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Noise-Induced Phenomena in Slow–Fast Dynamical Systems

Joint work with Nils Berglund (CPT–CNRS Marseille)



Leibniz
Gemeinschaft

Introduction: Small random perturbations of ODEs

Deterministic ODE $\dot{x}_t^{\text{det}} = f(x_t^{\text{det}})$ with $x_0^{\text{det}} \in \mathbb{R}^n$

Small random perturbation $dx_t = f(x_t) dt + \sigma F(x_t) dW_t$ with $x_0 = x_0^{\text{det}}$

where

- ▷ $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $F : \mathbb{R}^n \rightarrow \mathbb{R}^{k \times n}$
- ▷ $\{W_t\}_{t \geq 0}$ k -dim. standard Brownian motion
- ▷ $\sigma > 0$ small

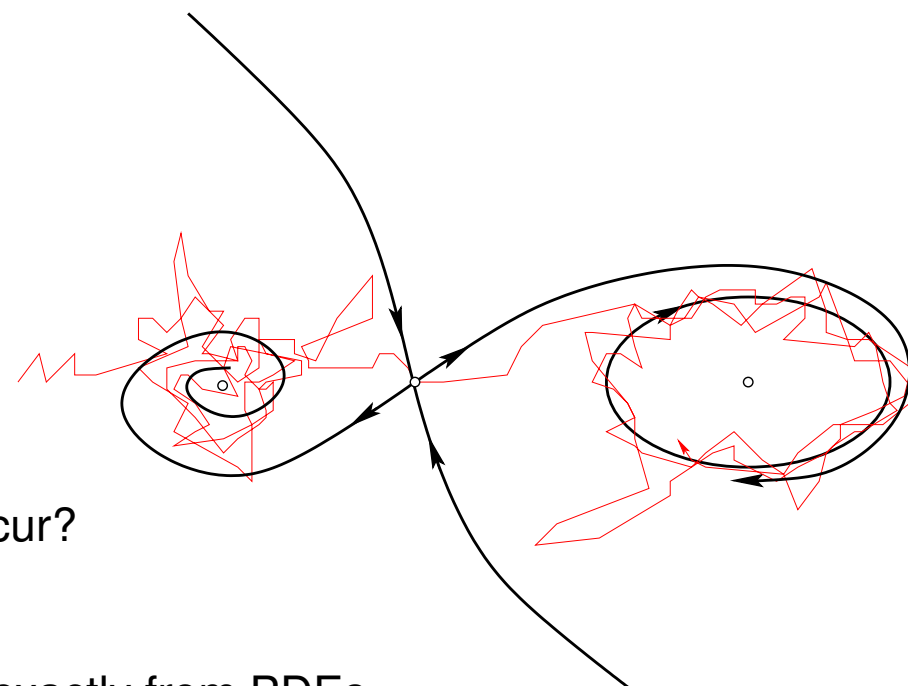
Noise enables transitions between basins of attraction

Questions

- ▷ Transition times? Where do transitions typically occur?

Answers

- ▷ *Mean* first-exit times and locations can be obtained exactly from PDEs (via infinitesimal generators of the diffusion)
- ▷ Exponential asymptotics provided by Wentzell–Freidlin theory (via variational principle)
- ▷ Rigorous proof of subexponential asymptotics for reversible diffusions is recent ($n > 1$) [Bovier, Eckhoff, Gayraud & Klein, 2004/05]



General slow-fast systems with noise: Definition

Fully coupled stochastic differential equations on two well-separated time scales

$$\begin{cases} dx_t = \frac{1}{\varepsilon} f(x_t, y_t) dt + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t) dW_t & \text{(fast variables } \in \mathbb{R}^n) \\ dy_t = g(x_t, y_t) dt + \sigma' G(x_t, y_t) dW_t & \text{(slow variables } \in \mathbb{R}^m) \end{cases}$$

where

- ▷ $\{W_t\}_{t \geq 0}$ k -dimensional standard Brownian motion
- ▷ $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$
- ▷ $f : \mathcal{D} \rightarrow \mathbb{R}^n$, $g : \mathcal{D} \rightarrow \mathbb{R}^m$ drift coefficients, $\in \mathcal{C}^2$
- ▷ $F : \mathcal{D} \rightarrow \mathbb{R}^{n \times k}$, $G : \mathcal{D} \rightarrow \mathbb{R}^{m \times k}$ diffusion coefficients, $\in \mathcal{C}^1$

Note diffusive scaling: $\text{Var}\left(\frac{\sigma}{\sqrt{\varepsilon}} \int_0^t F(x_s, y_s) dW_s\right) = \frac{\sigma^2}{\varepsilon} \int_0^t \mathbb{E}(F(x_s, y_s)^2) ds$ (for $k = n = 1$)

Small parameters

- ▷ $\varepsilon > 0$ adiabatic parameter (*no quasistatic* approach)
- ▷ $\sigma, \sigma' \geq 0$ noise intensities; may depend on ε : $\sigma = \sigma(\varepsilon)$, $\sigma' = \sigma'(\varepsilon)$ and $\sigma'(\varepsilon)/\sigma(\varepsilon) = \rho(\varepsilon) \leq 1$

Time scales

- ▷ Aiming at regime $T_{\text{relax}} = \mathcal{O}(\varepsilon) \ll T_{\text{driving}} = 1 \ll T_{\text{Kramers}} = \varepsilon e^{\bar{V}/\sigma^2}$ (in slow time)

Assumptions on deterministic fast variables

Existence of a slow manifold

$\exists \mathcal{D}_0 \subset \mathbb{R}^m \quad \exists x^* : \mathcal{D}_0 \rightarrow \mathbb{R}^n \quad \text{s.t.} \quad (x^*(y), y) \in \mathcal{D} \quad \text{and} \quad f(x^*(y), y) = 0 \quad \text{on} \quad \mathcal{D}_0$

Slow manifold is attracting

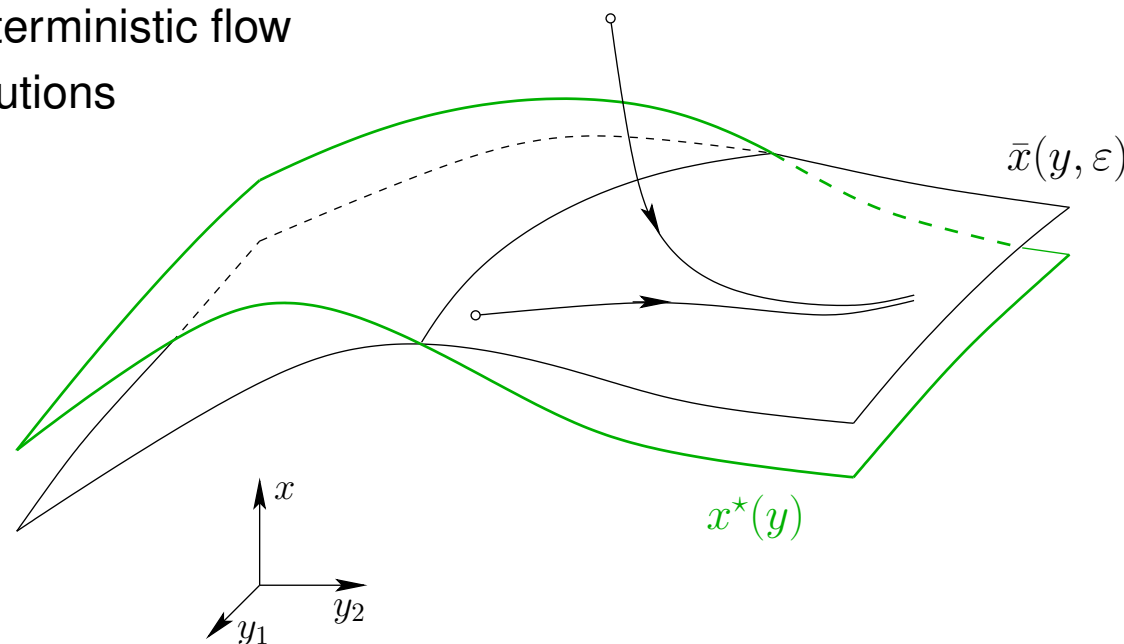
Eigenvalues of $A^*(y) := \partial_x f(x^*(y), y)$ satisfy $\text{Re } \lambda_i(y) \leq -a_0 < 0$, uniformly on \mathcal{D}_0

Theorem [Tihonov 1952, Fenichel 1979]

There exists an *adiabatic manifold*: $\exists \bar{x}(y, \varepsilon)$ s.t.

- ▷ $\bar{x}(y, \varepsilon)$ is an invariant manifold for the deterministic flow
- ▷ $\bar{x}(y, \varepsilon)$ attracts nearby (deterministic) solutions
- ▷ $\bar{x}(y, 0) = x^*(y)$ and $\bar{x}(y, \varepsilon) = x^*(y) + \mathcal{O}(\varepsilon)$

Consider now *stochastic* system under these assumptions



Noisy slow–fast systems: Defining typical neighbourhoods of adiabatic manifolds

- ▷ Consider deterministic process $(x_t^{\text{det}} = \bar{x}(y_t^{\text{det}}, \varepsilon), y_t^{\text{det}})$ on (invariant) adiabatic manifold
- ▷ Deviation $\xi_t := x_t - x_t^{\text{det}}$ of random fast variables from adiabatic manifold
- ▷ Linearize SDE for ξ_t
- ▷ Resulting process ξ_t^0 is Gaussian

Key observation

$\frac{1}{\sigma^2} \text{Cov } \xi_t^0$ is a particular solution of the deterministic slow–fast system

$$\begin{cases} \varepsilon \dot{X}(t) = A(y_t^{\text{det}})X(t) + X(t)A(y_t^{\text{det}})^T + F_0(y_t^{\text{det}})F_0(y_t^{\text{det}})^T \\ \dot{y}_t^{\text{det}} = g(\bar{x}(y_t^{\text{det}}, \varepsilon), y_t^{\text{det}}) \end{cases}$$

where $A(y) = \partial_x f(\bar{x}(y, \varepsilon), y)$ and F_0 is zeroth-order approximation to F

- ▷ System admits an adiabatic manifold $\bar{X}(y, \varepsilon)$

Typical neighbourhoods

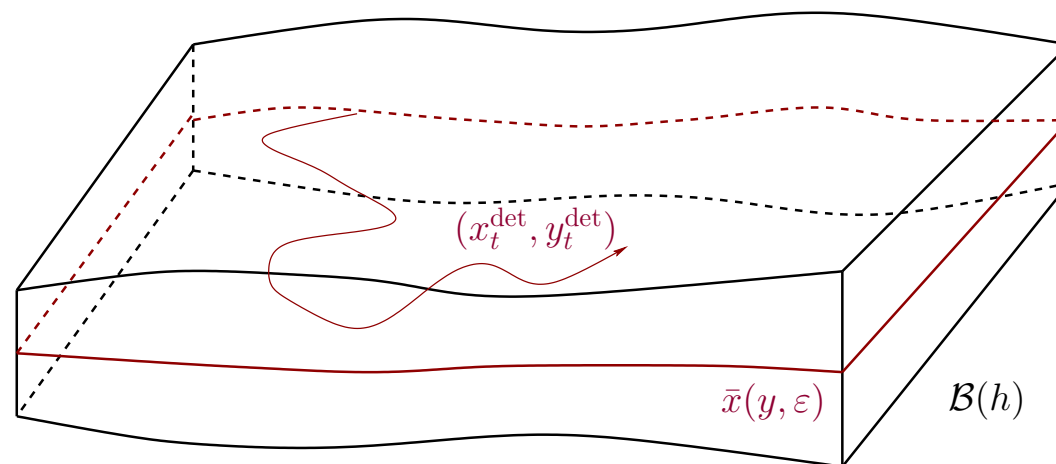
$$\mathcal{B}(h) := \{(x, y) : \langle [x - \bar{x}(y, \varepsilon)], \bar{X}(y, \varepsilon)^{-1}[x - \bar{x}(y, \varepsilon)] \rangle < h^2\}$$

Noisy slow-fast systems: Concentration of sample paths near adiabatic manifolds

Define (random) first-exit times

$$\tau_{\mathcal{D}_0} := \inf\{s > 0 : y_s \notin \mathcal{D}_0\}$$

$$\tau_{\mathcal{B}(h)} := \inf\{s > 0 : (x_s, y_s) \notin \mathcal{B}(h)\}$$



Theorem [Berglund & G, J. Differential Equations, 2003]

Assume: $\|\bar{X}(y, \varepsilon)\|$, $\|\bar{X}(y, \varepsilon)^{-1}\|$ are uniformly bounded in \mathcal{D}_0

Then: $\exists \varepsilon_0 > 0 \quad \exists h_0 > 0 \quad \forall \varepsilon \leq \varepsilon_0 \quad \forall h \leq h_0$

$$\mathbb{P}\{\tau_{\mathcal{B}(h)} < \min(t, \tau_{\mathcal{D}_0})\} \leq C_{n,m}(t) \exp\left\{-\frac{h^2}{2\sigma^2}[1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)]\right\}$$

where $C_{n,m}(t) = [C^m + h^{-n}] \left(1 + \frac{t}{\varepsilon^2}\right)$

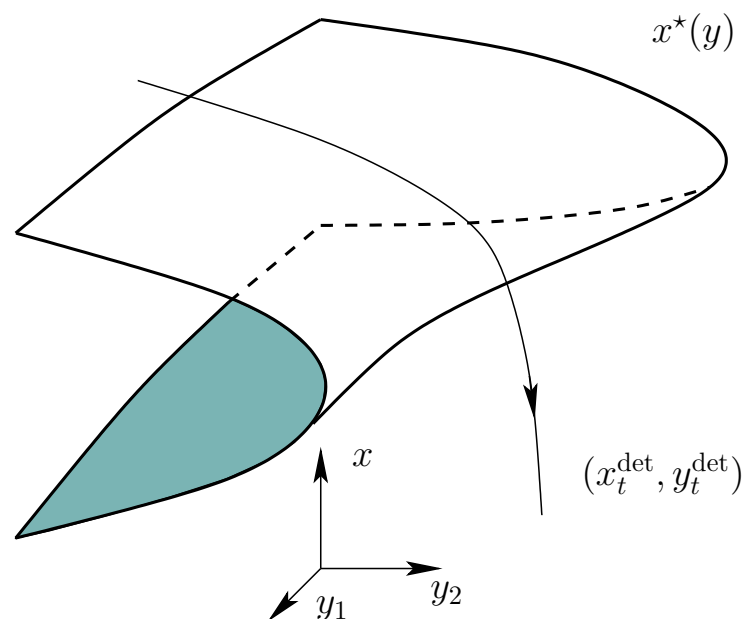
Remarks

- ▷ Bound is sharp: Lower bound similar
- ▷ If initial condition not on $\bar{x}(y, \varepsilon)$: additional transitional phase
- ▷ On longer time scales: Behaviour of slow variables becomes crucial (Assumptions on g)

Question

What happens if (x_t, y_t) approaches a bifurcation point (\hat{x}, \hat{y}) for the deterministic dynamics?

Example: Saddle-node bifurcation



General approach

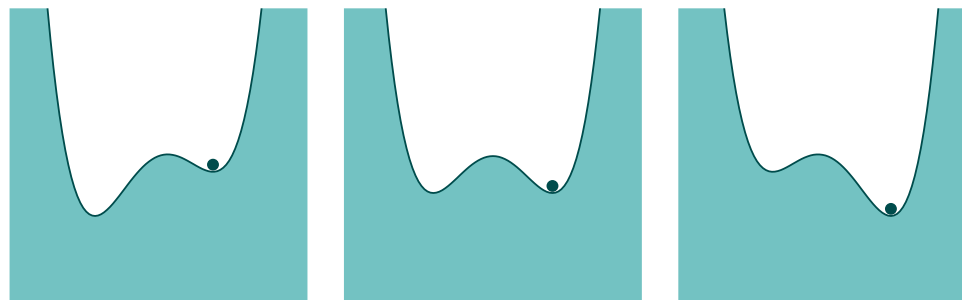
- ▷ Apply centre-manifold theorem
- ▷ Concentration results for deviation from centre manifold [Berglund & G, 2003]
- ▷ Consider reduced dynamics on centre manifold
- ▷ Concentration results for deviation of reduced system from original variables [Berglund & G, 2003]

Interesting phenomena for one-dimensional reduced dynamics

- ▷ Reduction of bifurcation delay due to noise
- ▷ **Stochastic resonance**
- ▷ Effect of noise on the size of hysteresis cycles

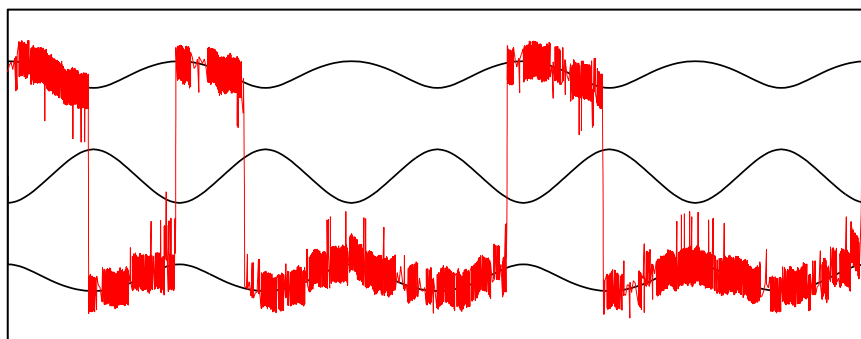
Example: Stochastic resonance

Overdamped motion of a Brownian particle in a periodically modulated double-well potential

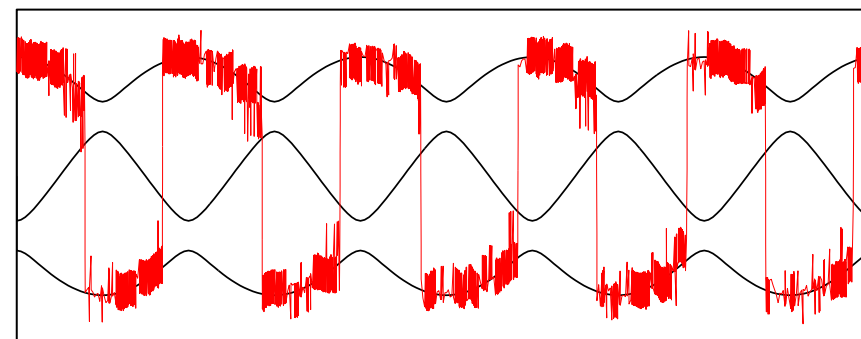


$$dx_t = -\frac{1}{\varepsilon} \frac{\partial}{\partial x} V(x_t, t) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t \quad \text{with} \quad V(x, t) = \frac{1}{4}x^4 - \frac{1}{2}x^2 - A \cos(t)x \quad (\text{where } A < A_c)$$

Simulations

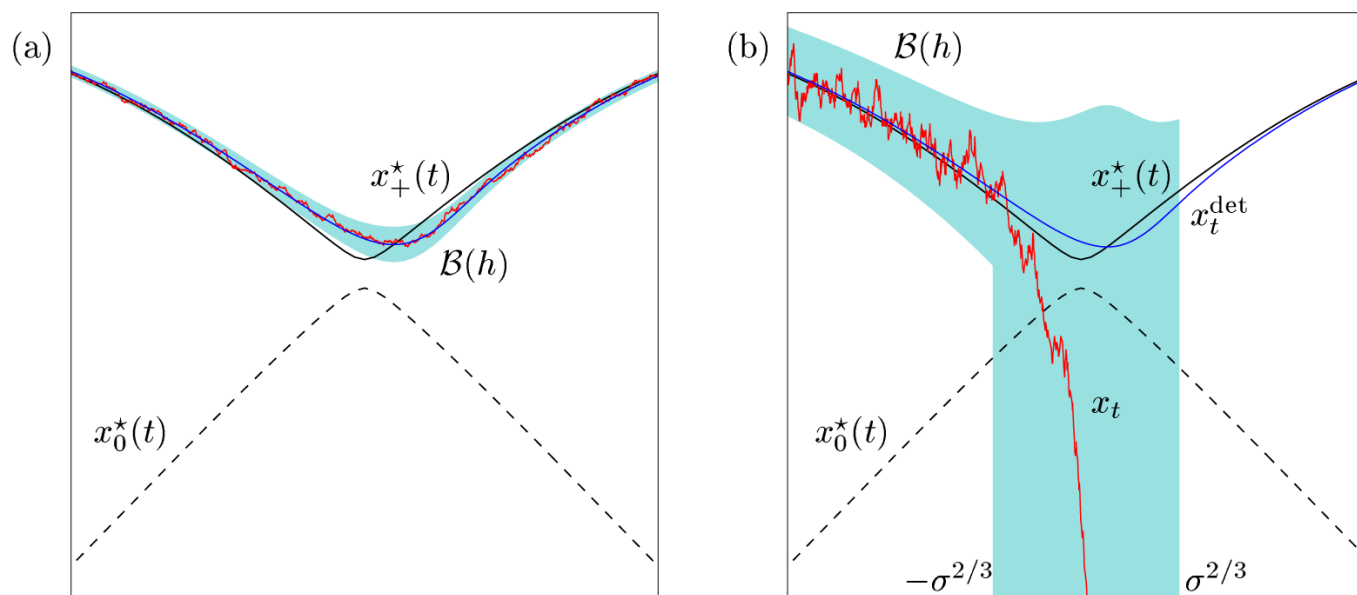


$$A = 0.24, \sigma = 0.20, \varepsilon = 0.001$$



$$A = 0.35, \sigma = 0.20, \varepsilon = 0.001$$

Sample paths behaviour for stochastic resonance



Theorem [Berglund & G, 2002]

There exists a threshold value $\sigma_c = (a_0 \vee \varepsilon)^{3/4}$ s.t.

Below threshold $\sigma \ll \sigma_c = (a_0 \vee \varepsilon)^{3/4}$:

- ▷ Transitions unlikely, probability for transition $\leq e^{-const \sigma_c^2 / \sigma^2}$
- ▷ Sample paths concentrated near bottom of well; typical spreading depends on local curvature

Above threshold $\sigma \gg \sigma_c = (a_0 \vee \varepsilon)^{3/4}$:

- ▷ 2 transitions per period likely (back and forth) with probability $\geq 1 - e^{-const \sigma^{4/3} / \varepsilon |\log \sigma|}$
- ▷ Transitions likely when barrier low; transition window has width $\asymp \sigma^{2/3}$

General results on sample-path behaviour in slow-fast systems

- ▷ *Noise-Induced Phenomena in Slow-Fast Dynamical Systems. A Sample-Paths Approach*, “Probability and its Applications”, Springer, London, 2005
- ▷ *Geometric singular perturbation theory for stochastic differential equations*, J. Differential Equations **191**, 1–54 (2003)

Case studies: Bifurcations in slowly driven systems

- ▷ *Pathwise description of dynamic pitchfork bifurcations with additive noise*, Probab. Theory Related Fields **122**, 341–388 (2002)
- ▷ *A sample-paths approach to noise-induced synchronization: Stochastic resonance in a double-well potential*, Ann. Appl. Probab. **12**, 1419–1470 (2002)
- ▷ *The effect of additive noise on dynamical hysteresis*, Nonlinearity **15**, 605–632 (2002)

Passage through an unstable periodic orbit

- ▷ *On the noise-induced passage through an unstable periodic orbit I: Two-level model*, J. Statist. Phys. **114**, 1577–1618 (2004)
- ▷ *Universality of first-passage- and residence-time distributions in non-adiabatic stochastic resonance*, Europhys. Lett. **70**, 1–7 (2005)

