

Mathematics and Statistics Colloquium

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The effect of noise on slow-fast systems

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Overview

Introduction: Classical results for autonomous systems

- ▷ Small random perturbations of dynamical systems
- ▷ Exponential asymptotics for first-exit times (Wentzell–Freidlin)
- ▷ Subexponential asymptotics

Slowly time-dependent systems and stochastic resonance

- ▷ The motion of a Brownian particle in a double-well potential
- ▷ Simulations
- ▷ Rigorous results
- ▷ Deterministic dynamics
- ▷ Stochastic dynamics for noise intensities below threshold
- ▷ Stochastic dynamics for noise intensities above threshold

General slow–fast systems

- ▷ Dynamics near slow manifolds
- ▷ Bifurcations and reduced dynamics

Autonomous dynamical systems: ODEs

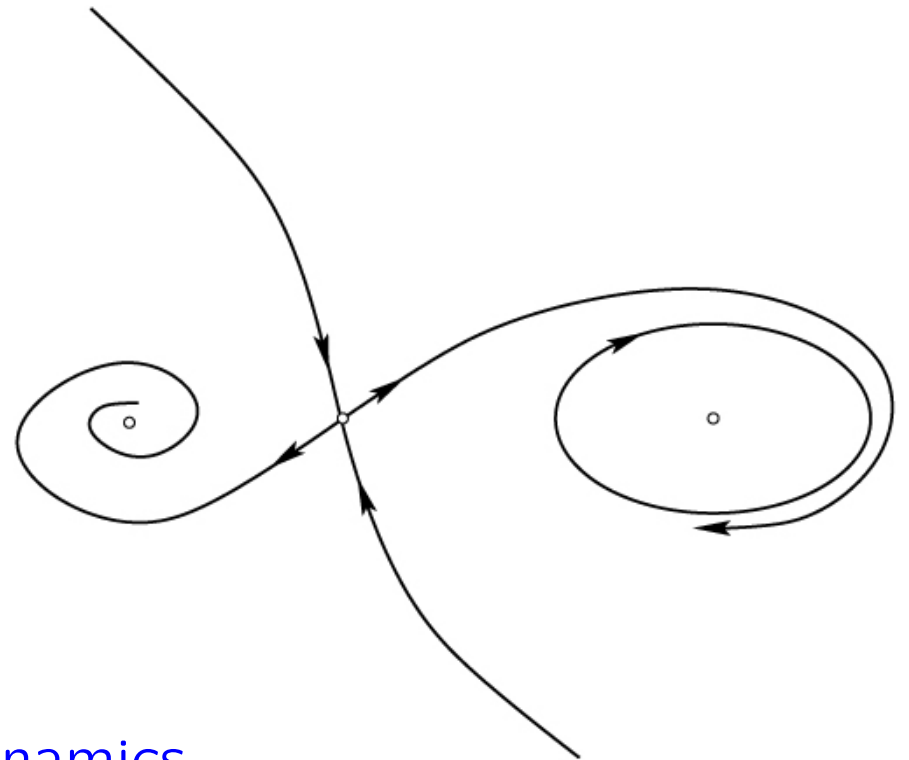
Deterministic ODE

$$\dot{x}_t^{\text{det}} = f(x_t^{\text{det}}), \quad x_0^{\text{det}} \in \mathbb{R}^d$$

with

$$f : \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad f \text{ "well-behaved"}$$

(Existence and uniqueness of solution)



Assumptions on deterministic dynamics

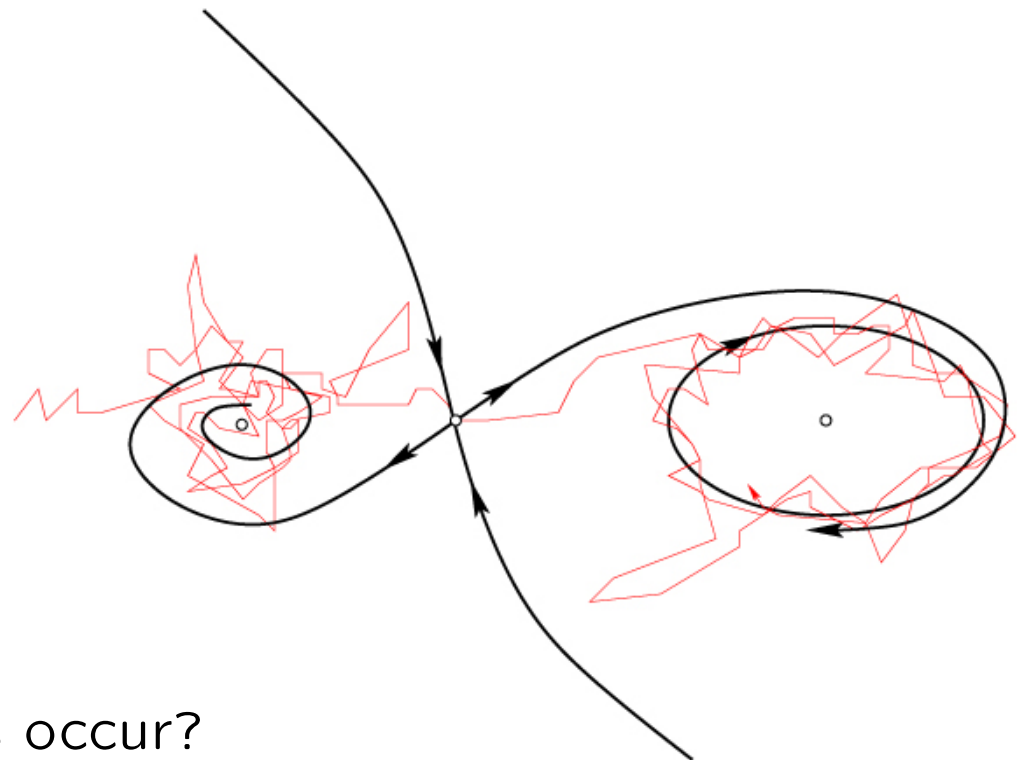
- ▷ Attractors $\mathcal{A}_1, \mathcal{A}_2, \dots$
- ▷ Domains of attraction $\mathcal{B}_1, \mathcal{B}_2, \dots$

Small random perturbations of autonomous systems

$$dx_t = f(x_t) dt + \sigma dW_t, \quad x_0 = x_0^{\text{det}} \in \mathbb{R}^d$$

- ▷ $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ “well-behaved”
- ▷ $\{W_t\}_{t \geq 0}$ d -dim. (standard-) Brownian motion
- ▷ $\sigma > 0$ small

Noise enables transitions
between
domains of attraction



Questions

Transition times?

Transition probabilities?

Where do typical transitions occur?

Diffusion exit from a domain: Exit problem

Bounded domain $\mathcal{D} \ni x_0$ (with smooth boundary)

- ▷ *first-exit time* $\tau = \tau_{\mathcal{D}} = \inf\{t > 0: x_t \notin \mathcal{D}\}$
- ▷ *first-exit location* $x_{\tau} \in \partial\mathcal{D}$

Distribution of τ and x_{τ} ?

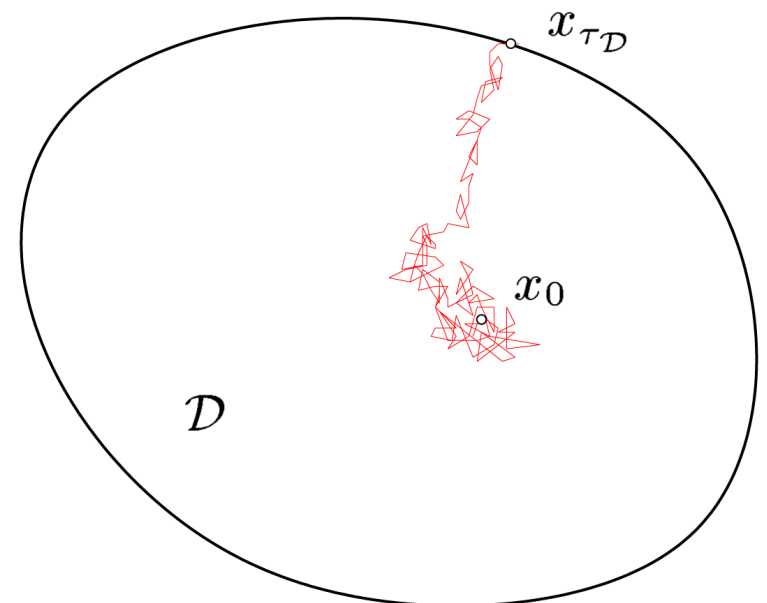
Interesting case

\mathcal{D} **positively invariant**

under deterministic flow

Approaches

- ▷ Mean first-exit times and locations via PDEs
- ▷ Exponential asymptotics via Wentzell–Freidlin theory



Exponential asymptotics: Large deviations

Large-deviation rate function

$$I_{[0,t]}(\varphi) = \frac{1}{2} \int_0^t \|\dot{\varphi}_s - f(\varphi_s)\|^2 ds \quad \text{for } \varphi \in H_1$$

$$I_{[0,t]}(\varphi) = +\infty \quad \text{otherwise}$$

Large-deviation principle

Probability $\sim \exp\{-I(\varphi)/\sigma^2\}$ to observe sample paths close to φ

Assumptions (for the next two slides)

- ▷ \mathcal{D} positively invariant
- ▷ unique, asymptotically stable equilibrium point at $0 \in \mathcal{D}$
- ▷ $\partial\mathcal{D} \subset$ basin of attraction of 0 (non-characteristic boundary)

Wentzell–Freidlin theory I

Quasipotential

- ▷ Quasipotential *with respect to* 0:

Cost to go **against the flow** from 0 to z

$$V(0, z) = \inf_{t > 0} \inf \{ I_{[0, t]}(\varphi) : \varphi \in \mathcal{C}([0, t], \mathbb{R}^d), \varphi_0 = 0, \varphi_t = z \}$$

- ▷ Minimum of quasipotential on boundary $\partial\mathcal{D}$:

$$\bar{V} := \min_{z \in \partial\mathcal{D}} V(0, z)$$

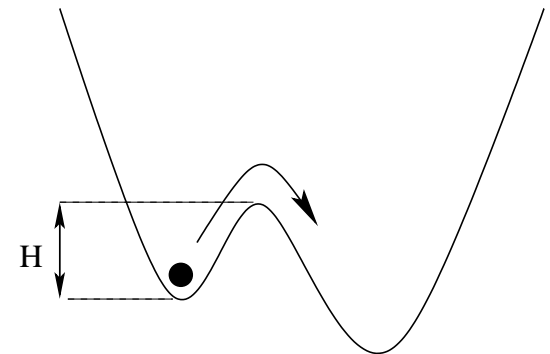
Gradient case (reversible diffusion)

Drift coefficient deriving from potential:

$$f = -\nabla V$$

- ▷ Cost for leaving potential well: $\bar{V} = 2H$
- ▷ Attained for paths against the flow:

$$\dot{\varphi}_t = -f(\varphi_t)$$



Wentzell–Freidlin theory II

Theorem [Wentzell & Freidlin \geq '70,'84] (under above assumptions)

For arbitrary initial condition $x_0 \in \mathcal{D}$

- ▷ Mean first-exit time

$$\mathbb{E}^{x_0}\{\tau\} \sim e^{\bar{V}/\sigma^2} \quad \text{as } \sigma \rightarrow 0$$

- ▷ Concentration of first-exit times

$$\mathbb{P}^{x_0}\left\{e^{(\bar{V}-\delta)/\sigma^2} \leq \tau \leq e^{(\bar{V}+\delta)/\sigma^2}\right\} \rightarrow 1 \quad \text{as } \sigma \rightarrow 0 \quad (\delta > 0)$$

- ▷ Concentration of exit locations near minima of quasipotential

$$\mathbb{P}^{x_0}\left\{\|x_\tau - z^*\| < \delta\right\} \rightarrow 1 \quad \text{as } \sigma \rightarrow 0 \quad (\delta > 0)$$

(z^* unique minimum of $z \mapsto V(0, z)$ on $\partial\mathcal{D}$)

Refined results in the gradient case

Simplest case: V double-well potential

First-hitting time τ^{hit} of deeper well

$$\triangleright \mathbb{E}^{x_1}\{\tau^{\text{hit}}\} = c(\sigma) e^{2[V(z)-V(x_1)]/\sigma^2}$$

$$\triangleright \lim_{\sigma \rightarrow 0} c(\sigma) = \frac{2\pi}{|\lambda_1(z)|} \sqrt{\frac{|\det \nabla^2 V(z)|}{\det \nabla^2 V(x_1)}} \quad \text{exists !}$$

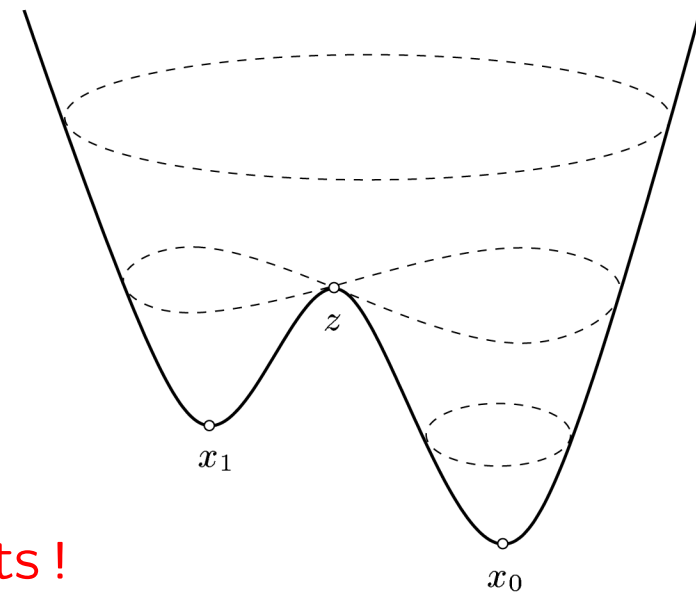
$\lambda_1(z)$ unique negative e.v. of $\nabla^2 V(z)$

([Eyring '35], [Kramers '40]; [Bovier, Gaynard, Eckhoff, Klein '02–'05],
[Helffer, Klein, Nier '04])

\triangleright **Subexponential** asymptotics known; related to geometry at well and saddle / small eigenvalues of the generator

$$\triangleright \tau^{\text{hit}} \approx \text{exp. distributed: } \lim_{\sigma \rightarrow 0} \mathbb{P}\{\tau^{\text{hit}} > t \mathbb{E} \tau^{\text{hit}}\} = e^{-t}$$

([Day '82], [Bovier *et al* '02])



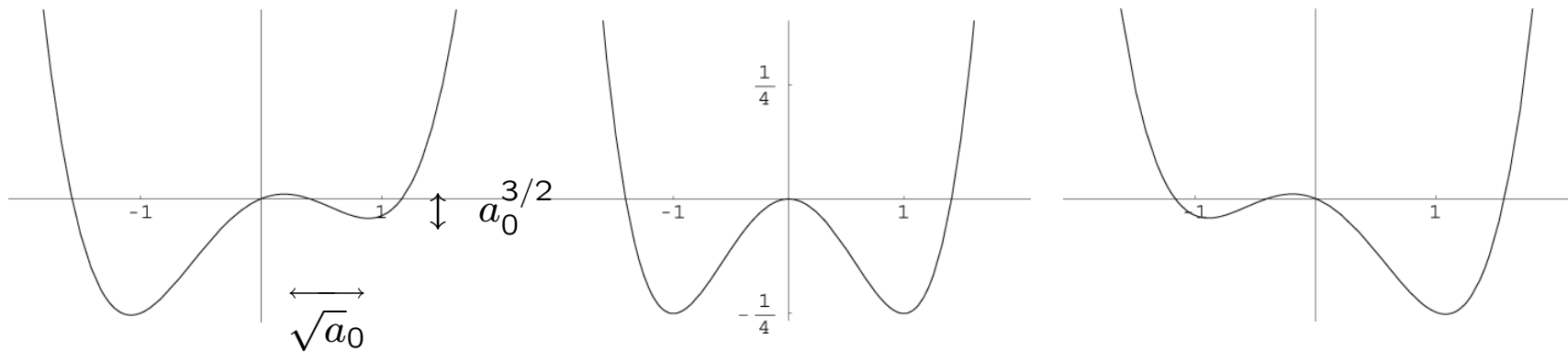
Slowly time-dependent systems

Overdamped motion of a Brownian particle

$$dx_s = -\frac{\partial}{\partial x} V(x_s, \varepsilon s) ds + \sigma dW_s$$

in a periodically modulated potential

$$V(x, \varepsilon s) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + (\lambda_c - a_0) \cos(2\pi\varepsilon s)x$$



$V(x, 0)$

$V(x, 1/4) = V(x, 3/4)$

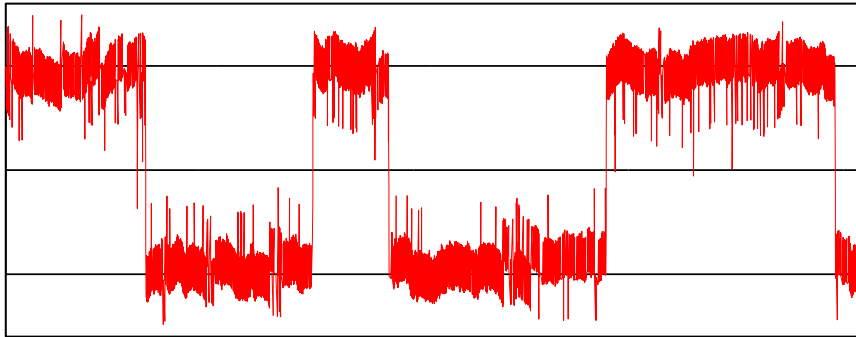
$V(x, 1/2)$

Sample paths

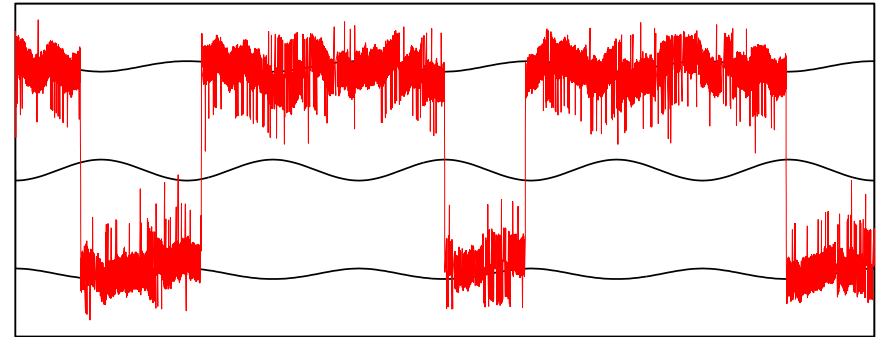
Amplitude of modulation $A = \lambda_c - a_0$

Speed of modulation ε

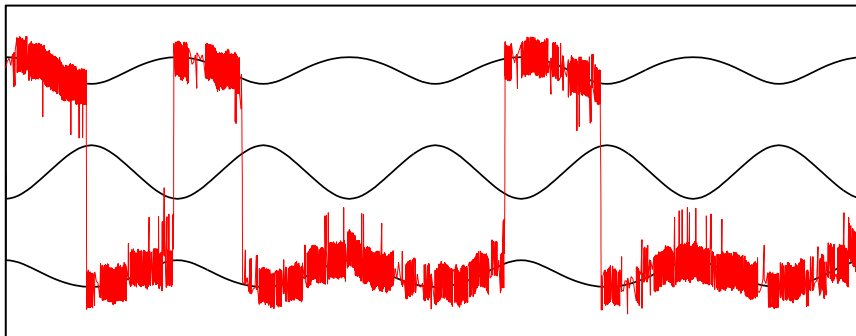
Noise intensity σ



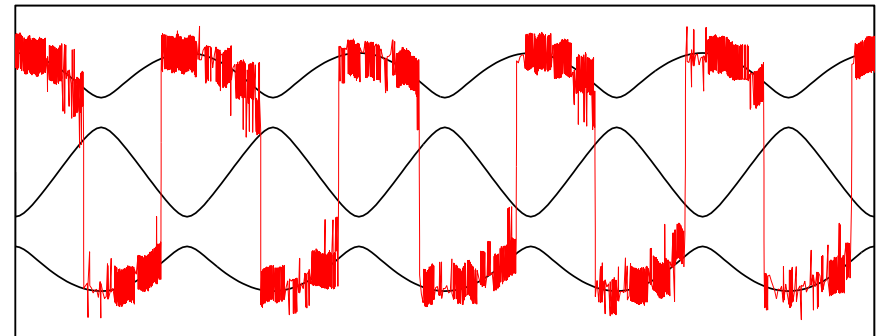
$A = 0.00, \sigma = 0.30, \varepsilon = 0.001$



$A = 0.10, \sigma = 0.27, \varepsilon = 0.001$



$A = 0.24, \sigma = 0.20, \varepsilon = 0.001$



$A = 0.35, \sigma = 0.20, \varepsilon = 0.001$

Different parameter regimes and stochastic resonance

Synchronisation I

- ▷ For matching time scales:
 $2\pi/\varepsilon = T_{\text{forcing}} = 2 T_{\text{Kramers}} \asymp e^{2H/\sigma^2}$
- ▷ Quasistatic approach: Transitions twice per period likely (physics' literature; [Freidlin '00], [Imkeller *et al*, since '02])
- ▷ Requires **exponentially long forcing periods**

Synchronisation II

- ▷ For intermediate forcing periods: $T_{\text{relax}} \ll T_{\text{forcing}} \ll T_{\text{Kramers}}$ and **close-to-critical** forcing amplitude: $A \approx A_c$
- ▷ Transitions twice per period with high probability
- ▷ Subtle dynamical effects: **Effective barrier heights** [Berglund & G '02]

SR outside synchronisation regimes

- ▷ Only occasional transitions
- ▷ But transition times localised within forcing periods
- ▷ **Cycling**: Distribution of first-exit locations doesn't converge ([Day '92], [Maier & Stein '96], [Berglund & G '04], [Berglund & G '05])

Synchronisation regime II

Characterised by 3 small parameters:

$$0 < \sigma \ll 1, \quad 0 < \varepsilon \ll 1, \quad 0 < a_0 \ll 1$$

Recall: Motion of a Brownian particle

$$dx_s = -\frac{\partial}{\partial x} V(x_s, \varepsilon s) ds + \sigma dW_s$$

$$V(x, \varepsilon s) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + (\lambda_c - a_0) \cos(2\pi\varepsilon s)x, \quad \lambda_c = \frac{2}{3\sqrt{3}}$$

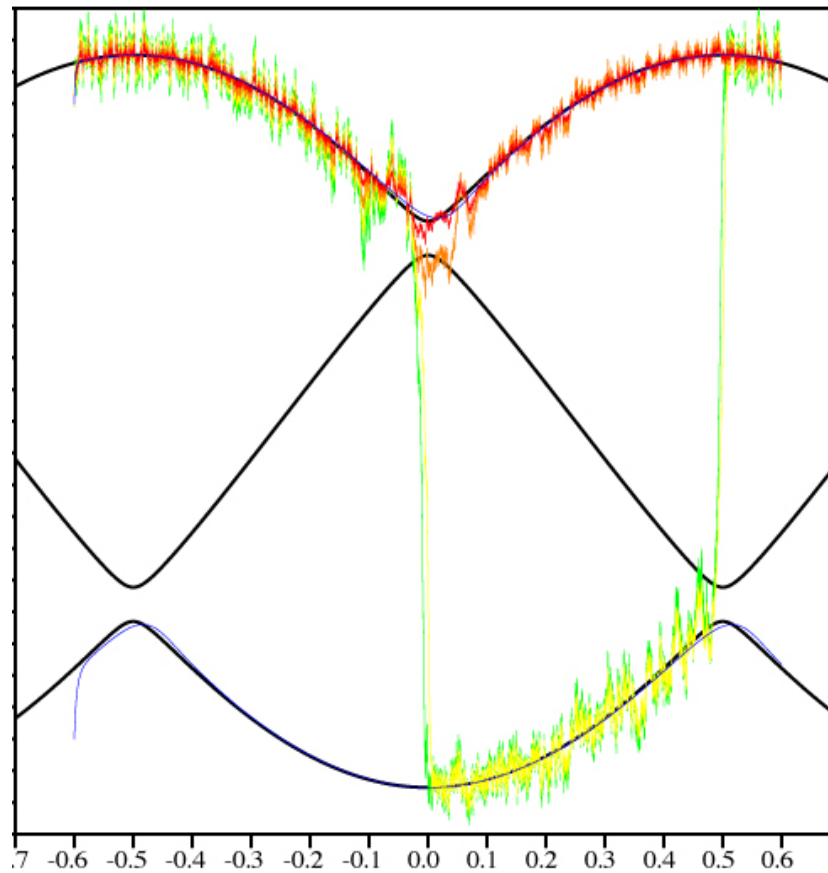
Rewrite in slow time $t = \varepsilon s$:

$$dx_t = \frac{1}{\varepsilon} f(x_t, t) dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t$$

with drift term

$$f(x, t) = -\frac{\partial}{\partial x} V(x, t) = x - x^3 - (\lambda_c - a_0) \cos(2\pi t)$$

Small-barrier-height regime



System Stochastic resonance

Epsilon	0.005	0.005	0.005	0.005	0.005
Sigma	0	0.03	0.06	0.09	0.12
Gap	0.005	0.005	0.005	0.005	0.005

Time step 0.001
Seeds 0.534154541 0.355564852

Effective barrier heights and scaling of small parameters

Theorem [Berglund & G, Annals of Appl. Probab. '02]

(informal version; exact formulation uses first-exit times from space–time sets)

$$\exists \text{ threshold value } \sigma_c = (a_0 \vee \varepsilon)^{3/4}$$

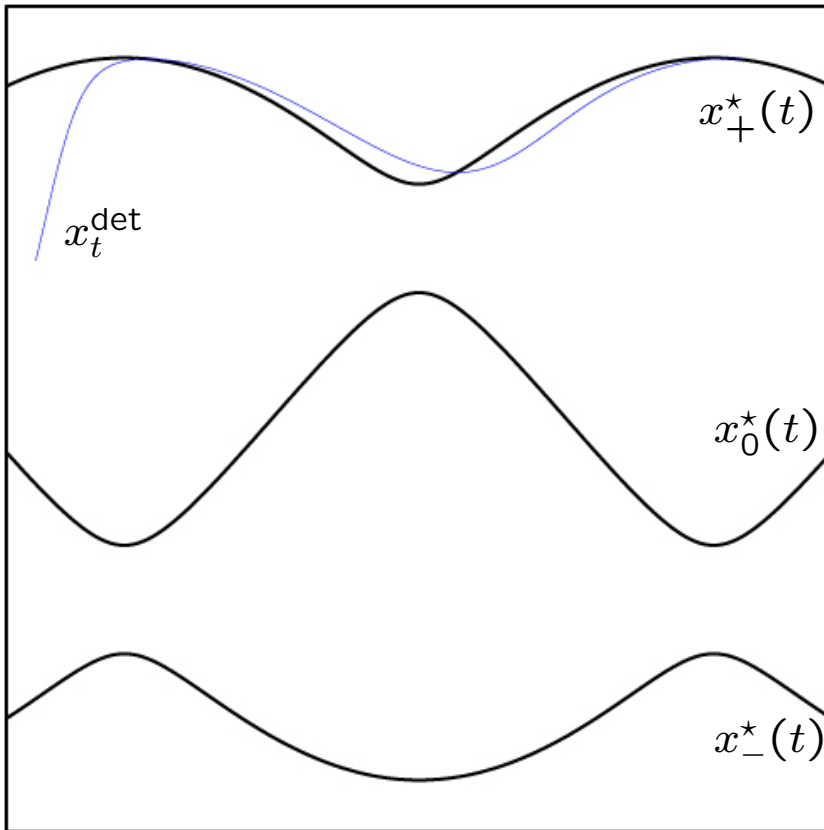
Below: $\sigma \leq \sigma_c$

- ▷ Transitions unlikely
- ▷ Sample paths concentrated in one well
- ▷ Typical spreading $\asymp \frac{\sigma}{(|t|^2 \vee a_0 \vee \varepsilon)^{1/4}} \asymp \frac{\sigma}{(\text{curvature})^{1/2}}$
- ▷ Probability to observe a transition $\leq e^{-const \sigma_c^2 / \sigma^2}$

Above: $\sigma \gg \sigma_c$

- ▷ 2 transitions per period likely (back and forth)
- ▷ with probability $\geq 1 - e^{-const \sigma^{4/3} / \varepsilon |\log \sigma|}$
- ▷ Transitions occur near instants of minimal barrier height;
- ▷ Transition window $\asymp \sigma^{2/3}$

Step 1: Deterministic dynamics



- ▷ For $t \leq -const$:
 x_t^{det} reaches ε -nbhd of $x_+^*(t)$
in time $\asymp \varepsilon |\log \varepsilon|$ (Tihonov '52)
- ▷ For $-const \leq t \leq -(a_0 \vee \varepsilon)^{1/2}$:
 $x_t^{\text{det}} - x_+^*(t) \asymp \varepsilon/|t|$
- ▷ For $|t| \leq (a_0 \vee \varepsilon)^{1/2}$:
 $x_t^{\text{det}} - x_0^*(t) \asymp (a_0 \vee \varepsilon)^{1/2} \geq \sqrt{\varepsilon}$
(effective barrier height)
- ▷ For $(a_0 \vee \varepsilon)^{1/2} \leq t \leq +const$:
 $x_t^{\text{det}} - x_+^*(t) \asymp -\varepsilon/|t|$
- ▷ For $t \geq +const$:
 $|x_t^{\text{det}} - x_+^*(t)| \asymp \varepsilon$

Step 2: Below threshold $\sigma \leq \sigma_c = (a_0 \vee \varepsilon)^{3/4}$

Behaviour of $y_t = x_t - x_t^{\text{det}}$?

Linearizing the drift coefficient \longrightarrow nonautonomous linear SDE

$$dy_t^0 = \frac{1}{\varepsilon} a(t) y_t^0 dt + \frac{\sigma}{\sqrt{\varepsilon}} dW_t, \quad y_0 = 0$$

$$a(t) = \partial_x f(x_t^{\text{det}}, t) = \text{curvature}; \quad \alpha(t, s) := \int_s^t a(u) du$$

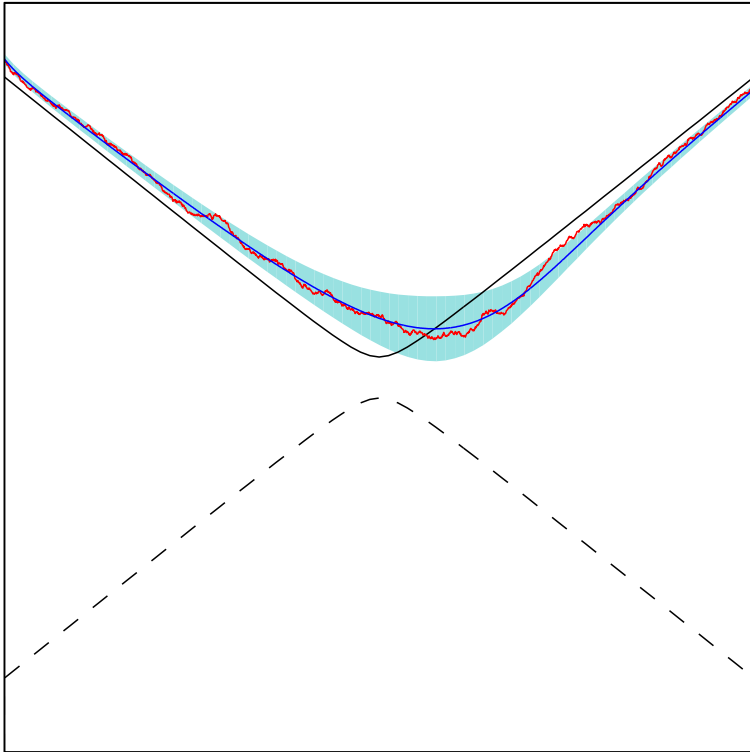
Solution $y_t^0 = \frac{\sigma}{\sqrt{\varepsilon}} \int_0^t e^{\alpha(t,s)/\varepsilon} dW_s$ is a Gaussian process

$$\text{Variance } v(t) = \frac{\sigma^2}{\varepsilon} \int_0^t e^{2\alpha(t,s)/\varepsilon} ds \sim \frac{\sigma^2}{\text{curvature}}$$

Concentration result for y_t^0 : $\mathbb{P}\{|y_t^0| > \delta\} \leq e^{-\delta^2/2v(t)}$

Aim: Analogous resultat for the whole sample path $\{y_t\}_{t \geq 0}$

Step 2: Below threshold $\sigma \leq \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



$$v(t) \sim \frac{\sigma^2}{\text{curvature}} \sim \frac{\sigma^2}{(|t|^2 \vee a_0 \vee \varepsilon)^{1/2}}$$

$$\zeta(t) := \frac{v(t)}{\sigma^2}$$

$$\mathcal{B}(h) := \left\{ (x, t) : |x - x_t^{\text{det}}| < h\sqrt{\zeta(t)} \right\}$$

$\tau_{\mathcal{B}(h)}$ = first-exit time of (x_t, t) from $\mathcal{B}(h)$

Step 2: Below threshold $\sigma \leq \sigma_c = (a_0 \vee \varepsilon)^{3/4}$

Theorem ([Berglund & G '02], [Berglund & G '05])

$\exists h_0, c_1, c_2, c_3 > 0 \quad \forall h \leq h_0$

$$C(h/\sigma, t, \varepsilon) e^{-\kappa_- h^2/2\sigma^2} \leq \mathbb{P}\{\tau_{\mathcal{B}(h)} < t\} \leq C(h/\sigma, t, \varepsilon) e^{-\kappa_+ h^2/2\sigma^2}$$

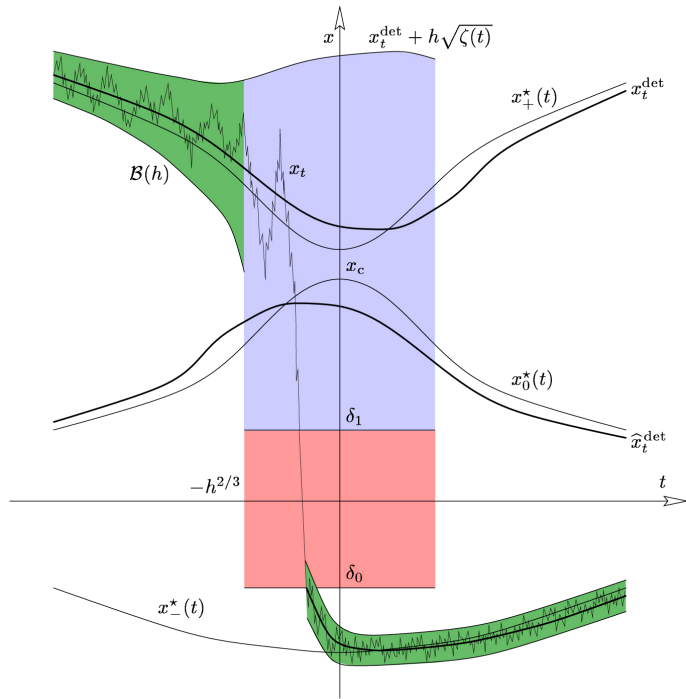
with $\kappa_+ = 1 - c_1 h$, $\kappa_- = 1 + c_1 h + c_1 e^{-c_2 t/\varepsilon}$;

$$C(h/\sigma, t, \varepsilon) = \sqrt{\frac{2}{\pi}} \frac{|\alpha(t)|}{\varepsilon} \frac{h}{\sigma} \left[1 + \mathcal{O}\left(\frac{\sigma}{h}\right) + \frac{t}{\varepsilon} e^{-c_3 h^2/\sigma^2} + e^{-c_1 t/\varepsilon} + \varepsilon \right]$$

Basic idea

local approximation of y_t by y_t^0 ; Gaussian process is a rescaled Brownian motion; results on the density of the first-passage time for BM through nonlinear curves

Step 3: Above threshold $\sigma \gg \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



Idea of the proof

With probability $\geq \delta > 0$, in time $\asymp \varepsilon |\log \sigma| / \sigma^{2/3}$, the path reaches

- ▷ x_t^{det} if above
- ▷ then the saddle
- ▷ finally the level δ_1

In time $\sigma^{2/3}$ there are $\frac{\sigma^{4/3}}{\varepsilon |\log \sigma|}$ attempts possible

During a subsequent time span of length ε , level δ_0 is reached (with probability $\geq \delta$)

Finally, the path reaches the new well

Result

$$\mathbb{P}\left\{x_s > \delta_0 \quad \forall s \in [-\sigma^{2/3}, t]\right\} \leq e^{-\text{const} \sigma^{4/3} / \varepsilon |\log \sigma|} \quad (t \geq -\gamma \sigma^{2/3}, \gamma \text{ small})$$

General slow–fast systems

Fully coupled SDEs on well-separated time scales

$$\begin{cases} dx_t = \frac{1}{\varepsilon} f(x_t, y_t) dt + \frac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t) dW_t & \text{(fast variables } \in \mathbb{R}^n) \\ dy_t = g(x_t, y_t) dt + \sigma' G(x_t, y_t) dW_t & \text{(slow variables } \in \mathbb{R}^m) \end{cases}$$

- ▷ $\{W_t\}_{t \geq 0}$ k -dimensional (standard) Brownian motion
- ▷ $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$
- ▷ $f : \mathcal{D} \rightarrow \mathbb{R}^n$, $g : \mathcal{D} \rightarrow \mathbb{R}^m$ drift coefficients, $\in \mathcal{C}^2$
- ▷ $F : \mathcal{D} \rightarrow \mathbb{R}^{n \times k}$, $G : \mathcal{D} \rightarrow \mathbb{R}^{m \times k}$ diffusion coefficients, $\in \mathcal{C}^1$

Small parameters

- ▷ $\varepsilon > 0$ adiabatic parameter (*no quasistatic* approach)
- ▷ $\sigma, \sigma' \geq 0$ noise intensities; may depend on ε :
 $\sigma = \sigma(\varepsilon)$, $\sigma' = \sigma'(\varepsilon)$ and $\sigma'(\varepsilon)/\sigma(\varepsilon) = \varrho(\varepsilon) \leq 1$

Near slow manifolds: Assumptions on the fast variables

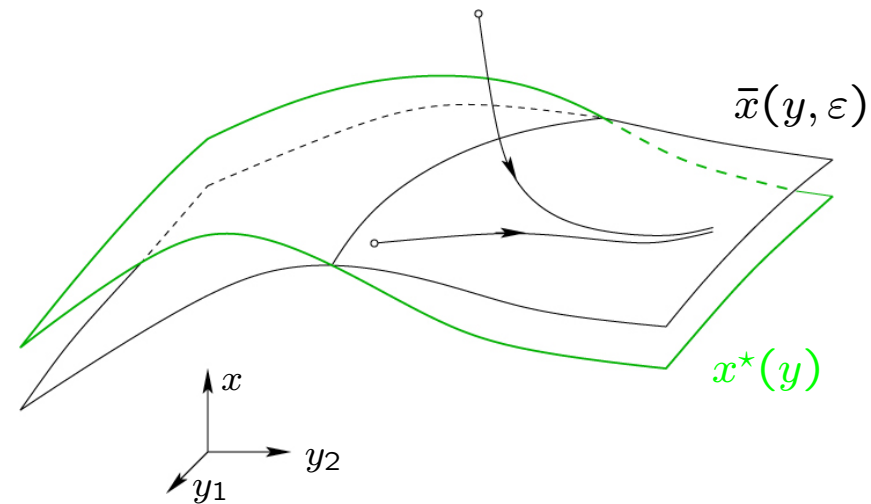
Existence of a slow manifold: $\exists \mathcal{D}_0 \subset \mathbb{R}^m \quad \exists x^* : \mathcal{D}_0 \rightarrow \mathbb{R}^n$
s.t. $(x^*(y), y) \in \mathcal{D}$ and $f(x^*(y), y) = 0$ for $y \in \mathcal{D}_0$

Slow manifold is attracting: Eigenvalues of $A^*(y) := \partial_x f(x^*(y), y)$
satisfy $\text{Re } \lambda_i(y) \leq -a_0 < 0$, uniformly in \mathcal{D}_0

Theorem ([Tihonov '52], [Fenichel '79])

There exists an *adiabatic manifold*:
 $\exists \bar{x}(y, \varepsilon)$ s.t.

- ▷ $\bar{x}(y, \varepsilon)$ is invariant manifold for deterministic dynamics
- ▷ $\bar{x}(y, \varepsilon)$ attracts nearby solutions
- ▷ $\bar{x}(y, 0) = x^*(y)$ and $\bar{x}(y, \varepsilon) = x^*(y) + \mathcal{O}(\varepsilon)$



Consider now *stochastic system* under these assumptions

Typical neighbourhoods of adiabatic manifolds

- ▶ Consider deterministic process $(x_t^{\text{det}} = \bar{x}(y_t^{\text{det}}, \varepsilon), y_t^{\text{det}})$ on (invariant) adiabatic manifold
- ▶ Dev. $\xi_t := x_t - x_t^{\text{det}}$ of **fast** variables from adiabatic manifold
- ▶ Linearize SDE for ξ_t ; resulting process ξ_t^0 is Gaussian

Key observation

$\frac{1}{\sigma^2} \text{Cov } \xi_t^0$ is a particular sol. of the det. slow-fast system

$$\begin{cases} \varepsilon \dot{X}(t) = A(y_t^{\text{det}})X(t) + X(t)A(y_t^{\text{det}})^\top + F_0(y_t^{\text{det}})F_0(y_t^{\text{det}})^\top \\ \dot{y}_t^{\text{det}} = g(\bar{x}(y_t^{\text{det}}, \varepsilon), y_t^{\text{det}}) \end{cases}$$

with $A(y) = \partial_x f(\bar{x}(y, \varepsilon), y)$ and F_0 0th-order approximation to F

- ▶ System admits an adiabatic manifold $\bar{X}(y, \varepsilon)$

Typical neighbourhoods

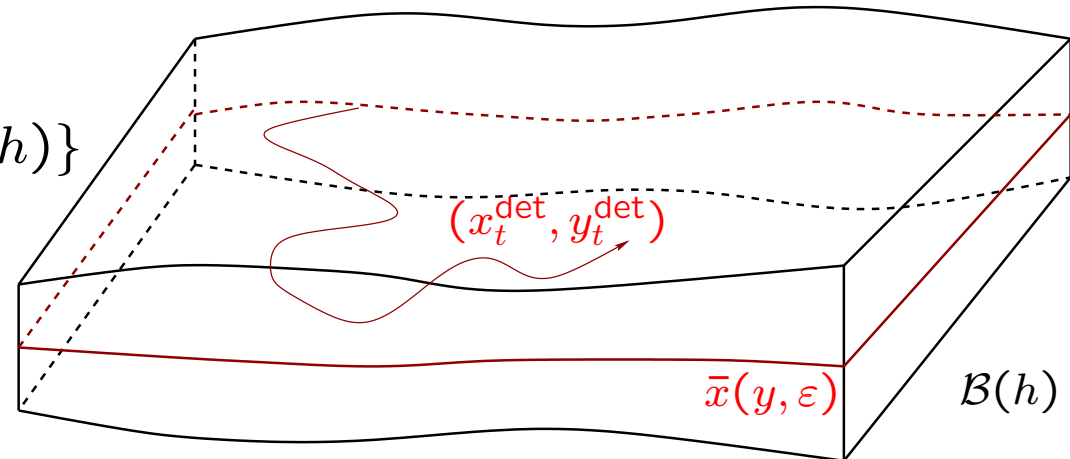
$$\mathcal{B}(h) := \left\{ (x, y) : \left\langle \left[x - \bar{x}(y, \varepsilon) \right], \bar{X}(y, \varepsilon)^{-1} \left[x - \bar{x}(y, \varepsilon) \right] \right\rangle < h^2 \right\}$$

Concentration of sample paths near adiabatic manifolds

Define (random) first-exit times

$$\tau_{\mathcal{D}_0} := \inf\{s > 0 : y_s \notin \mathcal{D}_0\}$$

$$\tau_{\mathcal{B}(h)} := \inf\{s > 0 : (x_s, y_s) \notin \mathcal{B}(h)\}$$



Theorem [Berglund & G, J. Differential Equations, 2003]

Assume: $\|\bar{X}(y, \varepsilon)\|, \|\bar{X}(y, \varepsilon)^{-1}\|$ uniformly bounded in \mathcal{D}_0

Then: $\exists \varepsilon_0 > 0 \quad \exists h_0 > 0 \quad \forall \varepsilon \leq \varepsilon_0 \quad \forall h \leq h_0$

$$\mathbb{P}\{\tau_{\mathcal{B}(h)} < \min(t, \tau_{\mathcal{D}_0})\} \leq C_{n,m}(t) \exp\left\{-\frac{h^2}{2\sigma^2} [1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)]\right\}$$

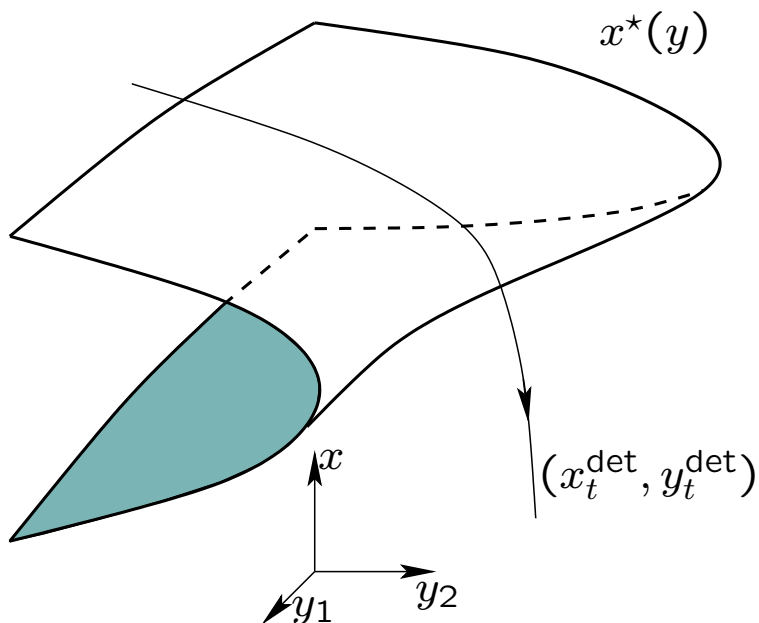
where $C_{n,m}(t) = [C^m + h^{-n}] \left(1 + \frac{t}{\varepsilon^2}\right)$

Bifurcations

Question

What happens if (x_t, y_t) approaches a bifurcation point (\hat{x}, \hat{y}) for the deterministic dynamics?

Ex.: Saddle–node bifurcation



General approach

- ▷ Apply centre-manifold theorem
- ▷ Concentration results for deviation from centre manifold ([Berglund & G, 2003])
- ▷ Consider reduced dynamics on centre manifold
- ▷ Concentration results for deviation of reduced system from original variables [Berglund & G, 2003]