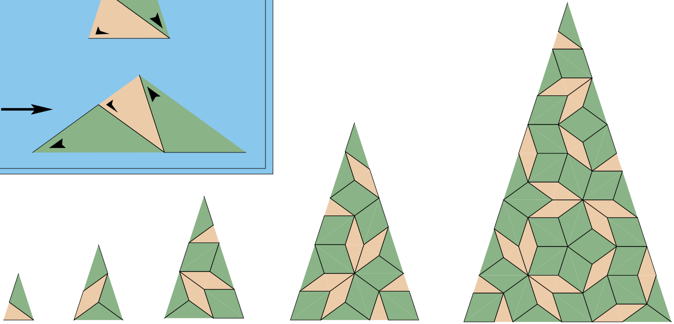
Two ways to generate a Penrose tiling:

Substitution Tilings



Penrose Substitution





Important: **substitution matrix** of some substitution σ (here for two tile types T_1, T_2):

$$M_\sigma = \begin{pmatrix} \#\{\text{tiles of type } T_1 \text{ in } \sigma(T_1)\} & \#\{\text{tiles of type } T_1 \text{ in } \sigma(T_2)\} \\ \#\{\text{tiles of type } T_2 \text{ in } \sigma(T_1)\} & \#\{\text{tiles of type } T_2 \text{ in } \sigma(T_2)\} \end{pmatrix}$$

First example: $M_\sigma = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$

Penrose tiling: $M_\sigma = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

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Facts:

- The leading eigenvalue η of M_σ equals λ^d (λ the inflation factor, d the dimension)
- The (right) eigenvector corr. to η contains the relative frequencies of the tile types.
- The left eigenvector corr. to η contains the areas of the tile types.

First example:

- Eigenvalues 4 and 1. Inflation factor 2, and $4 = 2^2$.
- Right eigenvector $\binom{1}{1}$, hence T_1 and T_2 have equal frequency.
- Left eigenvector $(1, 2)$: T_2 has twice the area as T_1.

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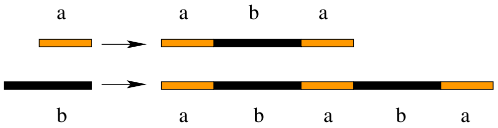
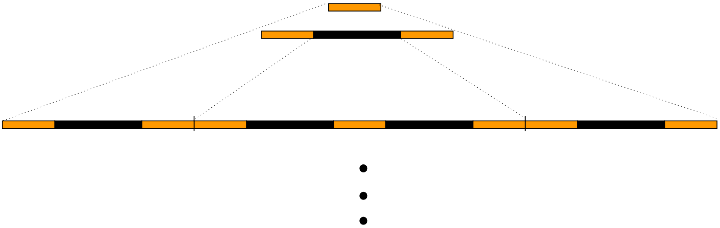
Penrose substitution:

- Eigenvalues $\frac{3+\sqrt{5}}{2} = (\frac{1+\sqrt{5}}{2})^2$ and $\frac{3-\sqrt{5}}{2}$.
  Inflation factor $\tau = \frac{1+\sqrt{5}}{2}$ (golden mean)
- Right eigenvector $\binom{1}{\tau}$,
  hence frequency(T_1) : frequency$(T_2) = \tau : 1$.
- Left eigenvector $(1, \tau)$: area$(T_2) = \tau$ area(T_1).

For much more examples visit the zoo of substitution tilings:
`tilings.math.uni-bielefeld.de`

A one-dimensional example:





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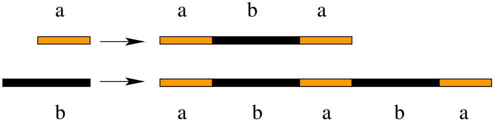

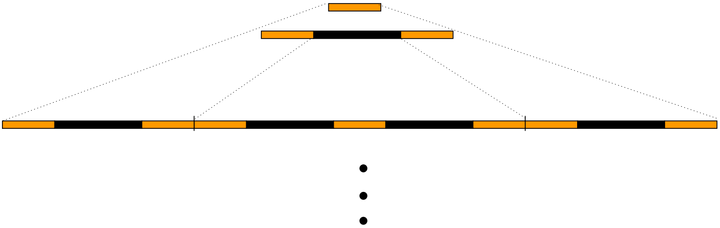

- $M_\sigma = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$
- Inflation factor $2 + \sqrt{3}$
- length$(a) = 1$, length$(b) = \sqrt{3}$
- frequency(a) : frequency$(b) = \sqrt{3} : 1$

Cut-and-Project Sets

$$\begin{array}{ccccc} E_{||} = \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{R}^e & \xrightarrow{\pi_2} & \mathbb{R}^e = E_\perp \\ \cup & & \cup & & \cup \\ \Lambda & & \Gamma & & W \end{array}$$

- Γ a *lattice* in $\mathbb{R}^d \times \mathbb{R}^e$
- π_1, π_2 *projections*
  - $\pi_1|_\Gamma$ injective
  - $\pi_2(\Gamma)$ dense
- W *compact* ("window", somehow nice, e.g. ∂W has zero measure)

Then $\Lambda = \{\pi_1(x) \,|\, x \in \Lambda, \pi_2(x) \in W\}$ is a (regular) *cut-and-project set* (CPS).

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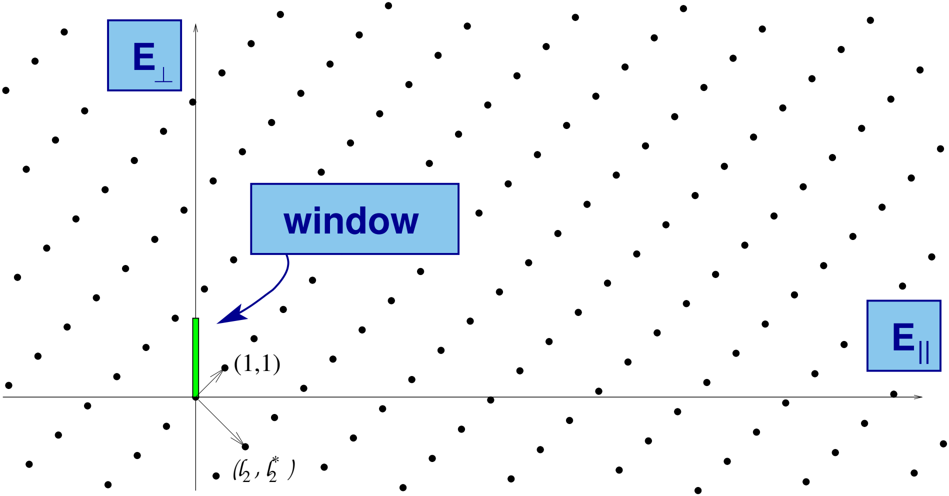
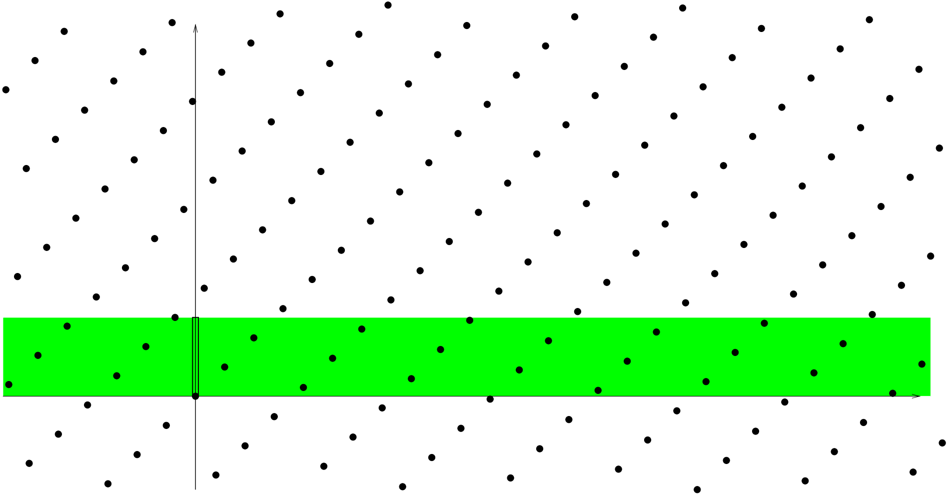
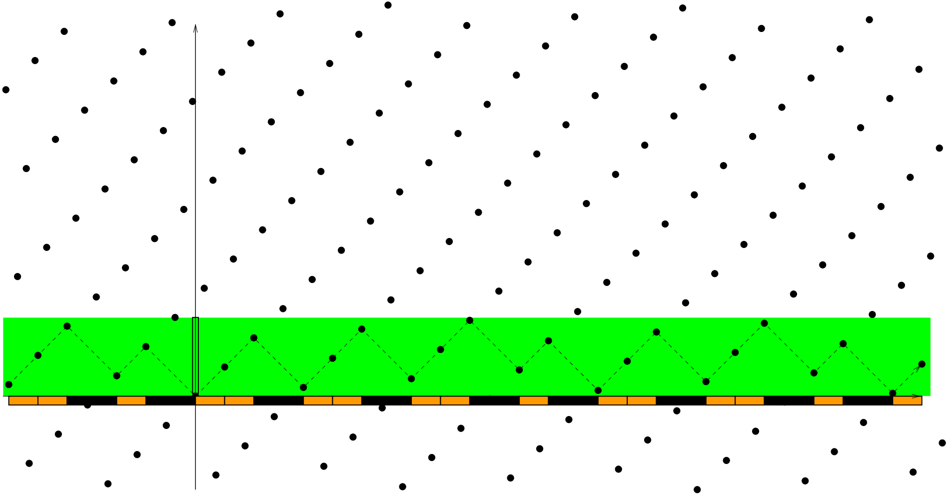
Then $\Lambda = \{\pi_1(x) \,|\, x \in \Lambda, \pi_2(x) \in W\}$ is a (regular) *cut-and-project set* (CPS).

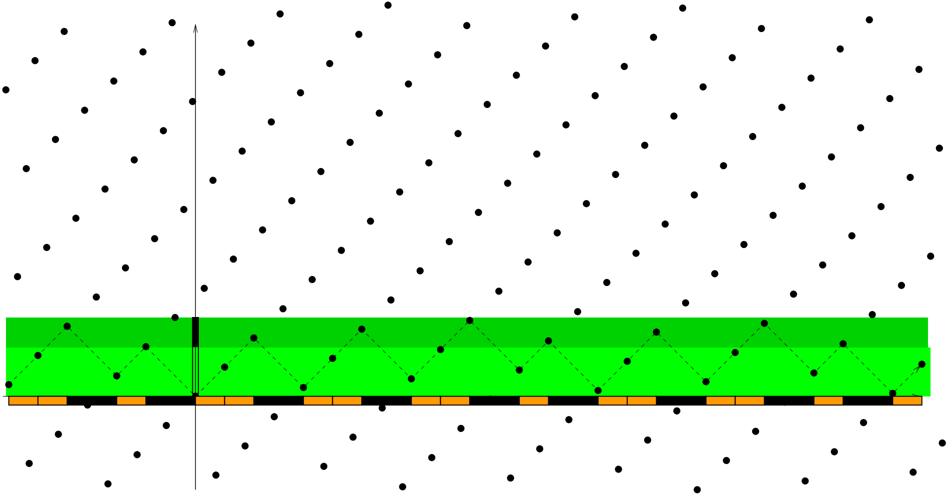
The *star map* : $\quad \star : \pi_1(\Lambda) \to \mathbb{R}^e, \; x^\star = \pi_2 \circ {\pi_1}^{-1}(x)$

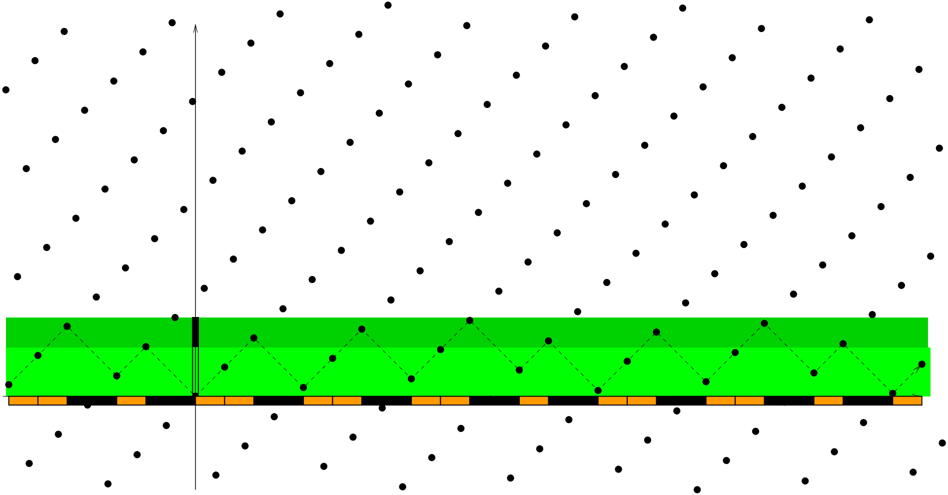


E⊥
window
(1,1)
E||







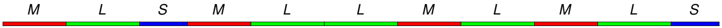


Star map: If x is some endpoint of some interval on the line, $x^\star$ is the preimage of x in W, on the vertical line.

The last one uses $d = e = 1$ ($E_{||} = \mathbb{R}^1, E_\perp = \mathbb{R}^1$).

An example with $d = 1$, $e = 2$:

$$\sigma : \quad S \to ML, \quad M \to SML, \quad L \to LML$$



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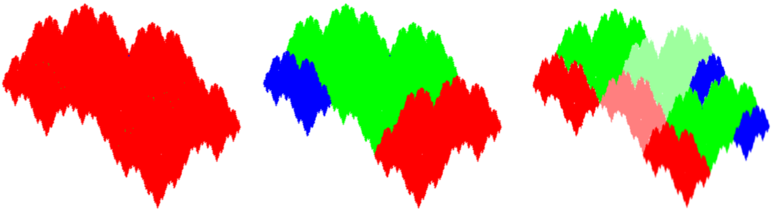
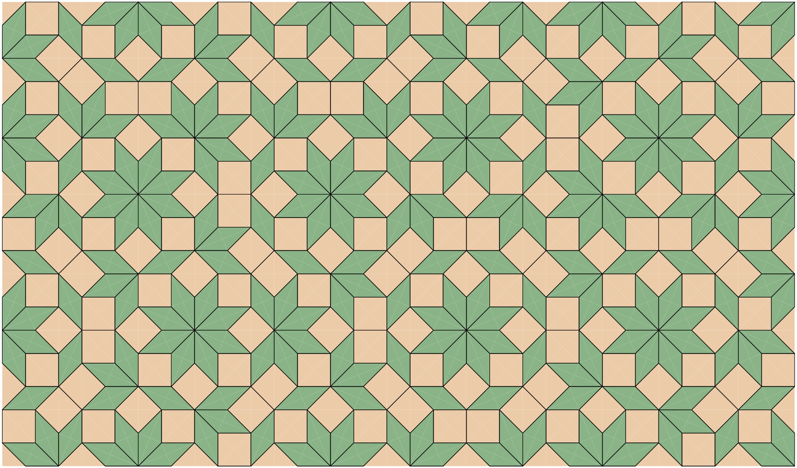
An example with $d = 1$, $e = 2$:

$$\sigma: \quad S \to ML, \quad M \to SML, \quad L \to LML$$

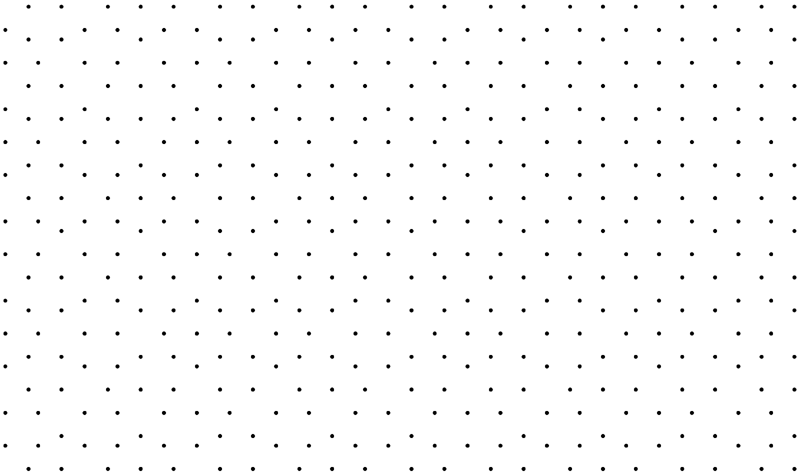
...uses a window W that looks like a fractal:



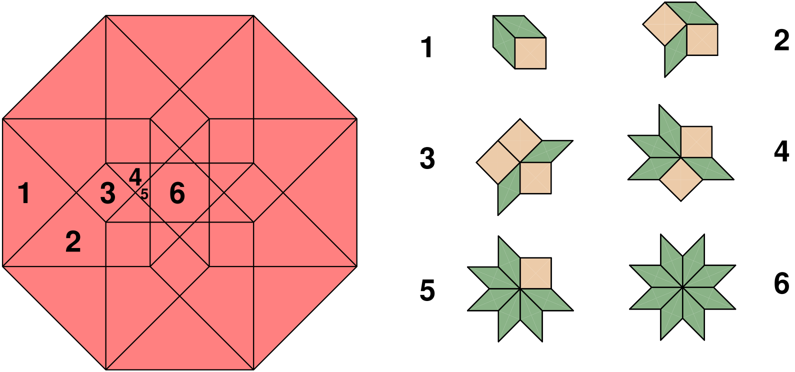
Several other substitution tilings can be obtained as CPS:



...respectively, the vertex set of the tiling:



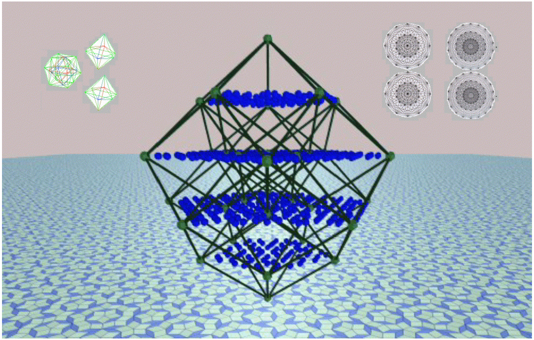
Here $d = 2$, $e = 2$. Window is an octagon:



For the Penrose pattern: slightly more complicated.

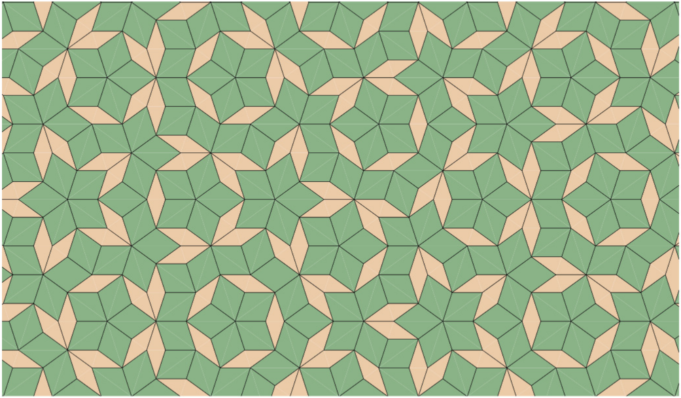
One can obtain it by projection from $R^2 \times \mathbb{R}^2$, but this requires some further techniques.

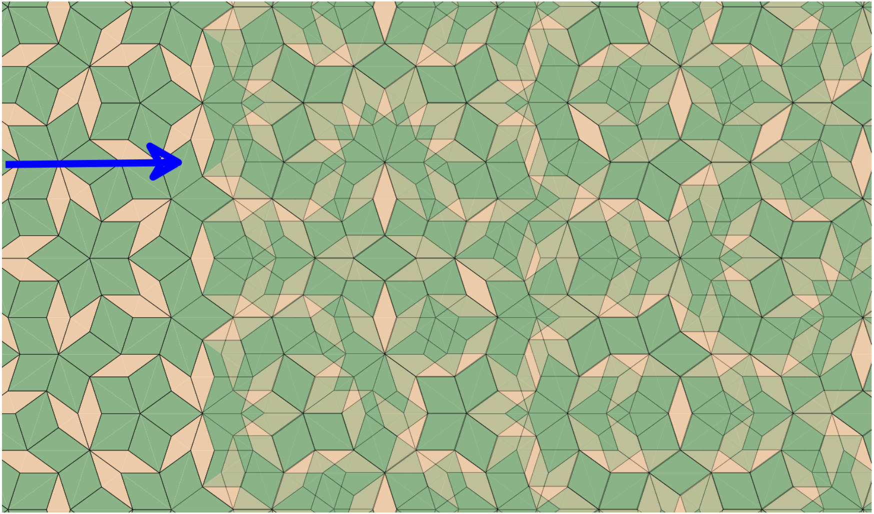
One obtains it by projection from $R^2 \times \mathbb{R}^3$ more easily.

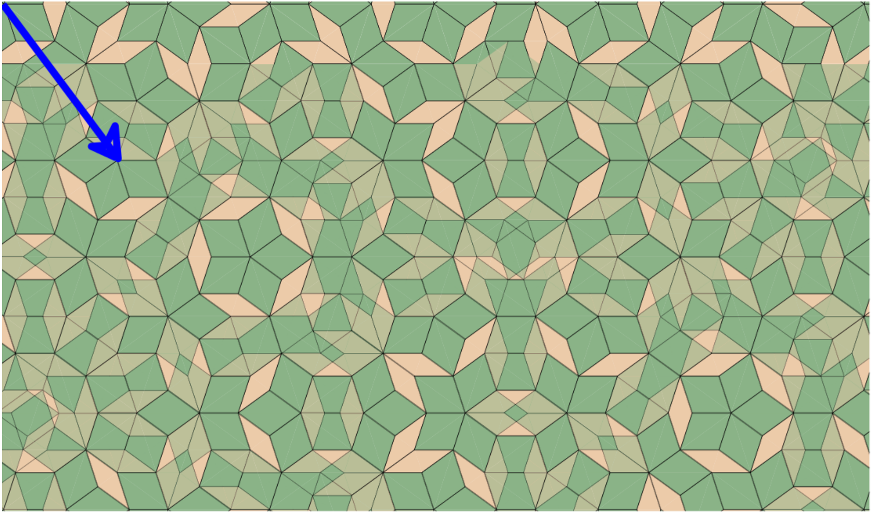


A Delone set Λ in $\mathbb{R}^d$ is **aperiodic**, of there is no $v \in \mathbb{R}^d$, $v \neq 0$, such that $\Lambda + t = \Lambda$.

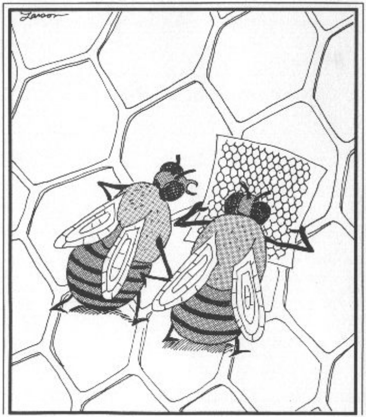
All previous examples are **aperiodic**. For instance the Penrose tiling:







Repetitive: there is $r > 0$ such that congruent copies of any local patch in Λ occur in every ball of radius r.



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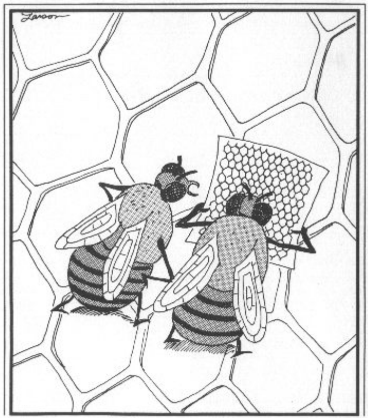

“Face it, Fred—you're lost!”

Linearly repetitive: r depends linearly on the diameter of the patch.

Theorem

If Λ is some CPS such that for all $x \in \Lambda$ holds: $x^\star \in interior(W)$, then Λ is repetitive.

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Not primitive: $\sigma : a \mapsto aa, \; b \mapsto aba$. Yields for instance

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Theorem

Primitive substitution tilings are linearly repetitive.

4. Average lattices

(Λ has an average lattice just means:
there is some lattice Γ such that $\Lambda \overset{\text{bd}}{\sim} \Gamma$.)

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(and btw: "quasicrystal" just means CPS)

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If Λ is a linearly repetitive Delone set in $\mathbb{R}^2$, then $\Lambda \stackrel{\text{bil}}{\sim} \mathbb{Z}^2$.

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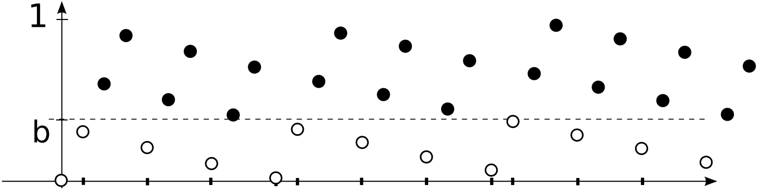

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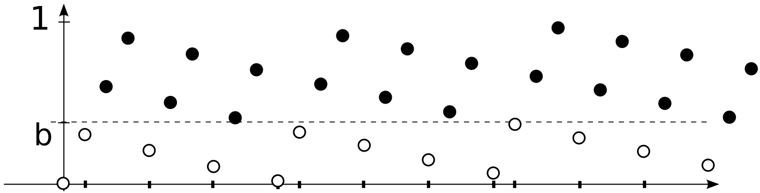

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One can state the argument in purely algebraic terms:

- X an $\mathbb{R}$-vector space (here $X = \mathbb{R}^2$),
- $X = V_p + V_i$ (here: horizontal + vertical), $W \subset V_i$ compact set (here $W = [0, b]$),
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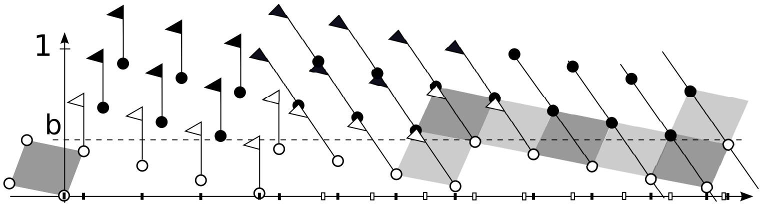

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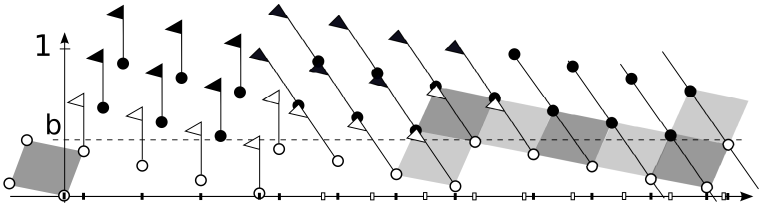

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- $Y = \pi_i^{-1}(W) \cap \Gamma$ (here: white points),
- $\Lambda = \pi_p(Y)$
- Z subgroup of X with $V_p + Z = X$, $Z/(Z \cap \Gamma)$ compact (here "lattice direction" for projection)
- π_Z corresponding projection etc...

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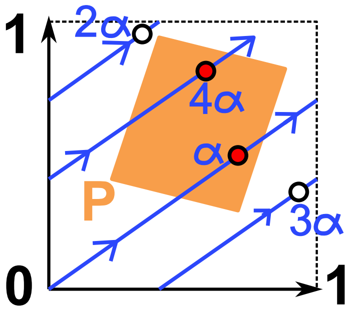
Last October I've learned from Alan Haynes that this was done already in

C. Godrèche and C. Oguey:
Construction of average lattices for quasiperiodic structures by the section method, *J. Phys. France* 51 (1990) 21-37

Grepstad and Lev (2015) obtained a generalisation of Kesten's theorem to d dimensions. Here for $d = 2$:



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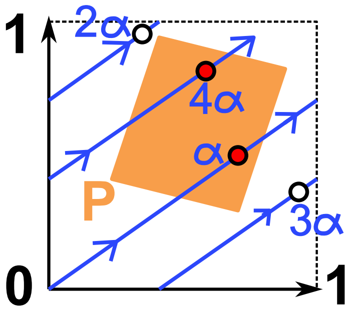

Slope $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1 \notin \mathbb{Q}$, $\alpha_2 \notin \mathbb{Q}$, $\frac{\alpha_1}{\alpha_2} \notin \mathbb{Q}$.

Consider $\alpha, 2\alpha, 3\alpha, \ldots$ mod 1. Colour those red that are in the parallelogram P. If the number of red points up to $n\alpha$ minus the expected value $n \cdot \text{area}(P)$ is bounded, then P is a **bounded remainder set** (BRS) with respect to α.

Theorem (Grepstad-Lev 2015)

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{R}^d$ such that $\alpha_i \notin \mathbb{Q}$ and $\frac{\alpha_i}{\alpha_j} \notin \mathbb{Q}$ for all $1 \le i < j \le d$.
If all edges of the parallelogram P are in $\mathbb{Z}^d + \alpha\mathbb{Z}^d$ then P is a BRS with respect to α.

For $d = 1$ this is the if-part of Kesten's theorem.

Grepstad and Lev obtain several further results, and in some sense also the only-if-part of Kesten's theorem.

One can translate it to CPS and use it to show that certain $1 \times e$-examples above are bounded distance equivalent to some lattice.

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A one-dimensional substitution tiling with inflation factor λ is a *Pisot substitution*, if all eigenvalues of M_σ other than λ are less than one in modulus. (E.g., many of the examples above: $a \to aba$, $b \to ababa$, or $S \to ML$, $M \to SML$, $L \to LML$)

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Unfortunately:

Theorem (Holton-Zamboni 1998)

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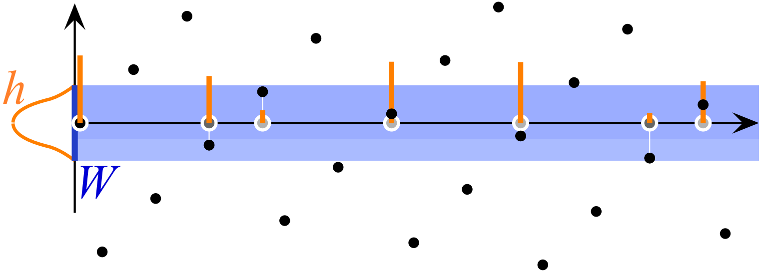
5. Weighted CPS

Take some CPS Λ and give each point a weight. One convenient way to write it: *Dirac comb*

$$\delta_{w,\Lambda} = \sum_{x \in \Lambda} w(x)\delta_x \qquad (w(x) \in \mathbb{R}, \delta_x \text{ the Dirac measure in } x)$$

If $w(x) = h(x^*)$ for $h : W \to \mathbb{R}$ continuous, then $\delta_{w,\Lambda}$ is called a *weighted CPS*.



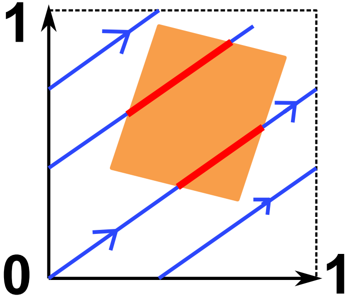
Theorem (F-Garber 2017 preprint)

Let $\delta_{w,\Lambda}$ be a weighted CPS with $e = d = 1$. Let $W = [a, b]$, $w(x) = h(x^*)$ and $h(a) = h(b) = 0$. If h is

1. *piecewise linear, or*
2. *twice differentiable,*

then $\delta_{w,\Lambda}$ is bounded distance equivalent to $c\mu$ for some $c > 0$, where μ denotes the one-dimensional Lebesgue measure.

This proof relies a lot on another result by Mrs Grepstad (this one with G. Larcher) on continuous BRS.



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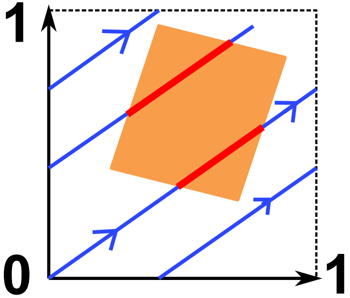
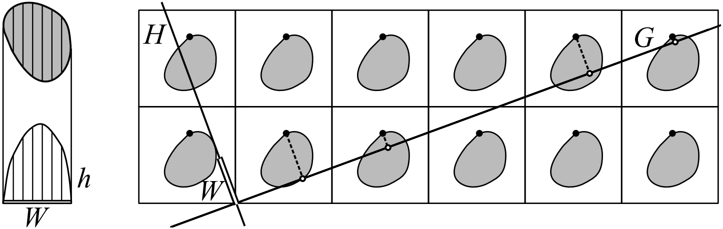


Theorem (Grepstad-Larcher 2017+)

For almost all $\alpha > 0$, every polygon $P \subset [0,1]^2$ with no edge of slope α is a BRS for the continuous irrational rotation with slope α.

In plain words: The length of the red part of the line segment $\{t\alpha \mod 1 \,|\, 0 \leq t \leq T\}$ does not deviate from the expected value $T/\operatorname{vol}(P)$ by more than some $C > 0$.

We use a certain weighted CPS tailored to the result above



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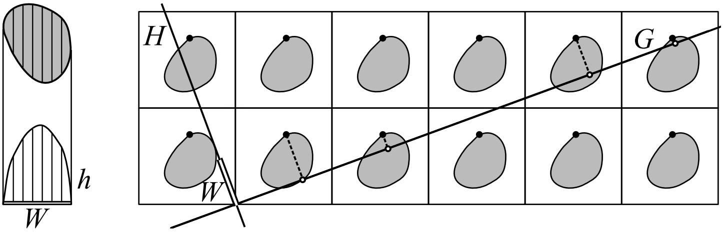

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Finally, our first new result! (at least we hope so...)

More in

D.F., Alexey Garber:
Bounded distance and bilipschitz equivalence of Delone sets, preprint,
www.math.uni-bielefeld.de/∼frettloe/papers/bilip-draft.pdf

and references therein (but it is already slightly outdated).

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Thank you!